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Lipschitz – Orlicz Spaces and the Laplace Equation

By A.G. AKSOY of Claremont and L. MALIGRANDA of Luleå

Dedicated to Professor LARS ERIK PERSSON *on the Occasion of his* 50^{th} *birthday*

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Abstract. STEIN and TAIBLESON gave a characterization for $f \in L_p(\mathbb{R}^n)$ to be in the spaces $Lip(\alpha, L_p)$ and $Zyg(\alpha, L_p)$ in terms of their Poisson integrals. In this paper we extend their results to Lipschitz-Orlicz spaces $Lip(\varphi, L_M)$ and Zygmund-Orlicz spaces $Zyg(\varphi, L_M)$ and to the general function $\varphi \in P$ instead of the power function $\varphi(t) = t^\alpha$. Such results describe the behavior of the Laplace equation in terms of the smoothness property of differences of f in Orlicz spaces $L_M(\mathbb{R}^n)$. More general spaces $\Lambda^k(\varphi, X, q)$ are also considered.

1. Introduction

The Poisson integral can be used to express the solution of the Dirichlet problem for the half-space $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}$: Let $f \in L_p(\mathbb{R}^n)$. Find a function $u(x,y)$ on $\mathbb{R}^{n+1}_+$ which is the solution of the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial y^2} + \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \quad (x \in \mathbb{R}^n,\ y > 0),$$

whose boundary values on $\mathbb{R}^n$ are $f(x)$. More precisely, if $f \in L_p(\mathbb{R}^n)$, then the Poisson integral of $f(x)$ is defined in $\mathbb{R}^{n+1}_+$ by

$$\begin{aligned} u(x,y) &= f(x) * P(x,y) = \int_{\mathbb{R}^n} f(x-z)\, P(z,y)\, dz \\ &= \int_{\mathbb{R}^n} f(z)\, P(x-z,y)\, dz\,. \end{aligned}$$

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The Poisson integral of $f(x)$ is the convolution of $f(x)$ with the Poisson kernel $P(x,y)$, which is defined by

$$P(x,y) = \frac{1}{c_n} \cdot \frac{y}{\left(|x|^2 + y^2\right)^{(n+1)/2}} \quad (x \in \mathbb{R}^n, \; y > 0),$$

where $c_n = \pi^{(n+1)/2}/\Gamma((n+1)/2)$ is chosen so that $\int_{\mathbb{R}^n} P(x,y)\,dx = 1$ for each $y > 0$. STEIN [19] and TAIBLESON [20] gave a characterization for f to be in the spaces $Lip(\alpha, L_p)$ and $Zyg(\alpha, L_p)$ in terms of their Poissin integrals. Such results correlate the smoothness properties of functions from L_p with the behavior of the solution of the Laplace equation. In their discussion they follow the earlier work of HARDY and LITTLEWOOD on the periodic spaces $Lip(\alpha, L_p^{2\pi})$, and of ZYGMUND [24] on 2π-periodic smooth functions $Zyg(\alpha, L_p^{2\pi})$ (c.f. BUTZER-BERENS [3]). It should also be mentioned that TAIBLESON's paper [20] includes a discussion of the Laplace equation as well as the heat equation to be in $L_p(\mathbb{R}^n)$-spaces. There are many papers which investigate $Lip(\alpha, L_p)$ in other directions, like Lorentz spaces L_{pq} instead of L_p-spaces or $Lip(\alpha, L_p)$ for negative α or $\Lambda(\alpha, p, q)$-spaces or $\Lambda^k(\varphi, X, q)$-spaces (see FLETT [5], HERZ [6], JANSON [7], JONES [8], PEETRE [16], STEIN [19], TAIBLESON [20] and TRIEBEL [21], [22]). These papers contain the problems of the duality, the equivalent norms and the interpolation spaces by the real and complex methods.

The purpose of this paper is to obtain the STEIN-TAIBLESON results for the Lipschitz-Orlicz spaces $Lip(\varphi, L_M)$ and the Zygmund-Orlicz spaces $Zyg(\varphi, L_M)$, with a general functions φ instead of the power function $\varphi(t) = t^\alpha$. A very rough description of the result would be that the derivative or the second derivative of a solution of the Laplace equation has a particular property if and only if f has a very precise smoothness property describable in terms of differences of f in the Orlicz spaces $L_M(\mathbb{R}^n)$.

The Orlicz space

$$L_M = L_M(\mathbb{R}^n) = \{f \in L_0(\mathbb{R}^n) \text{ such that }$$
$$I_M(\lambda f) := \int_{\mathbb{R}^n} M(\lambda\,|f(x)|\,)\,dx < \infty \quad \text{for some} \quad \lambda > 0\}$$

is a Banach space with the Luxemburg-Nakano norm

$$||f||_M = \inf\{\lambda > 0 : I_M(f/\lambda) \le 1\},$$

where $L^0(\mathbb{R}^n)$ denotes the space of all (equivalence classes of) Lebesgue measurable real functions on $\mathbb{R}^n$ and $M : [0,\infty) \to [0,\infty)$ is a Young function, i.e., a convex nondecreasing function vanishing at zero (not identically 0 or ∞ on $(0,\infty)$) (see [9], [13], [17]).

Let P be the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which are continuous nondecreasing and zero only at 0. For $\varphi \in P$, let us consider the Lipschitz-Orlicz space:

$$Lip(\varphi, L_M) = \{f \in L_M(\mathbb{R}^n) \text{ such that }$$
$$||f(x+h) - f(x)||_M \le C\varphi(|h|) \quad \text{for all} \quad |h| > 0\},$$

and the Zygmund-Orlicz space

$$Zyg(\varphi, L_M) = \{f \in L_M(\mathbb{R}^n) \text{ such that }$$
$$||f(x+h) + f(x-h) - 2f(x)||_M \le D\varphi(|h|) \text{ for all } |h| > 0\}.$$

Both spaces $Lip(\varphi, L_M)$ and $Zyg(\varphi, L_M)$ are Banach spaces with the norms

$$||f||_M + \sup_{|h|>0} \frac{||f(x+h) - f(x)||_M}{\varphi(|h|)}$$

and

$$||f||_M + \sup_{|h|>0} \frac{||f(x+h) + f(x-h) - 2f(x)||_M}{\varphi(|h|)},$$

respectively. Clearly $Lip(\varphi, L_M) \subset Zyg(\varphi, L_M)$.

We will need to put restrictions on the growth of the function $\varphi \in P$. For $\varphi \in P$ we can define the so called indices of φ (cf. [10], [12], [13]):

$$\alpha_\varphi = \lim_{t\to 0} \frac{\ln s_\varphi(t)}{\ln t}, \quad \beta_\varphi = \lim_{t\to\infty} \frac{\ln s_\varphi(t)}{\ln t},$$

where

$$s_\varphi(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)}.$$

Obviously $0 \le \alpha_\varphi \le \beta_\varphi$ for $\varphi \in P$. For the power function $\varphi(t) = t^a$ we have $\alpha_\varphi = \beta_\varphi = a$.

