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Abstract

Considering Banach spaces X_1 and X_2 continuously imbedded in a linear Hausdorff space and the function norms $K(t;f)$ and $J(t;f)$ on X_1+X_2 and $X_1\cap X_2$ respectively, we define two discrete intermediate spaces $[X_1,X_2]_{\Theta,t}$ and $[\![X_1,X_2]\!]_{\Theta,t}$. Some properties of these spaces as well as their interpolation property are established.

ARA UZAYLARI

Özet

Bu makalede doğrusal Hausdorff uzayı χ'in içine sürekli gömme olan iki Banach uzayı X_1 ve X_2 gözönüne alınarak, X_1+X_2 uzayında $K(t;f)$, $X_1\cap X_2$ uzayında $J(t;f)$ işlevsel düzgeleri konularak, Θ'nın bütün ve t'nin birden büyük değerleri için ayırtık ara uzayları $[X_1,X_2]_{\Theta,t}$ ve $[\![X_1,X_2]\!]_{\Theta,t}$ tanımlanmıştır. Bu ara uzaylarının özellikleri ve içdeğer biçimleri incelenmiştir. Ayrıca nükleer gönderimlerin içdeğer biçimine ait bir teorem, belli bir ara uzayı için ispatlanmıştır.

In this paper we consider two Banach spaces X_1 ve X_2, both continuously embedded in a linear Hausdorff space χ and the function norms $K(t;f)$ and $J(t;f)$ on X_1+X_2 and $X_1\cap X_2$ respectively and define discrete intermediate spaces

$$[X_1,X_2]_{\theta,t}=\{f\in X_1+X_2:\ t^{-n\theta}K(t^n;f)\in\Lambda_\infty(\alpha)\}$$

and

$$[\![X_1, X_2]\!]_{\theta,t} = \{f \in X_1 + X_2 : \exists (u_n)_0^\infty \in X_1 \cap X_2 \ni f = \sum_{n=0}^{\infty} u_n$$

$$\text{in} \qquad X_1 + X_2 \quad \text{and} \quad t^{-n\theta} J(t^n; f) \in \Lambda_\infty(\alpha)\}$$

where $-\infty < \theta < \infty$, $t > 1$ a real number $\alpha = (\alpha_n)$ is a stable nuclear exponent sequence and $\Lambda_\infty(\alpha)$ is the corresponding infinite type power series space.

Several properties of these intermediate spaces and their interpolation property are established. We also prove a theorem about an interpolation property of a nuclear map when the intermediate space is of a certain class.

Definitions 1. Let X_1 and X_2 be two Banach spaces contained in a linear Hausdorff space χ such that the identity mapping of X_i $(i = 1, 2)$ into χ is continuous. $X_1 + X_2$ is the algebraic sum of X_1 and X_2 defined as $X_1 + X_2 = \{f \in \chi : f = f_1 + f_2 \quad f_i \in X_i \ \ i = 1, 2\}$; the spaces $X_1 + X_2$ and $X_1 \cap X_2$ are Banach spaces under the norms

$$\|f\|_{X_1+X_2} = \inf_{f=f_1+f_2} (\|f_1\|_{X_1} + \|f_2\|_{X_2})$$

and

$$\|f\|_{X_1 \cap X_2} = \max(\|f\|_{X_1}, \ \|f\|_{X_2})$$

respectively. Furthermore, $X_1 \cap X_2 \subset X_i \subset X_1 + X_2 \subset \chi$, $i = 1, 2$ (see [1]).

We call a Banach space $X \subset \chi$ satisfying

$$X_1 \cap X_2 \subset X \subset X_1 + X_2$$

an *intermediate space* of X_1 and X_2. In the following, we shall discuss two general methods for generating intermediate spaces of X_1 and X_2. First we recall the definitions of the *function norms*;

$$K(t; f) = \inf_{f=f_1+f_2} (\|f_1\|_{X_1} + t\|f_2\|_{X_2}) \quad (0 < t < \infty) \ \text{ on } \ X_1 + X_2$$

and

$$J(t; f) = \max(\|f\|_{X_1}, t\|f\|_{X_2}) \ \ (0 < t < \infty) \ \text{ on } \ X_1 \cap X_2.$$

In the sequel, we need the following inequalities.

