J. Math. Anal. Appl. 398 (2013) 296–302

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Limit theorems for the numerical index

Asuman Güven Aksoy [a,*], Grzegorz Lewicki [b]

[a] Claremont McKenna College, Department of Mathematics, Claremont, CA 91711, USA

[b] Jagiellonian University, Department of Mathematics, Łojasiewicza 6, 30-348, Poland

ARTICLE INFO

Article history:
Received 16 June 2011
Available online 4 September 2012
Submitted by Martin Mathieu

Keywords:
Numerical index
Numerical radius
Local characterization condition

ABSTRACT

We improve on a limit theorem (see Martin et al. (2011) [13], Th. 5.1) for numerical index $n(\cdot)$ for large classes of Banach spaces including vector valued ℓ_p-spaces and ℓ_p-sums of Banach spaces where $1 \le p < \infty$. We introduce two conditions on a Banach space X, a local characterization condition (LCC) and a global characterization condition (GCC). We prove that if a norm on X satisfies the (LCC), then $n(X) = \lim_m n(X_m)$. An analogous result, in which $\mathbb{N}$ will be replaced by a directed, infinite set S will be proved for X satisfying the (GCC). We also present examples of Banach spaces satisfying the above mentioned conditions.

1. Introduction

Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. We write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $B(X)$. For a linear subspace Y of X we denote by $\mathcal{P}(X, Y)$ the set of all linear, continuous projections from X onto Y.

Definition 1.1. The *numerical range* of $T \in B(X)$ is defined by

$$W(T) = \{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$\nu(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Clearly, $\nu(\cdot)$ is a semi-norm on $B(X)$ and $\nu(T) \le \|T\|$ for all $T \in B(X)$. The *numerical index* of X is defined by

$$n(X) = \inf\{\nu(T) : T \in S_{B(X)}\}.$$

Equivalently, the numerical index $n(X)$ is the greatest constant $k \ge 0$ such that $k\|T\| \le \nu(T)$ for every $T \in B(X)$. The concept of numerical index was first introduced by Lumer [1] in 1968. Since then, much attention has been paid to this equivalence constant between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. It is known that $0 \le n(X) \le 1$ if X is a real space, and $\frac{1}{e} \le n(X) \le 1$ if X is a complex space. Furthermore, $n(X) > 0$ if and only if $\nu(\cdot)$ and $\|\cdot\|$ are equivalent norms. Calculation of the numerical index for some classical Banach spaces can be found in [2,3]. For more recent results we refer the reader to [4–14]. In [15] it is shown that

$$n(\ell_p) = n(L_p[0, 1])$$

* Corresponding author.
E-mail addresses: aaksoy@cmc.edu (A.G. Aksoy), Grzegorz.Lewicki@im.uj.edu.pl (G. Lewicki).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.08.055

for a fixed $1 < p < \infty$. In the same paper it is also established that $n(\ell_p^m) \neq 0$ for finite m in the real case. In [6] the numerical index of vector-valued function spaces is considered and a proof of

$$n(L_p(\mu, X)) = \lim_m n(\ell_p^m(X))$$

is provided for a Banach space X and for $1 \leq p < \infty$. Furthermore, it was recently proven in [16] that $n(L_p(\mu)) > 0$ for $p \neq 2$ and μ any positive measure, in the real case. In this paper we obtain the above type of limit theorem for a class of Banach spaces including vector valued ℓ_p or L_p spaces. Our main result is an improvement of the limit theorem presented in [13]. The study of numerical index of absolute sums of Banach spaces is given in [13], where under suitable conditions it is shown that the numerical index of a sum is greater than or equal to the limsup of the numerical index of the summands (see Theorem 5.1 of [13]). In this paper, we show that the liminf of the numerical index of the summands is greater than or equal to the numerical index of the sum if the Banach space satisfies a condition called the local characterization condition (LCC) or a condition called the global characterization condition (GCC). We show that if a norm on X satisfies the local characterization condition, then

$$n(X) = \lim_m n(X_m)$$

and

$$n(X) = \lim_{s \in S} n(X_s)$$

where S is any directed, infinite set and X satisfies the global characterization condition. We also provide examples of spaces where (LCC) or (GCC) is satisfied.

