The Spectral Presheaf of an Orthomodular Lattice

Some steps towards generalized Stone duality

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Abstract

In the topos approach to quantum physics, a functor known as the spectral presheaf of a von Neumann algebra plays the role of a generalized state space. Mathematically, the spectral presheaf also provides an interesting generalization of the Gelfand spectrum, which is only defined for abelian von Neumann algebras, to the nonabelian case. A partial duality result, analogous to Gelfand duality, exists for this spectral presehaf. This dissertation will begin to explore generalizations of the notion of a spectral presheaf and work towards a duality theory for certain non-distributive lattices. Specifically, we will define the spectral presheaf of an orthomodular lattice, which is a generalization of the Stone space of a Boolean algebra, and prove that it is a complete invariant: two orthomodular lattices are isomorphic if and only if their spectral presheaves are. The analogous result also holds for complete orthomodular lattices. We will map the elements of a complete orthomodular lattice L to the algebra of clopen subobjects of the spectral presheaf of L; using the right adjoint of this map, we show that these clopen subobjects, modulo an equivalence relation, form a complete lattice isomorphic to L. This can be seen as a generalization of Stone's representation theorem for Boolean algebras. We conclude by discussing some other possible generalizations of the spectral presheaf, including Lie groups.

Contents

Chapter 1

Introduction

1.1 Motivation and results

The spectral presheaf is a key object in the topos approach to quantum physics [9], where it serves as a generalized state space for a quantum system, an analogy to the state space of a classical system. The construction of the spectral presheaf is of mathematical interest as well, as it provides a new notion of spectrum for nonabelian von Neumann algebras. It generalizes the Gelfand spectrum, which is only defined for abelian von Neumann algebras, and more generally, abelian C^*- and Banach algebras. There is a dual equivalence between abelian von Neumann algebras and their Gelfand spectra; an partial duality type result exists for the spectral presheaf as well. Two von Neumann algebras with no type I_2 summand have isomorphic spectral presheaves if and only if they are isomorphic as weakly closed Jordan $*$-algebras [11].

The spectral presheaf of an orthomodular lattice. These results strongly suggest that we consider generalizations of the spectral presheaf to other noncommutative and nondistributive structures, in particular those that would yield potential generalizations of classical dualities. This direction has never been explored, and in this dissertation we take the first steps to generalize the spectral presheaf beyond the case of operator algebras. Concretely, the notion of a spectral presheaf is generalized to orthomodular lattices, which are a nondistributive analog of Boolean algebras. Orthomodular lattices have a connection to quantum logic and to von Neumann algebras; the projection lattice of a von Neumann algebra is a complete orthomodular lattice.

The first main result is Theorem 5.7.3, which states that two orthomodular lattices L and M are isomorphic if and only if their spectral presheaves $\underline{\Omega(L)}$ and $\underline{\Omega(M)}$ are isomorphic. The analogous result also holds for complete orthomodular lattices, see Theorem 6.3.6. These results show that the spectral presheaf is a complete invariant of an orthomodular lattice. Our results can be seen as steps towards a generalization of Stone duality to orthomodular lattices. Missing from a full duality result is an independent characterization of the category of spectral presheaves of orthomodular lattices. Currently, we regard them as presheaves over posets with values in Stone spaces (Stonean spaces in the complete case), but this category contains more objects than those of the form $\underline{\Omega(L)}$ for some orthomodular lattice L.

An analog of the Stone representation theorem. One may wonder if the Stone representation theorem, which states that each Boolean algebra is isomorphic to the algebra of clopen subsets of its Stone space [20], also has a generalization for orthomodular lattices. To this end, we consider clopen subobjects (subfunctors) of the spectral presheaf of a complete orthomodular lattice and show that they form a complete bi-Heyting algebra. We cannot expect to have a lattice isomorphism from a nondistributive orthomodular lattice L to the distributive bi-Heyting algebra $Sub_{cl}\underline{\Omega(L)}$ of clopen subobjects of its spectral presheaf. However, we define an injective, join-preserving map $\underline{\delta^o} : L \to Sub_{cl}\underline{\Omega(L)}$ and its meet-preserving right adjoint $\epsilon : Sub_{cl}\underline{\Omega(L)} \to L$ and show that $Sub_{cl}\underline{\Omega(L)}$, modulo an equivalence relation defined in terms of ϵ, is a complete lattice isomorphic to L. This is Theorem 7.5.4, our analog of Stone's representation theorem for orthomodular lattices.