Lemma 2: [1].

a) For each element $x \in X_1 + X_2$, $K(t; f)$ is a continuous, monotone increasing, concave function on $(0, \infty)$ and

$$\min(1, t)\|f\|_{X_1+X_2} \leq K(t; f) \leq \max(1, t)\|f\|_{X_1+X_2}.$$

b) For each element $f \in X_1 \cap X_2$, $J(t; f)$ is a continuous, monotone increasing, convex function on $(0, \infty)$ and

$$\min(1, t)\|f\|_{X_1 \cap X_2} \leq J(t; f) \leq \max(1, t)\|f\|_{X_1 \cap X_2}.$$

c) For each $f \in X_1 \cap X_2$,

$$K(t; f) \leq \min(1, \frac{t}{s}) J(s; f) \quad (0 < t, s < \infty).$$

Study of K-Methods:

Given two Banach spaces X_1 and X_2, we define

$$[X_1, X_2]_{\theta,t} = \{f \in X_1 + X_2 : t^{-n\theta} K(t^n; f) \in \Lambda_\infty(\alpha)\}$$

where $-\infty < \theta < \infty$, $t > 1$ is an arbitrary fixed real number and $\alpha = (\alpha_n)$ is a stable nuclear exponent sequence (see [2]). For $f \in [X_1, X_2]_{\theta,t}$ the ℓ-th seminorm of f is given by:

$$\|f\|_{\theta,t,\ell} = \sum_{n=0}^{\infty} t^{-n\theta} K(t^n; f) e^{\ell \alpha_n} < \infty \quad \forall \ell = 1, 2, \ldots.$$

Notation: $A \to B$ means A is continuously injected in B.

Proposition 3:

1) For $n/\alpha_n \to \infty$ and $\theta > 0$, we have $X_1 \cap X_2 \to [X_1, X_2]_{\theta,t} \to X_1 + X_2$.

2) For $n/\alpha_n \to \infty$ and $\theta > 1$, we have $[X_1, X_2]_{\theta,t} \cong X_1 + X_2$.

3) a. For $n/\alpha_n \to \infty$ and $\theta > 0$, we have $X_1 \to [X_1, X_2]_{\theta,t}$.
   b. For $n/\alpha_n \to \infty$ and $\theta > 1$, we have $X_2 \to [X_1, X_2]_{\theta,t}$.

4) If $\theta' < \theta$, then $[X_1, X_2]_{\theta',t} \to [X_1, X_2]_{\theta,t}$.

5) $[X_1, X_2]_{\theta,s} \cong [X_1, X_2]_{\theta,t}$.

Proof:

1) Let $f \in [X_1, X_2]_{\theta,t}$. Then $\|f\|_{\theta,t,\ell} < \infty \quad \forall \ell = 1, 2, \ldots$ using Lemma 2 a) we have

$$\|f\|_{X_1 + X_2} \leq K(t^n; f)$$

since $t > 1$; then

$$\sum_{n=0}^{\infty} t^{-n\theta}\, e^{\ell\alpha_n}\, \|f\|_{X_1+X_2} \leq \|f\|_{\theta,t,\ell} < \infty$$

for each fixed ℓ. However the summation on the left is finite, or equivalently $t^{-n\theta} \in \Lambda_\infty(\alpha)$ since

$$t^{-n\theta} \in \Lambda_\infty(\alpha) \Leftrightarrow (t^{-n\theta})^{1/\alpha_n} \to 0 \Leftrightarrow -\frac{n}{\alpha_n}\theta\, \ell n\, t \to -\infty \Leftrightarrow \frac{n}{\alpha_n} \to \infty.$$

Hence $[X_1, X_2]_{\theta,t} \to X_1 + X_2$.

Now let $f \in X_1 \cap X_2$. From Lemma 2 c) we know that for each $f \in X_1 \cap X_2$, $K(t; f) \leq \min(1, t/s) J(s; f)$, for $s, t \in (0, \infty)$. Setting $s = 1$ and observing $J(1, f) = \|f\|_{X_1 \cap X_2}$ we obtain $K(t^n; f) \leq \|f\|_{X_1 \cap X_2}$. Then

$$\|f\|_{\theta,t,\ell} \leq \|f\|_{X_1 \cap X_2} \sum_{n=0}^{\infty} t^{-n\theta} e^{\ell\alpha_n}.$$

The summation on the right is finite, as seen above, and we get $X_1 \cap X_2 \to [X_1, X_2]_{\theta,t}$.

