A GENERALIZATION OF N-WIDTHS

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Abstract

This paper is a study of the n-widths defined by Kolmogorov. In section I we give definitions of n-widths of a set in a Banach space and n-widths of an operator acting between Banach spaces. Several important well known results about this concepts are also included in section I. In section II, we introduce a refined concept of an approximation scheme with respect to which a refined concept of n-widths can be defined. Theorems about generalized n-widths illustrate the fact that this is a genuine generalization. We finish by the question of finding concept of n-widths in the context of Orlicz modular spaces.

I. N-Widths of a Set

Let X be a normed linear space and X_n be its n-dimensional subspace of X, for each $x \in X$ the distance, $d(x;X_n)$ of X_n to x is defined by:

$$d(x;X_n) = \text{Inf } \{\| x - y \| : y \in X_n \}.$$

If there is a $y^* \in X_n$ for which $d(x;Xn) = \| x - y^* \|$ holds then y^* is the best approximation to x from X_n. More than 100 years ago Weierstrass proved that given a continuous function f(x) on [a,b] and $\varepsilon > 0$, there exists a polynomial p(x) such that $\|f(x) - p(x)\| < \varepsilon$. Which tells us that $d(f; P_n) \to 0$ as $n \to \infty$ for each n, where $P_n = \text{span}(1, x^1, \cdots, x^n)$.

Now let us suppose instead of a single element x, we are given a subset A of X, then how well n-dimensional subspace X_n

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of X approximate the subset A? To answer this question one looks at the deviation of A from X_n, namely:

$$d(A;\ X_n) = \operatorname{Sup}\ \{\ d(a,X_n)\ :\ \ a\in A\ \}$$

Thus, $d(A;\ X_n)$ measures the extent to which the "worst element" of A can be approximated from X_n. If we take this process one step further by allowing n-dimensional subspaces X_n vary within X, then the question is how well one can approximate A by n-dimensional subspaces of X? The answer to this question was first given by Kolmogorov.

<u>Definition:</u> Let X be a normed linear space and A a subset of X, the <u>n-th width</u> or <u>n-diameter</u> (or Kolmogorov n-th diameter) of A in X is:

$$d_n(A;X)=\operatorname{Inf}\{\ d(A;X_n):\ X_n \text{ is n-dimensional subspace of } X\}$$

Thus $$d_n(A;X)=\operatorname*{Inf}_{X_n}\ \sup_{a\in A}\ \inf_{x\in X_n}\| a-x \| .$$

We often drop X and write $d_n\ (A)$.
A subspace X_n of X of dimension at most n,for which $d_n(A;X) = d(A;X_n)$ is called the optimal subspace for $d_n(A;X)$.

Besides defining the concept of n-widths, Kolmogorov also computed $d_n(A;X)$ for some particular cases. For example, he showed that [13]

$$d_o(A\ ;\ L_2) = \infty, \text{ and}$$

$$d_{2n-1}\ (A\ ;\ L_2) = d_{2n}\ (A\ ;\ L_2) = n^{-k}$$

where $L_2 = L_2\ [0;2\pi]$ square integrable functions on $[0;2\pi]$, and

$$A=\{\ f\ :\ \ f\in W_2^{(k)}\ ,\ \ \|f^{(k)}\|\leq 1\ \}$$

and $W_2^{(k)}$ is the space of 2π periodic, real valued, (k-1) times differentiable functions whose (k-1) st derivative is absolutely continuous and whose kth derivative is in L_2.

In general it is impossible to obtain $d_n(A\ ;\ X)$ for all A and X although there is a considerable effort devoted to calculate $d_n(A;X)$ for specific choices of A and X [See 13]. A usual method of calculation is to find an upper bound by calculating $d_n(A;X_n)$ for a "reasonable" choice of X_n, and then to show that the quantity obtained is infact the lower bound as well. It is also important to determine asymptotic behavior of $d_n(A;X)$ as $n \to \infty$. In many cases very simple n-dimensional subspaces may approximate A in an asymptotically optimal manner.