2. Main results

The following theorem, which is a direct consequence of ([14], Th. 2.5), plays a crucial role in our further investigations.

Theorem 2.1. *Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and let*

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}.$$

Denote by π_1 the natural projection from $\Pi(X)$ onto S_X defined by $\pi_1(x, x^*) = x$. Fix a set $\Gamma \subset \Pi(X)$ such that $\pi_1(\Gamma)$ is dense in S_X. Then for any $T \in B(X)$,

$$\nu(T) = \sup\{|x^*(Tx)| : (x, x^*) \in \Gamma\}.$$

Applying the above theorem one can prove the following.

Corollary 2.2. *Let X be an infinite-dimensional Banach space and let $Y \subseteq X$ be its linear subspace whose norm-closure is equal to X. Define for $L \in \mathcal{L}(X)$,*

$$\nu_Y(L) = \sup\{|x^*Lx| : x^* \in S_{X^*}, x \in S_Y, x^*(y) = 1\}. \tag{2.1}$$

Then $\nu(L) = \nu_Y(L)$.

Definition 2.3. Let X be a Banach space and $X_1 \subset X_2 \subset \cdots \subset X$ be its subspaces such that $X = \overline{\bigcup_{m=1}^{\infty} X_m}$. Suppose for any $m \in \mathbb{N}$ there exists $P_m \in \mathcal{P}(X_{m+1}, X_m)$ with $\|P_m\| = 1$. We say the norm on X, $\|.\|_X$ satisfies the *local characterization condition* (LCC) with respect to $\{P_m\}_{m=1}^{\infty}$ if and only if for any $m \in \mathbb{N}$ there exists D_m a dense subset of $S_{X_{m+1}}$ such that for any $x \in D_m$ there exist $x^* \in S_{X^*_{m+1}}$ a norming functional for x in X^*_{m+1} and a constant $b_m(x) \in \mathbb{R}_+$ such that $b_m(x)x^*|_{X_m}$ is a norming functional for $P_m x$ in X^*_m. (In fact, if $P_m(x) \neq 0$, then $b_m(x) = \|P_m(x)\|/x^*(P_m(x))$.)

We start by investigating some consequences of (LCC).

Proposition 2.4. *Let X be a Banach space satisfying (LCC) with respect to $\{P_m\}_{m=1}^{\infty}$. For a fixed $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$, define a sequence*

$$w_m(L) = \nu(L), w_{m+1}(L) = \nu(L \circ P_m), \ldots, w_{m+j}(L) = \nu(L \circ Q_{m,j}),$$

where $Q_{m,j} = P_m \circ \cdots \circ P_{m+j-1}$. (For $j \geq 1$ $\nu(L \circ Q_{m,j})$ denote the numerical radius of $L \circ Q_{m,j}$ with respect to X_{m+j}.) Then $\nu(L) = w_m(L) = w_{m+j}(L)$ for $j = 1, 2, \ldots$.

Proof. Since $X_m \subset X_{m+1}$ for any $m \in \mathbb{N}$, it is easy to see that $w_{m+j}(L)$ is an increasing sequence with respect to j, since

$$\begin{aligned} w_{m+j}(L) &= \sup\{|x^*L \circ Q_{m,j}x| : x \in S_{X_{m+j}}, x^* \in S_{X^*_{m+j}}, x^*(x) = 1\} \\ &\leq \sup\{|x^*LQ_{m,j}P_{m+j}x| : x \in S_{X_{m+j+1}}, x^* \in S_{X^*_{m+j+1}}, x^*(x) = 1\} = w_{m+j+1}(L). \end{aligned}$$