2) From 1) we know that $[X_1, X_2]_{\theta,t} \to X_1 + X_2$ when $n/\alpha_n \to \infty$ and $\theta > 0$. To show the other inclusion we again use Lemma 2 b) we have

$$K(t^n; f) \leq \max(1, t^n)\|f\|_{X_1+X_2} = t^n \|f\|_{X_1+X_2}$$

for $t > 1$ and

$$\|f\|_{\theta,t,\ell} \leq \left[\sum_{n=0}^{\infty} t^{n(1-\theta)} e^{\ell\alpha_n}\right] \|f\|_{X_1+X_2}.$$

As before, the summation on the right is finite by our assumptions on θ and α_n. Hence $X_1 + X_2 \to [X_1, X_2]_{\theta,t}$.

3) a. Let $f \in X_1$. Then $K(t; f) \leq \|f\|_{X_1}$ and we obtain

$$\|f\|_{\theta,t,\ell} \leq \left[\sum_{n=0}^{\infty} t^{-n\theta} e^{\ell\alpha_n}\right] \|f\|_{X_1}.$$

Since $n/\alpha_n \to \infty$ and $\theta > 0$ the summation on the right in the above inequality is finite and hence $f \in [X_1, X_2]_{\theta,t}$.

b. Let $f \in X_2$. Then $K(t^n; f) \leq t^n \|f\|_{X_2}$ and

$$\|f\|_{\theta,t,\ell} \leq \left[\sum_{n=0}^{\infty} t^{n(1-\theta)} e^{\ell\alpha_n}\right] \|f\|_{X_2}.$$

Under the given conditions

$$\sum_{n=0}^{\infty} t^{n(1-\theta)} e^{\ell\alpha_n} < \infty$$

and hence $f \in [X_1, X_2]_{\theta,t}$.

4) Since for $0 < \theta' < \theta$ and $t > 1$

$$t^{-n\theta} K(t^n; f) e^{\ell\alpha_n} \le t^{-n\theta'} K(t^n; f) e^{\ell\alpha_n},$$

the assertion is clear.

5) a. Let $s > t > 1$ and $f \in [X_1, X_2]_{\theta,s}$. Consider

$$\|f\|_{\theta,t,\ell} = \sum_{n=0}^{\infty} t^{-n\theta} K(t^n; f) e^{\ell\alpha_n} = \sum_{m=0}^{\infty} \sum_{n: s^m \le t^n < s^{m+1}} t^{-n\theta} K(t^n, f) \cdot e^{\ell\alpha_n} \quad (*)$$

We need the following observations:

i) $K(t; f)$ is increasing with t, therefore $K(t^n; f) \le K(s^{m+1}; f)$.

ii) $s^m \le t^n < s^{m+1}$ gives $m\frac{\log s}{\log t} \le n < (m+1)\frac{\log s}{\log t}$. Set $A = [\frac{\log s}{\log t}] + 1$ where $[\ldots]$ denotes the usual largest integer function. Since $\alpha = (\alpha_n)$ is increasing, we have $\alpha_n \le \alpha_{(m+1)A}$. Going back to equation (*) above we get

$$\begin{aligned} \|f\|_{\theta,t,\ell} &\le \sum_{m=0}^{\infty} \sum_{n: s^m \le t^n \le s^{m+1}} s^{-m\theta} K(s^{m+1}; f) e^{\ell\alpha_{(m+1)A}} \\ &\le A s^{\theta} \sum_{m=0}^{\infty} s^{-(m+1)\theta} K(s^{m+1}; f) e^{\ell\alpha_{(m+1)A}} \end{aligned}$$

Since $\alpha = (\alpha_n)$ is a stable nuclear exponent sequence $\alpha_{AM}/\alpha_M \le \delta$; therefore $\|f\|_{\theta,t,\ell} \le A s^{\theta} \|f\|_{\theta,t,\ell\delta}$ and hence $[X_1, X_2]_{\theta,s} \to [X_1, X_2]_{\theta,t}$.