N-widths of integral operators and n- widths of Soboloev spaces can be found in [13]. Let D be a fixed nxn matrix and the set A is

$$A = \{ Dx : \| x \|_{l_p^n} \le 1 \} \subset l_q^n \quad \text{where } p, q \in [1,\infty]$$

Very little are known about $d_n(A; l_q^n)$ unless p=q=2 or p=q=∞ and D is totally positive. Therefore one usually considers the case that D is a diagonal matrix. Following is such a result the proof of which can be found in [13] :

Let D = diag $\{a_1, a_2, \cdots, a_m\}$ be an mxm real diagonal matrix, assume that $a_1 \ge a_2 \ge \cdots \ge a_m > 0$. Given $1 \le q \le p \le \infty$. Let $1/r = 1/q - 1/p$. Then

$$d_n(D_p; l_q^m) = \Big(\sum_{k=n+1}^{m} a_k^r\Big)^{1/r}, \text{ where } D_p = \{ Dx : \| x \|_p \le 1 \}$$

It can be easily seen that the n-width $d_n(A;X)$ can also be written as

$$d_n(A ; X) = \operatorname*{Inf}_{X_n} \inf \{ \varepsilon > 0 : A \subset \varepsilon U_X + X_n \}$$

where U_X is the unit ball of X. This definition allows us the following generalization.

Let A, B be non-empty subsets of a normed linear space X. Assume that B absorbs A then n- width of A with respect to B, $d_n(A, B; X)$, is defined by

$$d_n(A, B; X) = \operatorname*{Inf}_{X_n} \inf \{ \varepsilon > 0 : A \subset \varepsilon B + X_n \}.$$

This definition is used in the concept of diametral dimension of nuclear spaces [3, 12].

The basic properties of n-widths can be found in [9,10,12,13]. It is easy to show that if X be a normed linear space and A be a closed subset of X, then

A is compact if and only if $d_n(A) \downarrow 0$ and A is bounded.

N-Widths of an Operator

Let $T : X \rightarrow Y$ be an operator between two normed linear spaces. The n-width of T:

$$d_n(T) = d_n(T(U_X); Y) = \text{Inf} \{ r > 0 : T(U_X) \subset r U_Y + Y_n \}.$$

It is known that

T is compact if and only if $d_n(T) \downarrow 0$.

<u>Notation:</u> Let F(X,Y) and K(X,Y) denote the closed subsets of L(X,Y) consist of finite rank and compact operators respectively. F(X,Y) is a subset of K(X,Y) and need not equal K(X,Y).The n-th approximation number $a_n(T)$ of $T \in L(X,Y)$ for n = 0,1,2,$\cdots$ defined as

$$a_n(T) = \text{Inf} \{ \| T - A \| : A \in F_n(X,Y) \}$$

where $F_n(X,Y)$ is the collection of all mappings whose range is at most n- dimensional. It is known that

$T \in F(X,Y)$ if and only if $\lim_{n \to \infty} a_n(T) = 0$

so, $a_n(T)$ provides a measure how well T can be approximated by finite mappings whose range is at most n-dimensional. Algebraic and analytic properties of $a_n(T)$ can be found in [9,12]. The following theorem [5] gives the relationship between the n-widths and the approximation numbers:

<u>Theorem:</u> For any $T \in L(X,Y)$, the following inequality is valid:

$$d_n(T) \leq a_n(T) < (\sqrt{n} + 1)\, d_n(T).$$

The best value p(n) for which $a_n(T) < p(n)\, d_n(T)$ is not known. But p(n) can not be replaced by a constant independent of n. There are spaces for which

$$\lim_n d_n(T) = 0 \quad \text{and} \quad \lim_n a_n(T) \neq 0.$$

It should be noted that if $T: H \to H$ is a compact operator on a Hilbert space H, then one can define $(d_n(T))$ as the sequence of eigenvalues of the positive operator $|T| = (TT^*)^{1/2}$. In this case:

i) $a_n(T) = d_n(T)$

ii) $\prod_{i=1}^{n} |\lambda_i(T)| \leq \prod_{i=1}^{n} d_i(T)$ (H.Weyl Inequality, 1949) [14]

where $(\lambda_i(T))$ is an eigenvalue sequence [6]. The last inequality can be viewed as relating the eigenvalues of T to those of $|T|$.