Now we prove that $w_m = w_{m+1}$. To do this for any $x \in D_m$ select $x_x^* \in S_{X_{m+1}^*}$ satisfying the requirements of Definition 2.3. Set

$$\Gamma_m = \{(x, x_x^*) \in \Pi(X_{m+1}) : x \in D_m\}.$$

Note that by Definition 2.3,

$$\frac{b_m(x)}{\|P_m(x)\|} = \frac{1}{x^*(P_m(x))} \geq 1.$$

Hence for any $(x, x_x^*) \in \Gamma_m$,

$$|x_x^* \circ L \circ P_m x| = |(x_x^*)|_{X_m} \circ L \circ P_m x| \leq \frac{b_m(x)}{\|P_m x\|} |(x_x^*)|_{X_m} \circ L \circ P_m x|$$
$$= |(b_m(x) x_x^*)|_{X_m} L\left(\frac{P_m x}{\|P_m x\|}\right)| \leq \nu(L).$$

Notice that by Definition 2.3, $\pi_1(\Gamma_m) = D_m$ and D_m is dense in $S_{X_{m+1}}$. By Theorem 2.1 applied to Γ_m and $L \circ P_m$,

$$w_{m+1}(L) = \nu(L \circ P_m) \leq \nu(L) = w_m(L)$$

and thus $w_m(L) = w_{m+1}(L)$. Induction on j results in $w_m(L) = w_{m+j}(L)$. □

Proposition 2.5. *Let $P_j \in \mathcal{P}(X_{j+1}, X_j)$ with $\|P_j\| = 1$. For a fixed $m \in \mathbb{N}$, define projections $Q_{m,j} \in \mathcal{P}(X_{m+j}, X_m)$ as $Q_{m,j} = P_m \circ P_{m+1} \circ \cdots \circ P_{m+j-1}$. Then*

$$\lim_{j\to\infty} Q_{m,j} = Q_m$$

where $Q_m \in \mathcal{P}(X, X_m)$ with $\|Q_m\| = 1$ and $X = \overline{\bigcup_{m=1}^{\infty} X_m}$.

Proof. Let $x \in \bigcup_{m=1}^{\infty} X_m$, then there is a minimal index k such that $x \in X_k$. Choose an index j_k such that $m + j_k - 1 \geq k$. Note that $Q_{m,j}x = Q_{m,j_k}x$ for all $j \geq j_k$. This follows from the very definition of

$$Q_{m,j}(x) = Q_{m,j_k} \circ (P_{m+j_k} \circ \cdots \circ P_{m+j-1})(x)$$

and the fact that P_{m+j_k} is a projection onto X_{m+j_k-1} with $X_k \subset X_{m+j_k-1}$ implying

$$(P_{m+j_k} \circ P_{m+j_k+1} \cdots \circ P_{m+j-1})(x) = x.$$

Define the limit of the almost constant sequence $\{Q_{m,j}x\}$ as $\lim_{j\to\infty} Q_{m,j}(x) = Q_m(x)$ for all $x \in \bigcup_{m=1}^{\infty} X_m$. Since a continuous, linear map defined on a dense subspace can be uniquely extended to the whole space, we can extend Q_m uniquely to $X = \overline{\bigcup_{m=1}^{\infty} X_m}$. It is clear that $Q_m \in \mathcal{P}(X, X_m)$ and $\|Q_m\| = 1$. □

Proposition 2.6. *For a fixed $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$ we have*

$$w_{m+j}(L) \leq \nu(L \circ Q_m)$$

for all j, where $\nu(L \circ Q_m)$ denotes the numerical radius of $L \circ Q_m$ with respect to X.