b. Let $s > t > 1$ and $f \in [X_1, X_2]_{\theta,t}$.
From ii) of 5) a., we have

$$1 \le \sum_{n: s^m \le t^n < s^{m+1}} 1 = [\frac{\log s}{\log t}] + 1.$$

Therefore

$$\begin{aligned} \|f\|_{\theta,s,\ell} &= \sum_{m=0}^{\infty} s^{-m\theta} K(s^m; f) e^{\ell\alpha_m} \\ &\le \sum_{m=0}^{\infty} s^{-m\theta} K(s^m; f) e^{\ell\alpha_m} \cdot [\sum_{n: s \le t^n < s^{m+1}} 1] \qquad (**) \end{aligned}$$

We again observe the following:

i) $K(t; f)$ is increasing with t, therefore $K(s^m; f) \leq K(t^n; f)$.

ii) $s^m \leq t^n < s^{m+1}$ gives $m \leq n \cdot \frac{\log t}{\log s} < m+1$. Let $B = [\frac{\log t}{\log s}] + 1$. $\alpha = (\alpha_n)$ is increasing therefore $\alpha_m \leq \alpha_{nB}$. Going back to the equation (**), we have

$$\|f\|_{\theta,s,\ell} \leq \sum_{m=0}^{\infty} \sum_{n:s^m \leq t^n < s^{m+1}} s^\theta t^{-n\theta} K(t^n; f) e^{\ell \alpha_{nB}}$$

Again using the stability of α_n, $\frac{\alpha_{nB}}{\alpha_n} \leq \delta'$; hence

$$\|f\|_{\theta,s,\ell} \leq s^\theta \sum_{n=0}^{\infty} t^{-n\theta} K(t^n; f) e^{\ell\delta' \alpha_n} = s^\theta \|f\|_{\theta,t,\ell\delta'}$$

and hence $[X_1, X_2]_{\theta,t} \to [X_1, X_2]_{\theta,s}$.
Notice that, if $s > t > 1$ from 5) a. and 5) b. we have $[X_1, X_2]_{\theta,t} \cong [X_1, X_2]_{\theta,s}$. Therefore from now on we shall write $[X_1, X_2]_\theta$ for the space $[X_1, X_2]_{\theta,t}$.

Definition 4. An *interpolation pair* (X_1, X_2) is a pair of Banach spaces X_1, X_2 continuously contained in a linear Hausdorff space χ. Let $L(\chi, \mathbf{Y})$ be the space of all linear transformations from $X_1 + X_2$ to $Y_1 + Y_2$ (where $\mathbf{Y}, Y_1, Y_2$ have a similar connotation) such that:

i) For $T \in L(\chi, \mathbf{Y})$, $f_i \in X_i \Rightarrow Tf_i \in Y_i$.

ii) $\|Tf_i\|_{Y_i} \leq M_i \|f_i\|_{X_i}$ $i = 1, 2$.

(i.e., the restriction of T to X_i is a bounded linear transformation from X_i to Y_i). Let X, Y be two intermediate spaces of X_1 and X_2 and of Y_1 and Y_2, respectively. We say X and Y have the *interpolation property* if for each $T \in L(\chi, \mathbf{Y})$ the restriction of T to X is a bounded linear transformation of X into Y.
We know by Proposition 3 that $[X_1, X_2]_\theta$ is an intermediate space when $0 < \theta < 1$ and $n/\alpha_n \to \infty$. The last condition is satisfied for example in case $\alpha_n = \sqrt{n}$ or $\alpha_n = \log n$. If $\alpha_n = \sqrt{n}$ then $\Lambda_\infty(\alpha)$ is isomorphic to the space of entire functions in two complex variables.

Theorem 5. Let (X_1, X_2) and (Y_1, Y_2) be two interpolation pairs of χ and $\mathbf{Y}$ respectively. Then the intermediate spaces $[X_1, X_2]_\theta$ and $[Y_1, Y_2]_\theta$ have the interpolation property.

Proof: Let $T \in L(\chi, \mathbf{Y})$ satisfy the conditions in Definition 4. Now consider

$$K(t^n; Tf) = \inf_{Tf = g_1 + g_2} (\|g_1\|_{Y_1} + t^n \|g_2\|_{Y_2})$$

$$\leq \inf_{f=f_1+f_2} (\|Tf_1\|_{Y_1} + t^n\|Tf_2\|_{Y_2})$$
$$\leq \max(M_1, M_2) \inf_{f=f_1+f_2} (\|f_1\|_{X_1} + t^n\|f_2\|_{X_2})$$

and hence

$$K(t^n; Tf) \leq \max(M_1, M_2) K(t^n; f)$$

and

$$\|Tf\|_{\theta,t,\ell} \leq \max(M_1, M_2)\|f\|_{\theta,t,\ell}.$$

Thus we see the restriction of T to $[X_1, X_2]_\theta$ into $[Y_1, Y_2]_\theta$.