This paper is organized as follows. In Section 2 we characterize the functions from the Lipschitz class $Lip(\varphi, L_M)$ in terms of the derivatives of their Poisson integrals. In Section 3 a similar characterization is given for the Zygmund class $Zyg(\varphi, L_M)$. In Section 4 we consider the more general spaces $\Lambda^k(\varphi, X, q)$, $k = 1, 2$, and prove some results about them. For example, for $0 < \alpha_\varphi \le \beta_\varphi < k$, $(k = 1, 2)$, $f \in \Lambda^k(\varphi, X, q)$ if and only if the solution u of the Laplace equation satisfies

$$\frac{y^k}{\varphi(y)} \left|\left| \frac{\partial^k u(x,y)}{\partial y^k} \right|\right|_X \in L_q\left((0,\infty), \frac{dy}{y}\right).$$

This section also contain some additional remarks.

2. The Lipschitz Condition

In the proof of the main theorem of this section we will need the following equivalence property between indices and integrals of $\varphi \in P$ (the proof of these equivalences can be found in [10], [12] or in [13], Th. 11.8):

Let $\varphi \in P$, $s_\varphi(t) < \infty$ for every $t > 0$, and $r > 0$. Then

$$\text{(2.1)} \qquad \alpha_\varphi > 0 \quad \text{if and only if} \quad \int_0^t \frac{\varphi(s)}{s}\, ds \le A\varphi(t) \quad \text{for all} \quad t > 0,$$

and

$$\text{(2.2)} \qquad \beta_\varphi < r \quad \text{if and only if} \quad \int_0^\infty \frac{\varphi(s)}{s^{r+1}}\, ds \le B\frac{\varphi(t)}{t^r} \quad \text{for all} \quad t > 0.$$

Theorem 2.1. *Let $f \in L_M(\mathbb{R}^n)$ and $u(x,y) = \int_{\mathbb{R}^n} f(x-z)\,P(z,y)\,dz$ be its Poisson integral. If $0 < \alpha_\varphi \le \beta_\varphi < 1$, then the following are equivalent:*

(i) $f \in Lip(\varphi, L_M)$,

(ii) $\left\| \frac{\partial}{\partial y} u(x,y) \right\|_M \le C\varphi(y)/y$ *for all* $y > 0$,

(iii) $\|u(x,y) - f(x)\|_M \le D\varphi(y)$ *for all* $y > 0$.

Proof. (i) $\Longrightarrow$ (ii). Let $f \in Lip(\varphi, L_M)$. Since

$$\begin{aligned}\frac{\partial P(x,y)}{\partial y} &= \frac{1}{c_n} \cdot \frac{|x|^2 - ny^2}{(|x|^2+y^2)^{(n+3)/2}} \\ &= \frac{1}{c_n}\left[\frac{1}{(|x|^2+y^2)^{(n+1)/2}} - \frac{(n+1)y^2}{(|x|^2+y^2)^{(n+3)/2}}\right],\end{aligned}$$

it follows that

$$\int_{\mathbb{R}^n} \frac{\partial P(x,y)}{\partial y}\,dx = 0,$$

$$\left|\frac{\partial P(x,y)}{\partial y}\right| \le \frac{n}{c_n} \min\left\{|x|^{-n-1}, y^{-n-1}\right\}$$

and

$$\left|\frac{\partial P(x,y)}{\partial y}\right| \le n\,\frac{P(x,y)}{y}.$$

Then, since the integral defining the convolution converges absolutely, we can write

$$\begin{aligned}\frac{\partial}{\partial y} u(x,y) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial y} P(z,y)\,f(x-z)\,dz \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial y} P(z,y)\,[f(x-z) - f(x)]\,dz\end{aligned}$$

and by the generalized Minkowski inequality (cf. [10], pp. 45 – 46)

$$\begin{aligned}\left\|\frac{\partial}{\partial y} u(x,y)\right\|_M &\le \int_{\mathbb{R}^n} \left|\frac{\partial}{\partial y} P(z,y)\right| \|f(x-z) - f(x)\|_M\,dz \\ &\le \frac{n}{c_n} y^{-n-1} \int_{|z|\le y} \|f(x-z) - f(x)\|_M\,dz \\ &\quad + \frac{n}{c_n} \cdot \int_{|z|>y} |z|^{-n-1} \|f(x-z) - f(x)\|_M\,dz.\end{aligned}$$

Next, set $z = r\xi \in \mathbb{R}^n$ with $r = |z|$ and $|\xi| = 1$. Then, with

$$\|f(x-z) - f(x)\|_M = \omega_M(z) = \omega_M(r\xi)$$

and

$$\Omega(r) = \int_{S^{n-1}} \omega_M(r\xi)\,d\sigma(\xi),$$

the inequality above becomes (because $dz = d\xi\, r^{n-1}\, dr$)

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M \le \frac{n}{c_n}\left[y^{-n-1}\int_0^y \Omega(r)\, r^{n-1}\, dr + \int_y^\infty \Omega(r)\, r^{-2}\, dr\right].$$

The assumption that $f \in Lip(\varphi, L_M)$ gives

$$\Omega(r) \le C\varphi(r)\,\sigma(S^{n-1}) = 2c_{n-1}C\varphi(r),$$

and by the assumption $\beta_\varphi < 1$, in the equivalent form (2.2), we obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M &\le \frac{2nCc_{n-1}}{c_n}\left[y^{-n-1}\int_0^y \varphi(r)\, r^{n-1}\, dr + \int_y^\infty \varphi(r)\, r^{-2}\, dr\right]\\ &\le \frac{2nCc_{n-1}}{c_n}\left[\frac{1}{n}\cdot\frac{\varphi(y)}{y} + B\frac{\varphi(y)}{y}\right]\\ &\le C_1\frac{\varphi(y)}{y}.\end{aligned}$$

P r o o f . (ii) $\Longrightarrow$ (iii). For a. e. $x \in \mathbb{R}^n$, it yields that

$$u(x,y) - f(x) = \int_0^y \frac{\partial u(x,s)}{\partial s}\, ds.$$

By the generalized Minkowski inequality and the assumption $\alpha_\varphi > 0$, in the equivalent form (2.1), we obtain

$$\begin{aligned}\|u(x,y) - f(x)\|_M &\le \int_0^y \left\|\frac{\partial u(x,s)}{\partial s}\right\|_M ds\\ &\le C\int_0^y \frac{\varphi(s)}{s}\, ds\\ &\le CA\varphi(y).\end{aligned}$$

P r o o f . (iii) $\Longrightarrow$ (ii). First, note that

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M = \lim_{m\to\infty}\left\|\frac{\partial}{\partial y}[u(x,y) - u(x,2^m y)]\right\|_M.$$

Thus, using the fact that the convolution operator is bounded from $L_M(\mathbb{R}^n) \times L_1(\mathbb{R}^n)$ into $L_M(\mathbb{R}^n)$ with norm less or equal than 1 (cf. Lemma 4.1), the above property of the Poisson kernel and the assumption $\beta_\varphi < 1$, in the equivalent form (2.2), we obtain

$$\begin{aligned}
&\left\|\frac{\partial}{\partial y}[u(x,y) - u(x,2^m y)]\right\|_M \\
&\le \sum_{k=1}^{m} \left\|\frac{\partial}{\partial y}\left[u(x,2^{k-1}y) - u(x,2^k y)\right]\right\|_M \\
&= \sum_{k=1}^{m} \left\|[f(x) - u(x,2^{k-1}y)] * \frac{\partial}{\partial y}(P,2^{k-1}y)\right\|_M \\
&\le \sum_{k=1}^{m} \|f(x) - u(x,2^{k-1}y)\|_M \left\|\frac{\partial}{\partial y}P(x,2^{k-1}y)\right\|_1 \\
&\le n\sum_{k=1}^{m} \{\|f(x) - u(x,2^{k-1}y)\|_M \ \|P(x,2^{k-1}y)\|_1 / (2^{k-1}y)\} \\
&\le n\sum_{k=1}^{m} \|f(x) - u(x,2^{k-1}y)\|_M / (2^{k-1}y) \\
&\le nD\sum_{k=1}^{m} \varphi(2^{k-1}y)/(2^{k-1}y) \ \le\ 2nD\sum_{k=1}^{m}\int_{2^{k-1}y}^{2^k y} \varphi(s)\, s^{-2}\, ds \\
&\le\ 2nD\int_y^\infty \varphi(s)\, s^{-2}\, ds \le 2nDB\varphi(y)/y\,.
\end{aligned}$$

Proof. (ii) $\Longrightarrow$ (i). First, we prove the following lemma.