Proof. Since $\nu(L \circ Q_m) = \sup\{|x^* L \circ Q_m x| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$, it is clear that

$$\nu(L \circ Q_m) \geq \sup\{|x^* L \circ Q_m x| : x \in S_{X_{m+j}}, x^* \in S_{X_{m+j}^*}, x^*(x) = 1\}$$

and that $Q_m(x) = Q_{m,j+1}(x)$ for any $x \in X_{m+j}$, implies $\nu(L \circ Q_m) \geq w_{m+j}(L)$. □

Proposition 2.7. *Let X satisfy (LCC) with respect to $\{P_m\}_{m=1}^{\infty}$. Then for any $m \in \mathbb{N}$ and $L \in \mathcal{L}(X_m)$,*

$$\nu(L) = \nu(L \circ Q_m),$$

where $Q_m \in \mathcal{P}(X, X_m)$ are defined in Proposition 2.5.

Proof. By Proposition 2.6, $\nu(L) \leq \nu(L \circ Q_m)$. To prove the converse, we apply Corollary 2.2. Let $Y = \bigcup_{m=1}^{\infty} X_m$. First we prove that for any $j \in \mathbb{N}$,

$$\nu_Y(L \circ Q_m) \leq w_{m+j}(L) \tag{2.2}$$

where ν_Y is defined in Corollary 2.2. To do that fix $\epsilon > 0$. By the definition of ν_Y there exist $j \in \mathbb{N}$, $x \in S_{X_{m+j}}$ and $x^* \in S_{X_{m+j}^*}$ with $x^*(x) = 1$ such that

$$\nu_Y(L \circ Q_m) \leq |x^*(LQ_m)x| + \epsilon \leq |x^*(LQ_{m,j+1})x| + \epsilon \leq w_{m+j}(L) + \epsilon,$$

which shows our claim. Note that by Proposition 2.4, $w_{m+j}(L) = w_m(L) = \nu(L)$ and by Corollary 2.2, $\nu_Y(L \circ Q_m) = \nu(L \circ Q_m)$, which completes our proof. □

Proposition 2.8. *Assume that $\|\cdot\|_X$ satisfies (LCC). Then for any $m \in \mathbb{N}$,*

$$n(X_m) \geq n(X).$$

Proof. Fix $\epsilon > 0$, $m \in \mathbb{N}$ and choose $L \in \mathcal{L}(X_m)$, $\|L\| = 1$ such that $n(X_m) + \epsilon > \nu(L)$. By (LCC) and Proposition 2.7,

$$\nu(L) = \nu(L \circ Q_m).$$

Since $\|Q_m\| = 1$, $n(X_m) + \epsilon \geq n(X)$ for any $\epsilon > 0$. Hence $n(X_m) \geq n(X)$, as required. □

Theorem 2.9. *Let X and X_m and P_m be as in Definition 2.3. Then*

$$n(X) = \lim_m n(X_m).$$

Proof. By Proposition 2.8, $n(X_m) \geq n(X)$ for any $m \in \mathbb{N}$. Hence,

$$\liminf_m n(X_m) \geq n(X).$$

By Theorem 5.1 of [13], we already know that

$$n(X) \geq \limsup_m n(X_m),$$

which proves the equality. □

Now we introduce another condition which permits us to prove an analogous result to Theorem 2.9 in a more general setting.

Definition 2.10. Let X be a Banach space and let $\{X_s\}_{s\in S}$ be a family of subspaces of X such that $X = \overline{\bigcup_{s\in S} X_s}$. Assume that for any $s_1, s_2 \in S$ there exists $s_3 \in S$ such that $X_{s_1} \cup X_{s_2} \subset X_{s_3}$, i.e. the family $\{X_s\}_{s\in S}$ forms a directed set. Suppose for any $s \in S$ there exists $P_s \in \mathcal{P}(X, X_s)$ with $\|P_s\| = 1$. We say the norm on X, $\|.\|_X$ satisfies the *global characterization condition* (GCC) with respect to $\{P_s\}_{s\in S}$ if and only if for any $s \in S$ there exists D_s a dense subset of S_X such that for any $x \in D_s$ there exist $x^* \in S_{X^*}$ a norming functional for x in X and a constant $b_m(x) \in \mathbb{R}_+$ such that $b_s(x)x^* |_{X_s}$ is a norming functional for $P_s x$ in X_s^*. (In fact, if $P_s(x) \neq 0$, then $b_s(x) = \|P_s(x)\|/x^*(P_s(x))$.)