We now give a theorem about an interpolation property of a nuclear map when the intermediate space is of a certain class. Given $0 < \theta < 1$, we say an intermediate space X of X_1 and X_2 is of *class* $K(\theta, X_1, X_2)$ if for each $f \in X$ and for each given $t > 0$, there exist $f_1 \in X_1$ and $f_2 \in X_2$ with $f = f_1 + f_2$ and $\|f_1\|_{X_1} \leq ct^\theta\|f\|_X$, $\|f_2\|_{X_2} \leq ct^\theta\|f\|_X$. The properties of intermediate spaces of class $K(\theta, X_1, X_2)$ and their relation to the study of K-methods are investigated in the classical work of J.L. Lions and J. Peetre [3].

Theorem 6. Let B be a Banach space, (X_1, X_2) be an interpolation pair and X be of class $K(\theta, X_1, X_2)$. Suppose $T : X_1 \to B$ is $\Lambda_\infty(\alpha)$-nuclear, $T : X_2 \to B$ is $\Lambda_\infty(\beta)$-nuclear, where α_n and β_n are stable, nuclear exponent sequences. Then $T : X \to B$ is $\Lambda_\infty(\gamma)$-nuclear, where $\gamma_n = (1-\theta)\alpha_n + \theta\beta_n$.

Proof: Let U be the unit ball of X and $\delta_n(T(U))$ be the n-th Kolmogorov diameter of T (see [4]). Given n, choose $t = \delta_n(T(U_2))/\delta_n(T(U_1))$. Let $u \in U$. Since X is of class $K(\theta, X_1, X_2)$, $\exists u_1 \in X_1$, $u_2 \in X_2$ such that $u = u_1 + u_2$ with $\|u_1\|_{X_1} \leq ct^\theta\|u\|_X$ and $\|u_2\|_{X_2} \leq ct^{\theta-1}\|u\|_X$. Let U_i be the unit ball of X_i. Then

$$T(U) \subset ct^\theta T(U_1) + ct^{\theta-1}T(U_2)$$

and consequently

$$\delta_{2n}(T(U)) \leq ct^\theta\delta_n(T(U_1)) + ct^{\theta-1}\delta_n(T(U_2)).$$

Then we have:

$$\delta_{2n}(T(U)) \leq C'\{\delta_n(T(U_1))\}^{1-\theta}\ \{\delta_n(T(U_2))\}^\theta \qquad (*)$$

Now $T : X_1 \to B$ is a $\Lambda_\infty(\alpha)$-nuclear, therefore $\{\delta_n(T(U_1))\}^{1/\alpha_n} \to 0$. So given $\epsilon > 0$, we can find an N_1 such that $\{\delta_n(T(U_1))\}^{1/\alpha_n} < \epsilon$ for $\forall n \geq N_1$ and an N_2 such that $\{\delta_n(T(U_2))\}^{1/\beta_n} < \epsilon$ for $\forall n \geq N_2$. Let $N = \min(N_1, N_2)$; then for $n \geq N$ we have from (*) observed that

$$\delta_{2n}(T(U)) < C'\ \epsilon^{(1-\theta)\alpha_n+\theta\beta_n}.$$

Let $\gamma_n = (1-\theta)\alpha_n + \theta\beta_n$. Since α_n and β_n are both stable, we have γ_n is also stable. Now

$$\sum_{n=0}^{\infty} \delta_{2n}(T(U))e^{k\gamma_{2n}} \leq \sum^{\infty} \delta_{2n}(T)e^{M\gamma_n} < \infty;$$

the proof is completed using the stability of (γ_n).

Corollary 7. Let B be a Banach space, (X_1, X_2) be an interpolation pair and X be of class $K(\theta, X_1, X_2)$. If $T: X_1 \to B$ is $\Lambda_\infty(\alpha)$-nuclear, where $(\alpha_n) = \alpha$ is a stable nuclear exponent sequence and $T: X_2 \to B$ is continuous, then $T: X \to B$ is $\Lambda_\infty(\alpha)$-nuclear.