Lemma 2.2. *Let $f \in L_M(\mathbb{R}^n)$ and $u(x,y) = \int_{\mathbb{R}^n} f(x-z)\,P(z,y)\,dz$ be its Poisson integral. If $0 < \alpha_\varphi \le \beta_\varphi < 1$, then for all $y > 0$*

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M \ \le\ C\varphi(y)/y \tag{2.3}$$

if and only if for all $y > 0$ and for each $i = 1, 2, \ldots, n$

$$\left\|\frac{\partial}{\partial x_i}u(x,y)\right\|_M \ \le\ C'\varphi(y)/y\,. \tag{2.4}$$

The smallest C in (2.3) is comparable to the smallest C' in (2.4).

Proof of Lemma 2.2. First we prove that if y_1, $y_2 > 0$, then

$$u(x,y_1+y_2) \ =\ u(x,y_2) * P(x,y_1) \tag{2.5}$$

and

$$\frac{\partial^2 u}{\partial y \partial x_i}(x,y_1+y_2) \ =\ \frac{\partial u}{\partial y}(x,y_2) * \frac{\partial P}{\partial x_i}(x,y_1)\,, \quad i = 1,\ 2,\ \ldots,\ n\,. \tag{2.6}$$

Since

$$\int_{\mathbb{R}^n} P(x,y_1+y_2)^p\,dx \ \le\ C_2 y_1^{-(p-1)n} \quad (p\ >\ 1)$$

for all $y_2 > 0$, it follows that the Poisson kernel $P(x, y_1 + y_2)$ has the Poisson integral in $\mathbb{R}^n \times (y_1, \infty)$

$$P(x, y_1 + y_2) = \int_{\mathbb{R}^n} P(s, y_1) P(s - z, y_2)\, ds = P(x, y_1) * P(x, y_2).$$

Then

$$\begin{aligned} u(x, y_1 + y_2) &= f(x) * P(x, y_1 + y_2) = \int_{\mathbb{R}^n} f(x)\, P(x - z, y_1 + y_2)\, dz \\ &= \int_{\mathbb{R}^n} f(z) \int_{\mathbb{R}^n} P(s, y_1)\, P(x - z - s, y_2)\, ds\, dz \\ &= \int_{\mathbb{R}^n} f(z) \int_{\mathbb{R}^n} P(x - t, y_1)\, P(t - z, y_2)\, dt\, dz \\ &= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(z)\, P(t - z, y_2)\, dz \right\} P(x - t, y_1)\, dt \\ &= \int_{\mathbb{R}^n} u(t, y_2)\, P(x - t, y_1)\, dt, \end{aligned}$$

that is, $u(x, y_1 + y_2) = u(x, y_2) * P(x, y_1)$, and the equality (2.5) is proved.

For fixed $y_1 > 0$, we have, according to the equality (2.5),

$$u(x, y_1 + y) = u(x, y) * P(x, y_1).$$

Differentiating we obtain

$$\frac{\partial}{\partial y} u(x, y + y_1) = \frac{\partial}{\partial y} u(x, y) * P(x, y_1),$$

which can be expressed as

$$\frac{\partial}{\partial y} u(x, y_1 + y_2) = \frac{\partial u}{\partial y}(x, y_1) * P(x, y_2) = \int_{\mathbb{R}^n} \frac{\partial}{\partial y} u(z, y_1)\, P(x - z, y_2)\, dz.$$

Therefore

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial x_i}(x, y_1 + y_2) &= \int_{\mathbb{R}^n} \frac{\partial u}{\partial y}(z, y_1) \frac{\partial P}{\partial x_i}(x - z, y_2)\, dz \\ &= \frac{\partial u}{\partial y}(x, y_2) * \frac{\partial P}{\partial x_i}(x, y_1), \end{aligned}$$

and also the equality (2.6) is proved. Taking $y_1 = y_2 = y/2$ in the equality (2.6), we obtain

$$\frac{\partial^2 u}{\partial y \partial x_i}(x, y) = \frac{\partial u}{\partial y}(x, y/2) * \frac{\partial P}{\partial x_i}(x, y/2),$$

and so (cf. Lemma 4.1)

$$\left\| \frac{\partial^2 u}{\partial y \partial x_i}(x, y) \right\|_M \le \left\| \frac{\partial u}{\partial y}(x, y/2) \right\|_M \left\| \frac{\partial P}{\partial x_i}(x, y/2) \right\|_1. \tag{2.7}$$

Now, if (2.3) holds, then

$$\left\|\frac{\partial u}{\partial y}(x,y/2)\right\|_M \le 2C\varphi(y/2)/y \le 2C\varphi(y)/y\,.$$

For the Poisson kernel we have

$$\frac{\partial P(x,y)}{\partial x_i} = \frac{1}{c_n}\cdot\frac{-(n+1)x_i y}{(|x|^2+y^2)^{(n+3)/2}}\,,\quad i=1,\,2,\,\ldots,\,n\,,$$

and so

$$\left|\frac{\partial P(x,y)}{\partial x_i}\right| \le \frac{n+1}{2}\cdot\frac{P(x,y)}{y}\quad i=1,\,2,\,\ldots,\,n\,,$$

which means that

$$\left\|\frac{\partial P}{\partial x_i}(x,y/2)\right\|_1 \le (n+1)\left\|\frac{P(x,y/2)}{y}\right\|_1 \le (n+1)\frac{1}{y}\,.$$

Substituting these estimates into (2.5) we obtain

$$\left\|\frac{\partial^2}{\partial y\partial x_i}u(x,y)\right\|_M \le 2C(n+1)\varphi(y)/y^2\,. \tag{2.8}$$

On the other hand, using the fact that the convolution operator is bounded from $L_M(\mathbb{R}^n)\times L_1(\mathbb{R}^n)$ into $L_M(\mathbb{R}^n)$ with the norm less or equal to 1 (cf. Lemma 4.1) and the above property of the Poisson kernel, we obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial x_i}u(x,y)\right\|_M &= \left\|f(x)*\frac{\partial P(x,y)}{\partial x_i}\right\|_M \le \|f(x)\|_M\left\|\frac{\partial P(x,y)}{\partial x_i}\right\|_1\\ &\le \frac{n+1}{2}\cdot\frac{1}{y}\|P(x,y)\|_1\|f(x)\|_M = \frac{n+1}{2}\cdot\frac{1}{y}\|f(x)\|_M\,,\end{aligned}$$

which implies that $\dfrac{\partial}{\partial x_i}u(x,y)\to 0$ as $y\to\infty$. Therefore

$$\frac{\partial}{\partial x_i}u(x,y) = -\int_y^\infty \frac{\partial^2 u}{\partial y\partial x_i}(x,s)\,ds$$

and, by (2.8), (2.2),

$$\begin{aligned}\left\|\frac{\partial}{\partial x_i}u(x,y)\right\|_M &\le \int_y^\infty\left\|\frac{\partial^2 u}{\partial y\partial x_i}(x,s)\right\|_M ds\\ &\le 2C(n+1)\int_y^\infty s^{-2}\varphi(s)\,ds\\ &\le 2BC(n+1)\,\varphi(y)/y\,.\end{aligned}$$