Remark 2.11. Note that if $S = \mathbb{N}$ and $X_s \subset X_z$ for $s, z \in \mathbb{N}$, $s \leq z$ then the global characterization condition (GCC) implies the local characterization condition (LCC).

The above definition is motivated by the space $X = \ell_p$ with $1 < p < \infty$, $X_m = \ell_p^{(m)}$ and a sequence of projections $\{P_m\}_{m=1}^{\infty}$ defined by

$$P_m(x_1, \ldots, x_m, x_{m+1}, \ldots) = (x_1, \ldots, x_m, 0, \ldots).$$

For $x \neq 0$ and $x \in \ell_p$, the form of the norming functional is $x^* = \frac{(|x_i|^{p-1} sgn(x_i))}{\|x\|_p^{p-1}}$ and clearly

$$x^* |_{X_m} = \frac{(|x_i|^{p-1} sgn(x_i))}{\|x\|_p^{p-1}} \quad \text{where } i \in \{1, 2, \ldots, m\}$$

and the norming functional for $P_m x$, $(P_m x)^*$ takes the form

$$(P_m x)^* = \frac{(|x_i|^{p-1} sgn(x_i))}{\|P_m x\|_p^{p-1}} \quad \text{where } b_m(x) = \frac{\|x\|_p^{p-1}}{\|P_m x\|_p^{p-1}}.$$

The above (GCC) is also satisfied for norms of ℓ_1 and c_0 (with the same sequence $\{P_m\}_{m=1}^{\infty}$). Now we prove the following.

Theorem 2.12. *Let X and X_s and P_s be as in Definition 2.10. Then*

$$n(X) = \lim_s n(X_s).$$

Proof. By Theorem 5.1 of [13], we already know that

$$n(X) \geq \limsup_s n(X_s).$$

Now we prove that

$$\liminf_{s} n(X_s) \geq n(X).$$

Fix $s \in S$ and $L \in \mathcal{L}(X_s)$. We show that $\nu(L) \geq \nu(L \circ P_s)$, where $\nu(L)$ denotes the numerical radius of L with respect to X_s and $\nu(L \circ P_s)$ denotes the numerical radius of $L \circ P_s$ with respect to X. To do that, for any $x \in D_s$ select $x_x^* \in S_{X^*}$ satisfying the requirements of Definition 2.10. Let

$$\Gamma_s = \{(x, x_x^*) \in \Pi(X) : x \in D_s\}.$$

Observe that for any $s \in S$,

$$\frac{b_s(x)}{\|P_s(x)\|} = \frac{1}{x^*(P_s(x))} \geq 1.$$

Note that by Definition 2.10, for any $(x, x_x^*) \in \Gamma_s$,

$$\begin{aligned} |x_x^* \circ L \circ P_s x| &= |(x_x^*)|_{X_s} \circ L \circ P_s x| \leq \frac{b_s(x)}{\|P_s x\|} |(x_x^*)|_{X_s} \circ L \circ P_s x| \\ &= |(b_s(x) x_x^*)|_{X_s} L\left(\frac{P_s x}{\|P_s x\|}\right)| \leq \nu(L). \end{aligned}$$

Notice that by Definition 2.10, $\pi_1(\Gamma_s) = D_s$ and D_s is dense in S_X. By Theorem 2.1 applied to Γ_s and $L \circ P_s$, $\nu(L \circ P_s) \leq \nu(L)$, as required. Hence we immediately get that

$$n(X_s) = \inf\{\nu(L) : L \in \mathcal{L}(X_s), \|L\| = 1\} \geq \inf\{\nu(W) : L \in \mathcal{L}(X), \|W\| = 1\} = n(X).$$

Consequently $\liminf_s n(X_s) \geq n(X)$ and finally $\lim_s n(X_s) = n(X)$, as required. □

Now we present an example of a Banach space X satisfying condition (GCC) given in Definition 2.10.