Proof: Recall the inequality (*) obtained in the proof of the previous theorem viz.,

$$\delta_{2n}(T(U)) \leq C' \{\delta_n(T(U_1))\}^{1-\theta} \{\delta_n(T(U_2))\}^{\theta}$$

$T: X_2 \to B$ is continuous, therefore $\delta_n(T(U_2)) \leq M$. $T: X_1 \to B$ is $\Lambda_\infty(\alpha)$-nuclear; therefore we can find an N such that $\delta_n(T(U_1)) < \epsilon^{\alpha_n}$ $\forall n > N$ or $\{\delta_n(T(U_1))\}^{1-\theta} \leq \epsilon^{(1-\theta)\alpha_n}$. Using the above equation again and the fact that $(\alpha_n) = \alpha$ is stable, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \delta_{2n}(T)e^{k\alpha_{2n}} &\leq \sum_{n=0}^{\infty} \delta_{2n}(T)e^{k(1-\theta)\alpha_{2n}} \\ &\leq \sum_{n=0}^{\infty} \delta_{2n}(T)e^{k'\alpha_n} < \infty. \end{aligned}$$

Study of J-Methods

Given two Banach spaces X_1 and X_2 we define

$$[\![X_1, X_2]\!]_{\theta,t} = \{f \in X_1 + X_2 : \exists (u_n)_0^\infty \in X_1 \cap X_2 \text{ such that } f = \sum_{n=0}^{\infty} u_n$$

in $X_1 + X_2$ and $t^{-n\theta}J(t^n; u_n) \in \Lambda_\infty(\alpha)\}$ where $-\infty < \theta < \infty$, $\alpha = (\alpha_n)$ is a stable nuclear exponent sequence, $t > 1$ is a fixed real number. Let $f \in [\![X_1, X_2]\!]_{\theta,t}$. Then

$$\|f\|_{\theta,t,\ell} = \inf_{f=\sum u_n} \sum_{n=0}^{\infty} t^{-n\theta}J(t^n; u_n)e^{\ell\alpha_n} \text{ for } \ell = 1, 2, \ldots.$$

Proposition 8.

1) If either $\theta \leq 1$ or n/α_n is bounded $(\theta > 1)$, then

$$X_1 \cap X_2 \to [\![X_1, X_2]\!]_{\theta,t} \to X_1 + X_2.$$

2) a. If either $\theta \leq 0$ or n/α_n is bounded $(\theta > 0)$, then

$$[\![X_1, X_2]\!]_{\theta,t} \cong X_1 \cap X_2.$$

b. $X_1 \cap X_2$ is a dense subspace of $[\![X_1, X_2]\!]_{\theta,t}$.

3) If $\theta' < \theta$, then $[\![X_1, X_2]\!]_{\theta',t} \to [\![X_1, X_2]\!]_{\theta,t}$.

4) a. If either $\theta < 0$ or n/α_n is bounded, $[\![X_1, X_2]\!]_{\theta,t} \to X_1$.

b. If either $\theta \leq 1$ or n/α_n is bounded, $[\![X_1, X_2]\!]_{\theta,t} \to X_2$.

Proof:

1) Let $f \in X_1 \cap X_2$ and let $\delta_{n,k}$ be the Kronecker symbol. Set $u_n = f \cdot \delta_{o,n}$. Then since only $u_0 \neq 0$, $f = \sum_{n=0}^{\infty} u_n$ holds; moreover $J(1; u_0) = J(1; f)$.

$$\|f\|_{\theta,t,\ell} \leq \sum_{n=0}^{\infty} t^{-n\theta} J(t^n; u_n) e^{\ell\alpha_n} = \|f\|_{X_1 \cap X_2}$$

and hence $X_1 \cap X_2 \to [\![X_1, X_2]\!]_{\theta,t}$.
Letting $f \in [\![X_1, X_2]\!]_{\theta,t}$ then $\exists (u_n)_0^\infty \in X_1 \cap X_2$ such that $f = \sum_{n=0}^{\infty} u_n$ in the $X_1 + X_2$-norm and $\|f\|_{\theta,t,\ell} < \infty$ for $\ell = 1, 2, \ldots$. Since $f = \sum_{n=0}^{\infty} u_n$ in the $X_1 + X_2$ norm,

$$\|f\|_{X_1+X_2} \leq \sum_{n=0}^{\infty} \|u_n\|_{X_1+X_2} \quad \text{and} \quad \|u_n\|_{X_1+X_2} = K(1; u_n).$$