Conversely, if (2.5) holds, then in the same way as before we obtain

$$\left\|\frac{\partial^2 u}{\partial {x_i}^2}\right\|_M \le C_3\varphi(y)/y^2\,,\quad i=1,\,2,\,\ldots,\,n\,.$$

Since u is harmonic, that is,

$$\frac{\partial^2}{\partial y^2}u(x,y) = -\sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2}u(x,y),$$

we therefore have

$$\left\|\frac{\partial^2}{\partial y^2}u(x,y)\right\|_M \le C_3\varphi(y)/y^2,$$

and a similar integration argument then shows that

$$\left\|\frac{\partial u}{\partial y}\right\|_M \le C_4\varphi(y)/y.$$

Proof. (ii) $\Longrightarrow$ (i). Assume that $f \in L_M$ and that the condition (2.3) holds. For $h \in \mathbb{R}^n$ and $0 < y < |h|$ we have

$$\begin{aligned} &u(x+h,y) - u(x,y)\\ = &\int_y^{|h|}\frac{\partial u(x,s)}{\partial s}\,ds + \int_0^{h_1}\frac{\partial u(x_1+z_1,x_2,x_3,\dots,x_n,|h|)}{\partial z_1}\,dz_1\\ + &\int_0^{h_2}\frac{\partial u(x_1+h_1,x_2+z_2,x_3,\dots,x_n,|h|)}{\partial z_2}\,dz_2 + \dots\\ + &\int_0^{h_n}\frac{\partial u(x_1+h_1,x_2+h_2,x_3+h_3,\dots,x_n+z_n,|h|)}{\partial z_n}\,dz_n\\ + &\int_{|h|}^{y}\frac{\partial u(x+h,s)}{\partial s}\,ds, \end{aligned}$$

and so

$$\begin{aligned} \|u(x+h,y)-u(x,y)\|_M \le\ & 2\int_y^{|h|}\left\|\frac{\partial u(x,s)}{\partial s}\right\|_M ds\\ & + \sum_{i=1}^{n}\int_0^{|h_i|}\left\|\frac{\partial u}{\partial x_i}(x,|h|)\right\|_M dz_i. \end{aligned}$$

Using the assumption (2.3), Lemma 2.2 and property (2.1), the last expression becomes less or equal to

$$2C\int_y^{|h|}\frac{\varphi(s)}{s}\,ds + C'\sum_{i=1}^{n}\int_0^{|h_i|}\frac{\varphi(|h|)}{|h|}\,dz_i \le 2AC\varphi(|h|) + C'\frac{\varphi(|h|)}{|h|}\sum_{i=1}^{n}|h_i|,$$

which is less or equal to

$$C_5\varphi(|h|).$$

Now, since $u(x,y) \to f(x)$ for almost all $x \in \mathbb{R}^n$ when $y \to 0^+$, we obtain (by the Fatou Lemma) that $f \in Lip(\varphi, L_M)$. This completes the proof of the theorem. □

Corollary 2.3. *If* $0 < \alpha_\varphi \le \beta_\varphi \le 1$, *then* $Lip(\varphi, L_M) = Zyg(\varphi, L_M)$.

Proof It is enough to prove the imbedding $Zyg(\varphi, L_M) \subset Lip(\varphi, L_M)$. Let $f \in Zyg(\varphi, L_M)$. Then for $u(x,y) = f(x) * P(x,y)$ we have

$$\begin{aligned}\frac{\partial}{\partial y}u(x,y) &= \int_{\mathbb{R}^n} \frac{\partial P}{\partial y}(z,y)\, f(x-z)\, dz \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial P}{\partial y}(z,y)\, [f(x+z) + f(x-z) - 2f(x)]\, dz\,,\end{aligned}$$

and, by the generalized Minkowiski inequality (cf. [10]),

$$\begin{aligned}\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M &\le \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\partial P}{\partial y}(z,y)\right| \|f(x+z) + f(x-z) - 2f(x)\|_M\, dz \\ &\le \frac{1}{2} D \int_{\mathbb{R}^n} \left|\frac{\partial P}{\partial y}(z,y)\right| \varphi(|z|)\, dz\,,\end{aligned}$$

and by using the estimates from the proof of Theorem 2.1, we obtain

$$\left\|\frac{\partial}{\partial y}u(x,y)\right\|_M \le C_6\, y^{-1}\varphi(y) \quad \text{for all} \quad y > 0\,,$$

which, according to Theorem 2.1, gives $f \in Lip(\varphi, L_M)$.

Remark 2.4. Theorem 2.1 in the case of $L_p(\mathbb{R}^n)$-space ($1 \le p \le \infty$) and with $\varphi(t) = t^\alpha$, where $0 < \alpha < 1$, was proved by STEIN ([19], Prop. 7, 7') and by TAIBLESON ([20], Th. 4).

Remark 2.5. Using the fact that

$$|P(x,y)| \le \frac{1}{c_n}\{\min\{y^{-n},\, y\,|x|^{-n-1}\}\}\,,$$

we can prove for $f \in Lip(\varphi, L_M)$, in a similar way as in the proof of Theorem 2.1, that

$$\begin{aligned}\|u(x,y) - f(x)\|_M &\le \int_{\mathbb{R}^n} \|f(x-z) - f(x)\|_M\, |P(z,y)|\, dz \\ &\le \frac{1}{c_n} y^{-n} \int_{|z|\le y} \|f(x-z) - f(x)\|_M\, dz \\ &\quad + \frac{1}{c_n}\int_{|z|\ge y} y|z|^{-n-1}\, \|f(x-z) - f(x)\|_M\, dz \\ &\le \frac{2nCc_{n-1}}{c_n}\left[y^{-n}\int_0^y \varphi(r)\, r^{n-1}\, dr + y\int_y^\infty \varphi(r)\, r^{-2}\, dr\right].\end{aligned}$$

3. The Zygmund Condition

The next result is the case of Zygmund condition in Orlicz spaces which gives the Zygmund- Orlicz spaces $Zyg(\varphi, L_M)$. ZYGMUND [23] introduced spaces of smooth functions $Zyg(1, L_p)$.