From (ii) it may be deduced that for all $n \in N$ and all $p \in (0,\infty)$,

$$\sum_{i=1}^{n} |\lambda_i(T)|^p \leq \sum_{i=1}^{n} a_i^p(T)$$

which implies that if $(a_i(T)) \in l_p$ then $(\lambda_i(T)) \in l_p$. This result can be used to obtain information about the distribution of eigenvalues of certain non-self-adjoint elliptic problems [see chapter XII of 4]. Although Weyl's inequality was given in Hilbert space setting, a simple proof of it in the context of Banach spaces can be found in [4].

II. Generalized N-Widths

Let X be a Banach space and $(A_n)_{n \in N}$ be a sequence of subsets of X satisfying the following conditions:

1) $(0) = A_0 \subset A_1 \subset A_2 \subset \cdots \subset X$

2) $\lambda A_n \subset A_n$ for all scalars λ and $n = 1,2,\cdots$

3) $A_n + A_m \subset A_{n+m}$ for $m,n = 1,2,\cdots$

then (X,A_n) is called an approximation scheme. The use of an approximation scheme on a Banach space and its use in approximation theory can be found in Butzer and Scherer [2] and in Pietsch [11].For example one can consider $X = l_p$ with $p > 1$ and A_n to be the set of all scalar sequences (a_m) such that $a_m = 0$ when $m > n$ or $X = L_p[0,1]$ $2 \le p \le \infty$ and $A_n = L_{p+1/n}[0,1]$.

Instead of looking at subset of X with the above properties, if we consider $Q = Q_n(X)$ a family of subsets of X with the same properties (replace A_n by Q_n in above 1,2,3) then it is possible to define a refined notion of approximation scheme. For example, for a given Banach space X, Q_n will be the set of all n-dimensional subspaces or for a given Banach space E, consider $X = L(E)$ and Q_n will be the set of all n-nuclear maps on E.

This refined approximation scheme allows us to define n- width $d_n(A ; Q)$ with respect to this approximation scheme as follows:

Definitions: 1) Let U_X be the closed unit ball of X and D be a bounded subset of X. Then the generalized n-th width of D with respect to U_X is defined by:

$$d_n(D;Q) = \text{Inf} \{ r > 0 : D \subset r U_X + A \quad A \in Q_n(X) \}.$$

The generalized n-th width $d_n(T;Q)$ of $T \in L(X)$ is defined as $d_n(T(U_X); Q)$. From the stated definition it follows that $(d_n(T;Q))$ is non- increasing sequence of non- negative numbers and

$$\|T\| = d_0(T\ Q) \ge d_1(T;Q) \ge \cdots \ge d_n(T;Q) \ge \cdots$$

Notice that if one choses Q_n to be the at most n- dimensional subspaces of X, then $d_n(T;Q)$ coincides with the usual definition of $d_n(T)$.

2) A bounded set D of X is said to be Q- compact set if $\lim_n d_n(D;Q) = 0$ and $T \in L(X)$ is said to be Q-compact operator if $\lim_n d_n(T;Q) = 0$. That is $T(Ux)$ is a Q- compact set.

We assume that each $A_n \in Q_n (n \in N)$ is separable, then it is immediate from the definitions that Q-compact sets are separable and Q-compact maps have separable range.

Q-Compactness Does Not Imply Compactness

We show that in $L_p[0,1]$, $2 \le p \le \infty$, with suitably defined approximation scheme, one can find a Q-compact map which is not compact.

Let $[r_n]$ be the space spanned by Radamacher functions and R_p be the closure of $[r_n]$ in L_p [0, 1]. Define an approximation scheme A_n on $L_p[0,1]$ as $A = L_{p+1/n}$. $L_{p+1/n} \subset L_{p+1/n+1}$ gives us $A_n \subset A_{n+1}$ for $n = 1, 2, \cdots$ and it is easily seen that $A_n + A_m \subset A_{n+m}$ for $n, m = 1,2, \cdots$ and that $\lambda A_n \subset A_n$.

Next we observe the existence of a projection

$$P : L_p[0,1] \text{ -----> } R_p \quad \text{for } p \ge 2 .$$

In fact $P = j \circ P_2 \circ i$ where i, j are isomorphisms shown in the diagram below and P_2 is the orthogonal projection.

$$L_p \xrightarrow{\ i\ } L_2 \xrightarrow{\ P_2\ } R_2 \xrightarrow{\ j\ } R_p .$$

<u>Theorem:</u> For $p \ge 2$ the projection $P: L_p[0, 1] \text{ ----> } R_p$ is Q-compact but not compact.