Example 2.13. Let S be a directed and infinite set. Fix $p \in [1, \infty)$. Let $X^p = (\oplus_{s \in S} X_s)_p$ be the direct, generalized l^p-sum of Banach spaces $(X_s, \|.\|_s)_{s \in S}$, defined as

$$X^p = \left\{(x_s)_{s \in S} : x_s \in X_s, card(\text{supp}\,((x_s)_{s \in S})) \leq \aleph_o \text{ and } \sum_{s \in S} (\|x_s\|_s)^p < \infty\right\},$$

where $\text{supp}\,((x_s)_{s \in S}) = \{s \in X_s : x_s \neq 0\}$. Clearly, the norm $x \in X^p$ is

$$\|x\| = \left(\sum_{s \in S} (\|x_s\|_s)^p\right)^{1/p}$$

and in case $S = \mathbb{N}$ and $X_i = X$ for all $i \in \mathbb{N}$, $X^p = \ell_p(X)$. Fix any finite set $W \subset S$. Next, we consider spaces $Z_W = \oplus_{s \in W} X_s$ and the projections

$$P_W((x_s)_{s \in S}) = (z_s)_{s \in S},$$

where $z_s = x_s$ for $s \in W$ and $z_s = 0$ otherwise. Let $F = \{W \subset S : card(W) < \infty\}$. Now we show that (GCC) is satisfied for X, $\{Z_W\}_{W \in F}$ and $\{P_W\}_{W \in F}$. It is obvious that $\|P_W\| = 1$ for any finite subset W of S and $p \in [1, \infty)$. Now assume that $1 < p < \infty$. To show that the characterization condition is satisfied for the norm on X, note that for any $x \in X^p \setminus \{0\}$ there exists a norming functional of the form

$$x^* = \frac{\left(\|x_s\|_s^{p-1} x_s^*(.)\right)_{s \in S}}{\left(\sum_{s \in S} (\|x_s\|_s)^p\right)^{\frac{p-1}{p}}}$$

where $x_s^* \in X_s^*$ is a norming functional for $x_s \in X_s$. Setting $C = (\sum_{s \in S} (\|x_s\|_s)^p)^{\frac{p-1}{p}}$, to see $\|x^*\| \leq 1$, let $y \in X$ be an element with $\|y\| = 1$, then

$$|x^*(y)| = \left|\frac{\sum_{s \in S} (\|x_s\|_s)^{p-1} x_s^*(y_s)}{C}\right| \leq \frac{1}{C} \sum_{s \in S} (\|x_s\|_s)^{p-1} |x_s^*(y_s)|.$$

Applying the Hölder inequality with conjugate pairs p and q:

$$|x^*(y)| \le \frac{1}{C}\left[\left(\sum_{s\in S}\|x_s\|_s^{p-1}\right)^q\right]^{\frac{1}{q}}\cdot\left[\sum_{s\in S}\|y_s\|_s^p\right]^{\frac{1}{p}}.$$

Since $q=\frac{p}{p-1}$ and $\|y\|=\left[\sum_{s\in S}\|y_s\|_s^p\right]^{\frac{1}{p}}=1$ we have $|x^*(y)|\le 1$. It is easy to see that x^* is a norming functional for x because

$$x^*(x)=\frac{1}{C}\sum_{s\in S}\|x_s\|_s^{p-1}x_s^*(x)=\frac{\|x\|^p}{\|x\|^{p-1}}=1.$$