Theorem 3.1. *Let $f \in L_M(\mathbb{R}^n)$ und $u(x,y) = \int_{\mathbb{R}^n} f(x-z)\,P(z,y)\,dz$ be its Poisson integral. If $0 < \alpha_\varphi \le \beta_\varphi < 2$, then $f \in Zyg(\varphi, L_M)$ if and only if*

$$\left\|\frac{\partial^2}{\partial y^2}u(x,y)\right\|_M \le C\varphi(y)/y^2 \quad \text{for all} \quad y > 0. \tag{3.1}$$

Proof (Necessity). Assume that $f \in Zyg(\varphi, L_M)$. Since

$$\begin{aligned}\frac{\partial^2 P(x,y)}{\partial y^2} &= \frac{n+1}{c_n}\cdot\frac{y(ny^2-3|x|^2)}{(|x|^2+y^2)^{(n+5)/2}} \\ &= \frac{n+1}{c_n}\left[\frac{(n+3)y^3}{(|x|^2+y^2)^{(n+5)/2}} - \frac{3y}{(|x|^2+y^2)^{(n+3)/2}}\right]\end{aligned}$$

it follows that

$$\int_{\mathbb{R}^n}\frac{\partial^2 P(x,y)}{\partial y^2}dx = 0,$$

$$\left|\frac{\partial^2 P(x,y)}{\partial y^2}\right| \le \frac{(n+1)(n+2)}{c_n}\cdot\left\{\min\left\{|x|^{-n-2}y, y^{-n-2}\right\}\right\},$$

and

$$\begin{aligned}\left|\frac{\partial^2 P(x,y)}{\partial y^2}\right| &\le \frac{(n+1)(n+2)}{c_n}\cdot\frac{y}{(|x|^2+y^2)^{(n+3)/2}} \\ &\le (n+1)(n+2)\,y^{-2}\,P(x,y).\end{aligned}$$

Then we can write

$$\begin{aligned}\frac{\partial^2}{\partial y^2}u(x,y) &= \int_{\mathbb{R}^n}\frac{\partial^2 P}{\partial y^2}(z,y)\,f(x-z)\,dz \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\frac{\partial^2 P}{\partial y^2}(z,y)\,[f(x+z)+f(x-z)-2f(x)]\,dz.\end{aligned}$$

Using the generalized Minkovski inequality we obtain

$$\begin{aligned}\left\|\frac{\partial^2}{\partial y^2}u(x,y)\right\|_M &\le \frac{1}{2}\int_{\mathbb{R}^n}\left|\frac{\partial^2 P}{\partial y^2}(z,y)\right|\,\|f(x+z)+f(x-z)-2f(x)\|_M\,dz \\ &\le \frac{1}{2}\cdot\frac{(n+1)(n+2)}{c_n}\,y^{-n-2} \\ &\quad\times\int_{|z|\le y}\|f(x+z)+f(x-z)-2f(x)\|_M\,dz \\ &\quad+\frac{1}{2}\cdot\frac{(n+1)(n+2)}{c_n}y \\ &\quad\times\int_{|z|>y}|z|^{-n-3}\,\|f(x+z)+f(x-z)-2f(x)\|_M dz.\end{aligned}$$

Next set $z = r\xi \in \mathbb{R}^n$, with $r = |z|$ and $|\xi| = 1$. Then, with

$$\|f(x+z)+f(x-z)-2f(x)\|_M = \omega_M(z) = \omega_M(r\xi)$$

and

$$\Omega(r) = \int_{S^{n-1}} \omega_M(r\xi)\, d\sigma(\xi)\,,$$

the inequality above becomes (because $dz = d\xi\, r^{n-1}\, dr$)

$$\left|\left|\frac{\partial^2}{\partial y^2}u(x,y)\right|\right|_M \le \frac{1}{2}\cdot\frac{(n+1)(n+2)}{c_n}\left[y^{-n-2}\int_0^y \Omega(r)\, r^{n-1}\, dr + y\int_y^\infty \Omega(r)\, r^{-4}\, dr\right].$$

The assumption $f \in Zyg(\varphi, L_M)$ gives

$$\Omega(r) \le D\,\varphi(r)\,\sigma(S^{n-1}) = 2c_{n-1}D\,\varphi(r)$$

and, by the assumption $\beta_\varphi < 2$, in the equivalent form (2.2), we obtain

$$\begin{aligned}\left|\left|\frac{\partial^2}{\partial y^2}u(x,y)\right|\right|_M &\le C_7\left[y^{-n-2}\int_0^y \Omega(r)\, r^{n-1}\, dr + y\int_y^\infty \varphi(r)\, r^{-4} dr\right]\\ &\le C_7\left[y^{-n-2}\int_0^y \Omega(r)\, r^{n-1}\, dr + \int_y^\infty \varphi(r)\, r^{-3}\, dr\right]\\ &\le C_8 y^{-2}\,\varphi(y)\,.\end{aligned}$$

Proof (Suficiency). First we prove the following lemma.

Lemma 3.2. *Let $f \in L_M(\mathbb{R}^n)$ und $u(x,y) = \int_{\mathbb{R}^n} f(x-z)P(z,y)\, dz$ be its Poisson integral. If $0 < \alpha_\varphi \le \beta_\varphi < 2$, then the following conditions are equivalent:*

(a) $\left|\left|\frac{\partial^2}{\partial y^2}u(x,y)\right|\right|_M \le C\varphi(y)/y^2$ *for all* $y > 0$,

(b) $\left|\left|\frac{\partial^2 u}{\partial y\partial x_i}(x,y)\right|\right|_M \le Dy^{-2}\,\varphi(y)$ *for all* $y > 0$ *and each* $i = 1, 2, \dots, n$,

(c) $\left|\left|\frac{\partial^2 u}{\partial x_i\partial x_j}(x,y)\right|\right|_M \le Ey^{-2}\,\varphi(y)$ *for all* $y > 0$ *and each* $i, j = 1, 2, \dots, n$.

Proof. (a) $\Rightarrow$ (b). Differentiating equality (2.5) we obtain

$$\frac{\partial^3 u}{\partial y^2\partial x_i}(x, y+y_1) = \frac{\partial^2 u}{\partial y^2}(x,y) * \frac{\partial P}{\partial x_i}(x,y_1)\,.$$

Then, by arguing in a similar way as in the proof of Lemma 2.2, we find that

$$\begin{aligned}\left|\left|\frac{\partial^3 u}{\partial y^2\partial x_i}(x,y)\right|\right|_M &= \left|\left|\frac{\partial^2 u}{\partial y^2}(x,y/2) * \frac{\partial P}{\partial x_i}(x,y/2)\right|\right|_M\\ &\le \left|\left|\frac{\partial^2 u}{\partial y^2}(x,y/2)\right|\right|_M \cdot \left|\left|\frac{\partial P}{\partial x_i}(x,y/2)\right|\right|_1\\ &\le Cy^{-2}\,\varphi(y)\,\frac{n+1}{2c_n}y^{-1} = C_9 y^{-3}\,\varphi(y)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\frac{\partial^2 u}{\partial y \partial x_i}(x,y)\right\|_M &= \left\|f(x) * \frac{\partial^2 P}{\partial y \partial x_i}(x,y)\right\|_M \\ &\le \|f(x)\|_M \left\|\frac{\partial^2 P}{\partial y \partial x_i}(x,y)\right\|_1,\end{aligned}$$

and the equality

$$\frac{\partial^2 P}{\partial y \partial x_i}(x,y) = \frac{(n+1)(n+2)}{c_n} \cdot \frac{x_i(y^2 - |x|^2)}{(|x|^2 + y^2)^{(n+5)/2}}$$

gives

$$\left|\frac{\partial^2 P}{\partial y \partial x_i}(x,y)\right| \le (n+1)(n+2)\,|x_i|\,y^{-2} P(x,y),$$

and so

$$\begin{aligned}\left\|\frac{\partial^2 u}{\partial y \partial x_i}(x,y)\right\|_M &\le (n+1)(n+2)\,|x_i|\,y^{-2}\,\|P(x,y)\|_1\,\|f(x)\|_M \\ &= C_{10}\,y^{-2}\,\|f(x)\|_M\end{aligned}$$

which, in its turn, implies that

$$\frac{\partial^2 u}{\partial y \partial x_i}(x,y) \to 0 \quad \text{as} \quad y \to \infty.$$

Therefore

$$\frac{\partial^2 u}{\partial y \partial x_i}(x,y) = -\int_y^\infty \frac{\partial^3 u}{\partial y^2 \partial x_i}(x,s)\,ds$$

which, by the assumption and equivalence (2.2), gives

$$\begin{aligned}\left\|\frac{\partial^2 u}{\partial y \partial x_i}(x,y)\right\|_M &\le \int_y^\infty \left\|\frac{\partial^3 u}{\partial y^2 \partial x_i}(x,s)\right\|_M ds \\ &\le C_9 \int_y^\infty s^{-3}\varphi(s)\,ds \le C_9 B\,y^{-2}\varphi(y)\end{aligned}$$

for all $y > 0$ and $i = 1, 2, \dots, n$.