It is easy to show that $P(U_{L_p}) \subset L_{p+1/n}$ thus $d_n(P;Q) \text{ ---> } 0$. To see P is not a compact operator observe that $\dim R_p = \infty$ and $I - P$ is a projection with kernel R_p, thus $I - P$ is not a Fredholm operator so, P can not be a compact operator. For details of the proof of the above theorem see [1].

<u>Definitions:</u> 1) A sequence $(x_{n,k})_k \subset A_n$ is said to be order c_o-sequence if followings hold:

i) For every $n \in N$, there exists an $A_n \in Q_n$ and $(x_{n,k})_k \subset A_n$.

ii) $\|x_{n,k}\|$ --> 0 as $n \text{--> } \infty$ uniformly in k.

2) Suppose $(x_{n,k})_k$ is an order- c_o-sequence in X. Then the set S_m associated with $(x_{n,k})_k$ is:

$$S_m = \{\sum_{n=1}^{m} \lambda_n x_{n,k(n)} \quad : \quad \sum_{n=1}^{m} |\lambda_n| \le 1 \}.$$

where $x_{1,k(1)} \in A_1$, $x_{2,k(2)} \in A_2$, $\cdots$, $x_{m,k(m)} \in A_m$.

Clearly $S_m \subset A_1 + A_2 + \cdots + A_m \in Q_{m^2}$.So if Q_n is n-dimensional, S_n is at most n^2-dimensional.

For a bounded set D in X, we define the <u>ball measure of non-Q-compactness</u> $\alpha(D;Q)$ of D by

$$\alpha(D, Q) = \inf\{r > 0 : \text{order-}c_o\text{-sequence } (x_{n,k})_k \text{ and associated } S_n \text{ such that } D \subset \bigcup_{x \in S_n} B(x, r) \text{ for some } n\}.$$

Following are the several results about Q- compact sets and Q- compact maps. The proofs of all are presented in [1].

<u>Theorems:</u> 1) Suppose (X, Q_n) is an approximation scheme with sets A_n Q_n assumed to be solid (i.e., $|\lambda|$ A_n A_n for $|\lambda| \le 1$). Then a bounded set D of X is Q- compact if and only if there exists an order co- sequence $(x_{n,k})_k$ A_n such that

$$D \subset \{\sum_{n=1}^{\infty} \lambda_n x_{n,k(n)} \quad : \quad x_{n,k(n)} \in (x_{n,k}), \ \sum_{n=1}^{\infty} |\lambda_n| \le 1\}.$$

This theorem can be considered an analogue of the Dieudonne-Schwartz lemma on compact sets in terms of standard Kolmogorov diameter. Again if one choses Q_n to be at most n- dimensional subspaces of X, one can show that Q- compactness of a bounded subset D coincides with the usual definition of compactness of D.

2) The uniform limit of Q-Compact maps is Q- compact and an ideal of Q- compact maps is equal to its surjective hull.

3) Given (X,Q_n), assume that each $A_n \in Q_n$ is a vector subspaces of X. Then, a bounded set D of X is Q-compact if and

only if $D \subset T(U_E)$ for a suitable Banach space E and a Q- compact map T on E into X.

4) Let X be a Banach space with approximation scheme Q_n and let D be a bounded subset of X; then

$$\alpha(D;Q) = \lim_{n \to \infty} d_n(D;Q)$$

Theorem (4) defines the ball measure of non-Q-compactness as a limit of generalized n-widths.

We finish by posing the following question: Suppose Orlicz function space L^Ψ is given (for definitions see [7]). If L^Ψ is considered with the norm $\| \ \|_\psi$. It is well known that $(L^\Psi, \| \ \|_\psi)$ is a Banach space [8]. Therefore n-widths $d_n(A)$ of a norm, bounded set A can be defined as usual. On the other hand it is more natural to consider L^Ψ with its Orlicz modular ρ where

$$\rho(f) = \int_X \psi(f(x))\ dx$$

after all $\|f\|_\psi = \text{Inf}\ \{\ \lambda > 0 : \rho(f/\lambda) \le 1\ \}$, defined in terms of this modular. Can one define an n-width of a modular bounded set A, say $d_n(A,\rho)$, such that $d_n(A,\rho) = d_n(A)$ and can this $d_n(A,\rho)$ be related with measures of non-compactness?

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