Furthermore, if $P_W\ne 0$, from

$$(P_Wx)^*=\frac{\left(\|x_w\|_w^{p-1}x_w^*(.)\right)_{w\in W}}{\left(\sum_{w\in W}\|x_w\|_w^p\right)^{\frac{p-1}{p}}}$$

and writing $x^*|_{Z_W}$ we obtain that $b_W(x)=\frac{\|x\|^{p-1}}{\|P_Wx\|^{p-1}}$. If $p=1$ then for any $x\in X^1\setminus\{0\}$ there exists a norming functional of the form $(x_s^*(.))_{s\in S}$ where $x_s^*\in X_s^*$ is a norming functional for $x_s\in X_s$. It is easy to see that

$$\|(x_s^*(.))_{s\in S}\|=\sup_{s\in S}\|x_s^*\|.$$

Reasoning as in the previous case we get that (GCC) is satisfied for $p=1$.

Now we present an example of a Banach space X satisfying the (LCC) given in Definition 2.3.

Example 2.14. Let for $n\in\mathbb{N}$ $(Y_n,\|\cdot\|_n)$ be a Banach space. Set $X_1=Y_1$ and $X_n=X_{n-1}\oplus Y_n$. Let for $n\in\mathbb{N}$, let $p_n\in[1,\infty)$. Define a norm $|\cdot|_1$ on X_1 by $|x|_1=\|x\|_1$ and a norm $|\cdot|_2$ on X_2 by

$$|(x_1,x_2)|_2=(\|x_1\|_1^{p_1}+\|x_2\|_2^{p_1})^{1/p_1},$$

where $x_i\in Y_i$ for $i=1,2$. Then having defined $|\cdot|_n$ for $x=(x_1,\dots,x_n)\in X_n$ we can define $|\cdot|_{n+1}$ on X_{n+1} by

$$|(x,x_{n+1})|_{n+1}=(|x|_n^{p_n}+\|x_{n+1}\|_{n+1}^{p_n})^{1/p_n}.$$

Note that if $x\in X_n$, and $m\ge n$, then $|x|_m=|x|_n$. Let

$$F=\{\{y_n\}: y_n\in Y_n \text{ and } y_n=0 \text{ whenever } n\ge m \text{ depending on } \{y_n\}\}.$$

One can identify F with $\bigcup_{n=1}^\infty X_n$, thus enabling us to define for $x\in F$, its norm as:

$$\|x\|_F=\lim_n |x|_n,$$

because for fixed $x\in F$ the sequence $|x|_n$ is constant from some point on by the above mentioned property. Notice that the completion of F (we will denote it by X) is equal to the space of all sequences $\{x_n\}$ such that $x_n\in X_n$ and

$$\lim_n\|Q_nx\|_F=\sup_n\|Q_nx\|_F<+\infty,$$

where for $n\in\mathbb{N}$ and $x=(x_1,x_2,\dots)$

$$Q_n(x)=(x_1,\dots,x_n,0,\dots).$$

Indeed, let $\{x^s\}$ be a Cauchy sequence in X. Notice that by definition of $\|\cdot\|_F$, $\|Q_n|_X\|=1$. Hence for any $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for any $s,k\ge N$ and $n\in\mathbb{N}$,

$$\|Q_n(x^s-x^k)\|_n\le\epsilon.$$

Consequently, for any $n\in\mathbb{N}$, $Q_n(x^s)$ converges to some point in X_n. Hence for any $i\in\mathbb{N}$ $(x^s)_i\to x_i\in Y_i$. Set $x=(x_1,x_2,\dots)$. Then, it is easy to see that $x\in X$, since any Cauchy sequence is bounded and

$$\|Q_n(x)\|_F=\lim_s\|Q_n(x^s)\|_F\le\sup_s\|x^s\|_F<+\infty.$$

Moreover, for fixed $\epsilon>0$, for $s,k\ge N$ and any $n\in\mathbb{N}$,

$$\|Q_n(x^k-x^s)\|_F\le\|x^s-x^k\|_F\le\epsilon.$$

Hence fixing $k \geq N$ and taking limit over s we get for any $n \in \mathbb{N}$,

$$\|Q_n(x^k - x)\|_F \leq \epsilon,$$

and consequently $\|x - x^k\|_X \leq \epsilon$ for $k \geq N$, which shows that $\{x^k\}$ converges to $x \in X$. Hence X is a Banach space. Since for any $x \in X$, $\lim_n \|Q_n(x) - x\| = 0$, F is a dense subset of X. Note that for any $n \in \mathbb{N}$ a map $P_n : X_{n+1} \to X_n$ given by

$$P_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, 0),$$

is a linear projection of norm one. By Definition 2.3 and the proof from Example 2.13, the (LCC) is satisfied for the norm on X.