(b) $\Rightarrow$ (c). The proof is similar to the proof of (a) $\Rightarrow$ (b) but here the equality

$$\frac{\partial^3 u}{\partial x_j \partial y \partial x_i}(x, y + y_1) = \frac{\partial^2 u}{\partial y \partial x_i}(x,y) * \frac{\partial P}{\partial x_j}(x, y_1)$$

is essential.

(c) $\Rightarrow$ (a). The proof is the same as that of Lemma 2.2.

Proof of Theorem 3.1 (Sufficiency). Assume that $f \in L_M$ and the condition (3.1) holds. Let $h \in \mathbb{R}^n$ and $0 < y < |h|$. Integrating by parts we find that

$$\begin{aligned}
&\int_y^{|h|} s\left[\frac{\partial^2 u}{\partial y^2}(x+h,s) + \frac{\partial^2 u}{\partial y^2}(x-h,s) - 2\,\frac{\partial^2 u}{\partial y^2}(x,s)\right] ds \\
&= |h|\left[\frac{\partial u}{\partial y}(x+h,|h|) + \frac{\partial u}{\partial y}(x-h,|h|) - 2\,\frac{\partial u}{\partial y}(x,|h|)\right] \\
&\quad - y\left[\frac{\partial u}{\partial y}(x+h,y) + \frac{\partial u}{\partial y}(x-h,y) - 2\,\frac{\partial u}{\partial y}(x,y)\right] \\
&\quad - \int_y^{|h|}\left[\frac{\partial u}{\partial y}(x+h,y) + \frac{\partial u}{\partial y}(x-h,y) - 2\,\frac{\partial u}{\partial y}(x,y)\right] ds \\
&= |h|\left[\frac{\partial u}{\partial y}(x+h,|h|) + \frac{\partial u}{\partial y}(x-h,|h|) - 2\,\frac{\partial u}{\partial y}(x,|h|)\right] \\
&\quad - y\left[\frac{\partial u}{\partial y}(x+h,y) + \frac{\partial u}{\partial y}(x-h,y) - 2\,\frac{\partial u}{\partial y}(x,y)\right] \\
&\quad - \left[u(x+h,|h|) + u(x-h,|h|) - 2u(x,|h|)\right] \\
&\quad + u(x+h,y) + u(x-h,y) - 2u(x,y)
\end{aligned}$$

and so,

$$\begin{aligned}
&u(x+h,y) + u(x-h,y) - 2u(x,y) \\
&= u(x+h,|h|) + u(x-h,|h|) - 2u(x,|h|) \\
&\quad + y\left[\frac{\partial u}{\partial y}(x+h,y) + \frac{\partial u}{\partial y}(x-h,y) - 2\,\frac{\partial u}{\partial y}(x,y)\right] \\
&\quad - |h|\left[\frac{\partial u}{\partial y}(x+h,|h|) + \frac{\partial u}{\partial y}(x-h,|h|) - 2\,\frac{\partial u}{\partial y}(x,|h|)\right] \\
&\quad + \int_y^{|h|} s\left[\frac{\partial^2 u}{\partial y^2}(x+h,s) + \frac{\partial^2 u}{\partial y^2}(x-h,s) - 2\,\frac{\partial^2 u}{\partial y^2}(x,s)\right] ds \\
&= I_1 + I_2 - I_3 + I_4
\end{aligned}$$

Thus, by the generalized Minkowski inequality and the assumption $\alpha_\varphi > 0$ in the equivalent form (2.1),

$$\begin{aligned} \|I_4\|_M &\le \int_y^{|h|} s \left\| \frac{\partial^2 u}{\partial y^2}(x+h,s) + \frac{\partial^2 u}{\partial y^2}(x-h,s) - 2\,\frac{\partial^2 u}{\partial y^2}(x,s) \right\|_M ds \\ &\le 4C \int_y^{|h|} s^{-1}\,\varphi(s)\,ds \\ &\le 4CA\varphi(|h|). \end{aligned}$$

Similarly, as in the sufficiency part of the proof of Theorem 1.1 (by using the generalized Minkowski inequality, Lemma 3.2 and the assumption $\alpha_\varphi > 0$ in the equivalent form (2.1)), we find that

$$\begin{aligned} \|I_2\|_M &\le y \left\| \frac{\partial u}{\partial y}(x+h,y) - \frac{\partial u}{\partial y}(x,y) \right\|_M + y \left\| \frac{\partial u}{\partial y}(x-h,y) - \frac{\partial u}{\partial y}(x,y) \right\|_M \\ &\le 4y \int_y^{|h|} \left\| \frac{\partial^2 u}{\partial y^2}(x,s) \right\|_M ds + 2\sum_{i=1}^n y \int_0^{|h_i|} \left\| \frac{\partial^2 u}{\partial y \partial x_i}(x,|h|) \right\|_M dz_i \\ &\le C_{11}\varphi(|h|). \end{aligned}$$

Similarly,

$$\|I_3\|_M \le C_{12}\varphi(|h|).$$

For the estimate of I_1 we first prove that for any real function u on $\mathbb{R}^n$ of class C^2 and any $h \in \mathbb{R}^n$ we have

$$\begin{aligned} & u(x+h) + u(x-h) - 2u(x) \\ & = \sum_{i=1}^n \sum_{j=1}^n \int_{-1}^1 (1-|t|)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x+th)\,h_i\,h_j\,dt. \end{aligned} \tag{3.2}$$

In fact, by the chain rule $\frac{d}{dt}u(x+th) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x+th)h_i$, we can integrate both sides from 0 to 1, and then integrate by parts to obtain

$$\begin{aligned} & u(x+h) - u(x) \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial u}{\partial x_i}(x+th)\,h_i\,dt \\ &= \sum_{i=1}^n \left[(t-1)\frac{\partial u}{\partial x_i}(x+th)\,h_i \Big|_0^1 - \sum_{j=1}^n \int_0^1 (t-1)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x+th)\,h_i\,h_j\,dt \right] \\ &= \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\,h_i + \sum_{i=1}^n \sum_{j=1}^n \int_{-1}^1 (1-t)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x+th)\,h_i\,h_j\,dt. \end{aligned}$$

Similarly,

$$\begin{aligned} & u(x-h) - u(x) \\ &= -\sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\,h_i + \sum_{i=1}^n \sum_{j=1}^n \int_{-1}^1 (1-t)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x-th)\,h_i\,h_j\,dt. \end{aligned}$$

If we add the above identities we obtain (3.2).