Using Lemma 2 part c) we obtain $K(1; u_n) \leq \min(1, t^{-n}) J(t^n; u_n)$; $t > 1$ now gives

$$\|u_n\|_{X_1+X_2} \leq t^{-n} J(t^n; u_n),$$

$$\|f\|_{X_1+X_2} \leq \sum_{n=0}^{\infty} t^{-n} J(t^n; u_n) \leq \sum_{n=0}^{\infty} t^{n(\theta-1)} t^{-n\theta}\, J(t^n; u_n) e^{\ell\alpha_n}.$$

Now if $\theta - 1 \leq 0$ $(\theta \leq 1)$, then

$$\|f\|_{X_1+X_2} \leq \sum_{n=0}^{\infty} t^{-n\theta} J(t^n; u_n)^{\ell\alpha_n}$$

giving $[\![X_1, X_2]\!]_{\theta,t} \to X_1 + X_2$.
Or if n/α_n is bounded and $\theta > 1$, then $t^{n(\theta-1)} \le e^{k\alpha_n}$.

$$\|f\|_{X_1+X_2} \le \sum_{n=0}^{\infty} t^{-n\theta} J(t^n; u_n) e^{(k+\ell)\alpha_n} \text{ implies } [\![X_1, X_2]\!]_{\theta,t} \to X_1 + X_2.$$

2) a. By part 1) we always have $X_1 \cap X_2 \to [\![X_1, X_2]\!]_{\theta,t}$. To show the other inclusion, take $f \in [\![X_1, X_2]\!]_{\theta,t}$ and then $f = \sum_{n=0}^{\infty} u_n$ and hence

$$\|f\|_{X_1+X_2} \le \sum_{n=0}^{\infty} \|u_n\|_{X_1+X_2} = \sum_{n=0}^{\infty} J(1; u_n).$$

Since $J(t; f)$ increasing in t and $t > 1$, $J(1; u_n) \le J(t^n; u_n)$ and

$$\|f\|_{X_1 \cap X_2} \le \sum_{n=0}^{\infty} J(t^n; u_n).$$

Now if $\theta \le 0$ we have $t^{n\theta} \le 1$ and so

$$\|f\|_{X_1 \cap X_2} \le \sum_{n=0}^{\infty} t^{-n\theta} J(t^n; u_n) e^{\ell\alpha_n},$$

or if $\theta > 0$ and n/α_n is bounded ($n \le M\alpha_n$), letting $\theta\ \ell n\ t = S > 0$, we have $t^{n\theta} = e^{nS} \le e^{MS\alpha_n}$ and so

$$\|f\|_{X_1 \cap X_2} \le \sum_{n=0}^{\infty} t^{-n\theta}\ J(t^n; u_n) e^{(MS+\ell)\alpha_n}.$$

In both cases $[\![X_1, X_2]\!]_{\theta,t} \to X_1 \cap X_2$.

b. To show $X_1 \cap X_2$ is dense in $[\![X_1, X_2]\!]_{\theta,t}$, take $f \in [\![X_1, X_2]\!]_{\theta,t}$; then $\exists (u_n)_0^{\infty} \in X_1 \cap X_2$ such that $f = \sum_{n=0}^{\infty} u_n$ and $\|f\|_{\theta,t,\ell} < \infty\ \forall \ell$. Define $f_N = \sum_{n=0}^{N} u_n$ then $f_N \in X_1 \cap X_2$ and

$$f - f_N = \sum_{N+1}^{\infty} u_n.$$

So given $\epsilon, \ell\ \exists N_\ell = N \ni \|f - f_N\|_{\theta,t,\ell} \le \sum_{N+1}^{\infty} t^{-n\theta} J(t^n; u_n) e^{\ell\alpha_n} < \epsilon$.

3) Consider $t^{-n\theta} J(t^n; u_n) e^{\ell\alpha_n} = t^{n(\theta'-\theta)} t^{-n\theta'} J(t^n; u_n) e^{\ell\alpha_n}$; but $\theta' < \theta$ and hence $t^{n(\theta'-\theta)} \le 1$.