Remark 2.15. If for any $n \in \mathbb{N}$, $Y_n = \mathbb{R}$ and $p_n = p \in [1, \infty)$ then the space X from Theorem 2.9 is equal to l^p. If $p_n = p \in [1, \infty)$ for any $n \in \mathbb{N}$ and the Banach spaces Y_i are arbitrary then

$$X = Y_1 \otimes_p Y_2 \otimes_p Y_3 \otimes_p \cdots$$

If $Y_n = Y$ for any $n \in \mathbb{N}$, then $X = l^p(Y)$.

Acknowledgment

The second author was supported by the State Committee for Scientific Research, Poland (grant no. N N201 541 348).

References

[1] G. Lumer, Semi-inner-product spaces, Trans. Amer. Math. Soc. 100 (1961) 29–43.
[2] F.F. Bonsall, J. Duncan, Numerical ranges of operators on normed spaces and of elements of normed algebras, in: London Math. Soc., Lecture Note Ser., vol. 2, Cambridge University Press, 1971.
[3] F.F. Bonsall, J. Duncan, Numerical ranges II, in: London Math. Soc., Lecture Note Ser., vol. 10, Cambridge University Press, 1971.
[4] A.G. Aksoy, B.L. Chalmers, Minimal numerical radius extension of operators, Proc. Amer. Math. Soc. 135 (4) (2007) 1039–1050.
[5] A.G. Aksoy, G. Lewicki, Best approximation in numerical radius, Numer. Funct. Anal. Optim. 32 (6) (2011) 593–609.
[6] E. Ed-dari, M.A. Khamsi, A.G. Aksoy, On the numerical index of vector-valued function spaces, Linear Multilinear Algebra 55 (6) (2007) 507–513.
[7] E. Ed-dari, On the numerical index of Banach spaces, Linear Algebra Appl. 403 (2005) 86–96.
[8] C. Finet, M. Martín, R. Payá, Numerical index and renorming, Proc. Amer. Math. Soc. 131 (3) (2003) 871–877.
[9] K.E. Gustafson, D.K.M. Rao, Numerical range: The Field of Values of Linear Operators and Matrices, Springer-Verlag UTX, New York, 1997.
[10] Vladimir Kadets, Miguel Martín, Javier Merí, Rafel Payá, Convexity and smootheness of Banach spaces with numerical index 1, Illinois J. Math. 53 (1) (2009) 163–182.
[11] G. López, M. Martín, R. Payá, Real Banach spaces with numerical index 1, Bull. London Math. Soc. 31 (1999) 207–212.
[12] M. Martín, A survey on the numerical index of Banach space, Extracta Math. 15 (2000) 265–276.
[13] M. Martin, J. Meri, M. Popov, B. Randrianantoanina, Numerical index of absolute sums of Banach spaces, J. Math. Anal. Appl. 375 (1) (2011) 207–222.
[14] A.R. Palacios, Numerical ranges of uniformly continuous functions on the unit sphere of a Banach space, J. Math. Anal. Appl. 297 (2004) 472–476.
[15] E. Ed-dari, M.A. Khamsi, The numerical index of the L_p space, Proc. Amer. Math. Soc. 134 (7) (2005) 2019–2025.
[16] M. Martin, J. Meri, M. Popov, On the numerical index of $L_p(\mu)$-spaces, Israel J. Math. 184 (2011) 183–192.