Now using the identity (3.2) we have for our expression (I_1)

$$\begin{aligned} I_1 &= u(x+h,|h|) + u(x-h,|h|) - 2u(x,|h|) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-1}^{1}(1-|t|)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x+th,|h|)\,h_i\,h_j\,dt\,. \end{aligned}$$

Thus, by the generalized Minkowski inequality and Lemma 3.2, it yields

$$\begin{aligned} \|I_1\|_M &\le \|u(x+h,|h|) + u(x-h,|h|) - 2u(x,|h|)\|_M \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-1}^{1}(1-|t|)\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}(x+th,|h|)\right\|_M |h_i|\,|h_j|\,dt \\ &\le E\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-1}^{1}(1-|t|)\,|h|^{-2}\,\varphi(|h|)\,|h_i|\,|h_j|\,dt \\ &\le nE\varphi(|h|)\,. \end{aligned}$$

Putting the above estimates togheter we obtain

$$\|u(x+h,y) + u(x-h,y) - 2u(x,y)\|_M \le C_{13}\varphi(|h|)$$

for any $0 < y < |h|$. Now, since $u(x,y) \to f(x)$ for almost all $x \in \mathbb{R}^n$ when $y \to 0^+$, we obtain (by Fatou Lemma) that $f \in Zyg(\varphi, L_M)$. This completes the proof. □

Remark 3.3. Theorem 3.1 in the case of $L_p(\mathbb{R}^n)$-space $(1 \le p \le \infty)$ and with $\varphi(t) = t^\alpha$, where $0 < \alpha < 2$, was proved by STEIN ([19], Prop. 8, 8') and TAIBLESON ([20], Th. 4).

Remark 3.4. For $\varphi \in P$ we have always the imbedding

$$Lip(\varphi, L_M) \subset Zyg(\varphi, L_M)$$

but Corollary 2.3 states that if $0 < \alpha_\varphi \le \beta_\varphi < 1$, then $Lip(\varphi, L_M) = Zyg(\varphi, L_M)$. Already ZYGMUND [23] (cf. [19], p. 148-149) observed that the space $Lip(1, L_\infty)$ is strictly smaller than the space $Zyg(1, L_\infty)$. More examples of functions giving the strict inclusions

$$Lip(1, L_p) \subset Zyg(1, L_p) \quad \text{and} \quad Zyg(2, L_p) \subset Zyg(1, L_p)$$

can be found in [19], p. 161 and [20], pp. 470-474.

Remark 3.5. In the definition of the space $Lip(\varphi, L_M)$ we have the inequality

$$\|f(x+h) - f(x)\|_M \le C\varphi(|h|) \quad \text{for all} \quad |h| > 0\,.$$

It is enough to have such an inequality only for small $|h|$, i.e.,

$$\begin{aligned} Lip(\varphi, L_M) = \{&f \in L_M(\mathbb{R}^n) \text{ such that} \\ &\|f(x+h) - f(x)\|_M = O(\varphi(|h|)) \text{ as } |h| \to 0\}\,. \end{aligned}$$

Similarly

$$\begin{aligned} Zyg(\varphi, L_M) &= \{f \in L_M(\mathbb{R}^n) \text{ such that} \\ &\qquad \|f(x+h) + f(x-h) - 2f(x)\|_M = O(\varphi(|h|)) \text{ as } |h| \to 0\} . \end{aligned}$$

This observation suggests the possibility of considering the closed subspaces (analogues to the spaces $lip(1, L_p)$ and $zyg(1, L_p)$ considered by ZYGMUND [24]):

$$\begin{aligned} lip(\varphi, L_M) &= \{f \in L_M(\mathbb{R}^n) \text{ such that} \\ &\qquad \|f(x+h) - f(x)\|_M = o(\varphi(|h|)) \text{ as } |h| \to 0\} . \end{aligned}$$

and

$$\begin{aligned} zyg(\varphi, L_M) &= \{f \in L_M(\mathbb{R}^n) \text{ such that} \\ &\qquad \|f(x+h) + f(x-h) - 2f(x)\|_M = o(\varphi(|h|)) \text{ as } |h| \to 0\} . \end{aligned}$$

4. Some generalizations and additional remarks

In the proof of Lemma 2.2 we used the following result, in the case when X is the Orlicz space $L_M(\mathbb{R}^n)$:

Lemma 4.1. *Let $X = X(\mathbb{R}^n)$ be a Banach function space with the Fatou property. Then the convolution operator $(f * g) = \int_{\mathbb{R}^n} f(x-z)\, g(z)\, dz$ is a bounded operator from $X(\mathbb{R}^n) \times L_1(\mathbb{R}^n)$ into $X = X(\mathbb{R}^n)$ and*

$$\|f * g\|_X \leq \|f\|_X \|g\|_1 .$$

Proof. For any $h \in X'$ with $\|h\|_{X'} \leq 1$ we have, by the Fubini and Hölder inequalities, that

$$\begin{aligned} \int_{\mathbb{R}^n} |(f*g)(x)\, h(x)|\, dx &\leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-z)\, g(z)\, h(x)|\, dz \right] dx \\ &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-z) h(x)|\, dx \right] |g(z)|\, dx \\ &\leq \int_{\mathbb{R}^n} \|f\|_X \|h\|_{X'} |g(z)|\, dz \\ &\leq \|f\|_X \|g\|_1 , \end{aligned}$$

and, by the Fatou property of the norm,

$$\begin{aligned} &\|f*g\|_X = \|f*g\|_{X''} \\ &= \sup\left\{ \int_{\mathbb{R}^n} |(f*g)(x)\, h(x)| dx \text{such that} \quad h \in X',\ \|h\|_{X'} \leq 1 \right\} \leq \|f\|_X \|g\|_1 . \end{aligned}$$

□

Note that O'NEIL [15] proved the lemma above for the Orlicz spaces $L_M(\mathbb{R}^n)$ instead of $X(\mathbb{R}^n)$ but with the constant 2 in the estimate of the norms.

Remark 4.2. Using Lemma 4.1 we can prove Theorems 2.1 and 3.1 not only for the Orlicz spaces $L_M = L_M(\mathbb{R}^n)$ but even for general Banach function spaces $X = X(\mathbb{R}^n)$ with the Fatou property.

We consider now more general spaces $\Lambda^k(\varphi, X, q)$ which contain the $Lip(\varphi, X)$-spaces, $Zyg(\varphi, X)$-spaces, the Stein-Taibleson $\Lambda(\alpha, p, q)$-spaces and the Herz $\Lambda(\alpha, X, q)$-spaces.

Let $\varphi \in P$, $1 \le q \le \infty$ and let $X = X(\mathbb{R}^n)$ be a Banach function space with the Fatou property. The spaces $\Lambda^k(\varphi, X, q)$, $k = 1, 2$, are the spaces of all $f \in X(\mathbb{R}^n)$ for which

$$I^k_{\varphi,X,q}(f) = \left\{ \int_{\mathbb{R}^n} \left(\frac{||\Delta^k_h f(x)||_X}{\varphi(|h|)} \right)^q \frac{dh}{|h|^n} \right\}^{1/q} < \infty ,$$

with $\Delta^1_h f(x) = f(x+h) - f(x)$, $\Delta^2_h f(x) = f(x+h) + f(x-h) - 2f(x)$, and with the norm

$$||f||_{\Lambda^k(\varphi,X,q)} = ||f||_X + I^k_{\varphi,X,q}(f) .$$

Note that $\Lambda^1(\varphi, L_M, \infty) = Lip(\varphi, L_M)$ and $\Lambda^2(\varphi, L_M, \infty) = Zyg(\varphi, L_M)$.