4) a. Let $f \in [\![X_1, X_2]\!]_{\theta,t}$ then $\|f\|_{X_1} \le \sum_{n=0}^{\infty} \|u_n\|_{X_1}$. On the other hand, $J(t^n, f) = \max(\|f\|_{X_1}, t^n\|f\|_{X_2})$ gives $\|f\|_{X_1} \le J(t^n; f)$; hence

$$\|f\|_{X_1} \le \sum_{n=0}^{\infty} J(t^n; u_n) = \sum_{n=0}^{\infty} t^{-n\theta} t^{n\theta} J(t^n; u_n) e^{\ell\alpha_n}.$$

If $\theta < 0$, $t^{n\theta} \le 1$ then

$$\|f\|_{X_1} \le \sum_{n=0}^{\infty} t^{-n\theta}\, J(t^n; u_n) e^{\ell\alpha_n}.$$

Or if $\theta > 0$ but n/α_n is bounded, we can write $t^{n\theta} = e^{n\theta}\ \ell n\ t$. For $\theta/nt = S > 0$ and $n/\alpha_n \le M$.

$$\|f\|_{X_1} \le \sum_{n=0}^{\infty} t^{-n\theta}\, J(t^n; u_n) e^{(MS+\ell)\alpha_n}.$$

Hence if either $\theta < 0$ or $n/\alpha_n \le M$, then $[\![X_1, X_2]\!]_{\theta,t} \to X_1$.

b. We know $t^n\|f\|_{X_2} \le J(t^n; f)$. In particular $t^n\|u_n\|_{X_2} \le J(t^n; u_n)$. Let $f \in [\![X_1, X_2]\!]_{\theta,t}$ then $f = \sum_{n=0}^{\infty} u_n$ and

$$\begin{aligned} \|f\|_{X_2} &\le \sum_{n=0}^{\infty} \|u_n\|_{X_2} \le \sum_{n=0}^{\infty} t^{-n} J(t^n, u_n) \\ &\le \sum_{n=0}^{\infty} t^{n(\theta-1)} t^{-n\theta}\, J(t^n; u_n) e^{\ell\alpha_n}. \end{aligned}$$

Now if $\theta - 1 < 0$ then $t^{n(\theta-1)} \le 1$. Or if $\theta - 1 > 0$ and n/α_n is bounded then $t^{n(\theta-1)} \le e^{MS\alpha_n}$ where $S = (\theta - 1)\cdot \ell n\ t \ge 0$ and $n/\alpha_n \le M$. Hence

$$\|f\|_{X_2} \le \sum_{n=0}^{\infty} t^{-n\theta}\, J(t^n; u_n) e^{(MS+\ell)\alpha_n}$$

gives $[\![X_1, X_2]\!]_{\theta,t} \to X_2$.

Theorem 9. Let (X_1, X_2) and (Y_1, Y_2) be two interpolation pairs of χ and $\mathbf{Y}$ respectively. The intermediate spaces $[\![X_1, X_2]\!]_{\theta,t}$ and $[\![Y_1, Y_2]\!]_{\theta,t}$ have the interpolation property.

Proof:

$$\begin{aligned} J(t; Tf) &= \max(\|Tf\|_{X_1}, t\|Tf\|_{X_2}) \le \max(M_1, M_2)(\|f\|_{X_1}, t\|f\|_{X_2}) \\ &\le \max(M_1, M_2) J(t; f) \\ T(1; Tf) &= \|Tf\|_{Y_1 \cap Y_2} \le \max(M_1, M_2)\|f\|_{X_1 \cap X_2} \end{aligned}$$

therefore T is a continuous linear map form $X_1 \cap X_2$ into $Y_1 \cap Y_2$. From the proof of Theorem 5 we have $K(t, Tf) \leq \max(M_1, M_2)K(t; f)$. Letting $t = 1$ we get $\|Tf\|_{Y_1+Y_2} \leq \max(M_1, M_2)\|f\|_{X_1+X_2}$, (i.e. T is continuous linear map form $X_1 + X_2$ into $Y_1 + Y_2$). For $f \in [\![X_1, X_2]\!]_{\theta,t}$ we have $(u_n) \in X_1 \cap X_2 \ni$

$$f = \sum_{n=0}^{\infty} u_n$$

in $X_1 + X_2$-norm; so we also have $Tf = \sum_{n=0}^{\infty} Tu_n$ in $Y_1 + Y_2$ norm where $\{Tu_n\} \in Y_1 \cap Y_2$. Finally $J(t; Tu_n) \leq \max(M_1, M_2)J(t; u_n)$ completes the proof.

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