Theorem 4.3. *(a) The spaces $\Lambda^k(\varphi, X, q)$ are Banach spaces.*

(b) $\Lambda^1(\varphi, X, q) \subset \Lambda^2(\varphi, X, q)$, and if $0 < \alpha_\varphi \le \beta_\varphi < 1$, then $\Lambda^1(\varphi, X, q) = \Lambda^2(\varphi, X, q)$.

(c) Let $f \in X(\mathbb{R}^n)$ and let u be its Poisson integral. If $0 < \alpha_\varphi \le \beta_\varphi < k$, then $f \in \Lambda^k(\varphi, X, q)$ if and only if

$$\left\{ \int_0^\infty \left[\frac{y^k}{\varphi(y)} \left|\left| \frac{\partial^k u(x,y)}{\partial y^k} \right|\right|_X \right]^q \frac{dy}{y} \right\}^{1/q} < \infty, \quad k = 1,\ 2 .$$

Proof (a) If (f_n) is a Cauchy sequence in $\Lambda^k(\varphi, X, q)$, then (f_n) is obviously a Cauchy sequence in X, and therefore it converges in X to a function f. Hence

$$||\Delta^k_h f_n(x)||_X \longrightarrow ||\Delta^k_h f(x)||_X$$

as $n \to \infty$, and therefore , by Fatou's Lemma,

$$I^k_{\varphi,X,q}(f) \le \liminf_{n\to\infty} I^k_{\varphi,X,q}(f_n) \le \liminf_{n\to\infty} ||f_n||_{\Lambda^k_{\varphi,X,q}} < \infty ,$$

so that $f \in_{\Lambda^k_{\varphi,X,q}}$. Further, for $m = 1, 2, \ldots$,

$$||\Delta^k_h f_n(x) - \Delta^k_h f_m(x)||_X \longrightarrow ||\Delta^k_h f(x) - \Delta^k_h f_m(x)||_X$$

as $n \to \infty$, whence, again by Fatou's Lemma,

$$\begin{aligned} ||f - f_m||_{\Lambda^k(\varphi,X,q)} &\le \liminf_{n\to\infty} \Big[||f_n - f_m||_X + I^k_{\varphi,X,q}(f_n - f_m) \Big] \\ &= \liminf_{n\to\infty} ||f_n - f_m||_{\Lambda^k(\varphi,X,q)} . \end{aligned}$$

Since the expression on the right-hand side is arbitrarily small for all sufficiently large m, it follows that $f_m \to f$ in $\Lambda^k(\varphi, X, q)$, so $\Lambda^k(\varphi, X, q)$ is complete.

(b) The proof is the same as the proof of Corollary 2.3. We can also prove the statement by using the following equality

$$\Delta_h f(x) = 2^{-n} \Delta_{2^n h} f(x) - \sum_{k=0}^{n-1} 2^{-k-1} \Delta^2{}_{2^k h} f(x + 2^k h).$$

(c) Similarly, as in the proof of Th. 2.1 and Th. 3.2, for $f \in \Lambda^k(\varphi, X, q)$ and for $k = 1, 2$, we have that

$$\left\| \frac{\partial^k u(x,y)}{\partial y^k} \right\|_X \leq C \left[y^{-n-k} \int_0^y \Omega(r)\, r^{n-1}\, dr + \int_y^\infty \Omega(r)\, r^{-k-1}\, dr \right],$$

where

$$\Omega(r) = \int_{S^{n-1}} \omega_X(r\xi)\, d\sigma(\xi) = \int_{S^{n-1}} \|f(x - r\xi) - f(x)\|_X\, d\sigma(\xi).$$

By the Hardy inequalities proved in [11] it yields that

$$\left\{ \int_0^\infty \left[\frac{y^k}{\varphi(y)} \left\| \frac{\partial^k u(x,y)}{\partial y^k} \right\|_X \right]^q \frac{dy}{y} \right\}^{1/q} \leq C' \left\{ \int_0^\infty \left[\frac{\Omega(r)}{\varphi(r)} \right]^q \frac{dr}{r} \right\}^{1/q}$$

and, by the Hölder inquality,

$$\Omega(r) \leq \sigma(S^{n-1})^{1/q'} \left\{ \int_{S^{n-1}} \omega_X(r\xi)^q\, d\sigma(\xi) \right\}^{1/q}$$

so that we obtain

$$\begin{aligned} &\left\{ \int_0^\infty \left[\frac{y^k}{\varphi(y)} \left\| \frac{\partial^k u(x,y)}{\partial y^k} \right\|_X \right]^q \frac{dy}{y} \right\}^{1/q} \\ &\leq C'' \left\{ \int_{S^{n-1}} \int_0^\infty \left[\frac{\omega_X(r\xi)}{\varphi(r)} \right]^q \frac{dr}{r} d\sigma(\xi) \right\}^{1/q} \\ &= C'' \left\{ \int_{\mathbb{R}^n} \left[\frac{\|f(x-z) - f(x)\|_X}{\varphi(|z|)} \right]^q \frac{dz}{|z|^n} \right\}^{1/q}. \end{aligned}$$

In the same way as in Theorems 2.1 and 3.1, we can prove the reverse inequalities by first proving the results similar to Lemmas 2.2 and 3.2.

Remark 4.4. In STEIN [19] there are misprints in Proposition 7' and Lemma 4': conditions (61) and (62) should have $y^{1-\alpha}$ instead of $y^{\alpha-1}$.

Remark 4.5. Considering the modulus of continuity $\omega_1(t,f)_X$ and the modulus of smoothness $\omega_2(t,f)_X$ of the function $f \in X(\mathbb{R}^n)$, that is,

$$\omega_k(t,f)_X = \sup_{0<|h|\leq t} \left|\left|\Delta_h^k f(x)\right|\right|_{X'}, \quad k = 1,\ 2,$$

we can easily prove (cf. TAIBLESON [20]) that

$$\Lambda^k(\varphi, X, q) = \left\{ f \in X(\mathbb{R}^n) : \left\{ \int_0^\infty (\omega_k(t,f)_X/\varphi(t))^q dt/t \right\}^{1/q} < \infty \right\}.$$

These are the generalized Besov-Nikolskii spaces (cf. [14]). The more general spaces $\Lambda^k(B, X)$ were investigated by CALDERÓN [4] and BRUDNYI-SHALASHOV [2].

We conclude this paper remarking that results about the convolution operator and the pointwise multiplication for Lipschitz-Orlicz $\Lambda(\varphi, M, q)$- spaces (which will contain the theorems proved in [6] and [20]) is possilble to prove by using our Theorem 4.3 (c) and the appropriate results proved by O'NEIL [15] (see also [13] and [17]) for the convolution operator and the pointwise multiplication in Orlicz spaces.

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Department of Mathematics
Claremont McKenna College
Claremont, CA 91711
USA
e-mail: aaksoy@cmcvax.claremont.edu

Department of Mathematics
Luleå University
S-971 87 Luleå
Sweden
e-mail: lech@sm.luth.se