These results show that it is of interest to generalize the construction of the spectral presheaf to further nonabelian and nondistributive structures. The techniques developed here will be useful in such future explorations. In the concluding section, we briefly discuss some potential generalizations.

In the rest of this introductory section, we first sketch the physical motivation behind the topos approach to quantum theory, which provides a new mathematical formalism to describe quantum systems. We then briefly describe some aspects of the topos approach including its main tool, the spectral presheaf of a von Neumann algebra. Finally, we consider quantum logic and its relation to orthomodular lattices in order to justify our choice of orthomodular lattices as a first generalization of the spectral presheaf.

1.2 State spaces in classical and quantum physics

In classical physics, including mechanics, electromagnetism, and relativity theory, any physical system can be described by its state space. The state space can be thought of as a set, with some additional structure, containing all possible states that a system

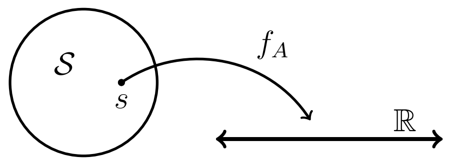


Figure 1.1: A classical state space $\mathcal{S}$, for which observable A is represented by function $f_A : \mathcal{S} \to \mathbb{R}$.

can be in. Measurable physical properties of the system, such as position, momentum, energy, etc., are called *observables*. Observables can be represented by functions from the state space to the real numbers; these functions map each state to the value of the observable when the system is in that state. A graphical interpretation of this can be seen in Figure 1.1, where f_A is the function corresponding to observable A.

In classical physics, any observable of a given system can be measured concurrently with any other observable; when a physical system is in some given state s, it is possible to simultaneously know the value of every one of its observables. These values are simply the numbers $f_A(s)$, for A varying over the set of observables. Though this may seem quite obvious, such statements do not hold for quantum systems.

During the late 19th and early 20th centuries, various researchers began to notice that certain unexplainable effects did not fit in with the traditional concepts of classical physics; these results are summarized in the first chapter of [25]. While Young's 1802 double-slit experiment demonstrated the wave-like properties of electromagnetic radiation, studies in the late 1800s of black-body radiation and atomic spectra, as well Einstein's 1905 investigation of the photoelectric effect, all indicated that electromagnetic radiation also behaved as though it consisted of particles. Further diffraction experiments indicated that quantum particles, such as electrons, also exhibited wave-like behavior. This wave-particle duality contradicted many of the accepted assumptions of classical physics. For example, wave-particle duality as interpreted mathematically by the Heisenberg uncertainty principle indicates that it is not possible to know both the position and momentum of a quantum particle at the same time.

Such results suggested that classical physics was not sufficient to describe certain physical phenomena, and led to the development of quantum theory and the Hilbert space formalism. This mathematical formalism, originally presented in full by von Neumann in 1932 [32], replaces the classical state space picture with the more abstract Hilbert space $\mathcal{H}$, a complex linear space with an inner product. States are unit vectors in $\mathcal{H}$, or, more generally, density matrices. Observables are represented by linear operators $A : \mathcal{H} \to \mathcal{H}$. In general, an observable A does not have a definite value in a given state, unless that state happens to be an eigenstate of A. However, definite values arise when measurements are taken. The eigenvalues of a self-adjoint operator A are the possible outcomes of the measurement, and when the measurement is made the state changes into the eigenstate corresponding to the outcome of the measurement.

For quite a while, the question remained open whether quantum theory could be explained by, or absorbed into, an underlying classical state space theory, in analogy to the step from thermodynamics to statistical mechanics. The impossibility of such a construction was shown explicitly by Kochen and Specker in 1967; they proved that a quantum system cannot have a state space that is a set [23].

The Hilbert space formalism has served as the mathematical underpinning of quantum mechanics since its introduction by von Neumann, and it is extremely useful in describing many processes of quantum systems. However, one drawback of the Hilbert space formalism is that it does not actually describe quantum systems as they are, but merely gives predictive probability distributions for measurements. An external observer and the measurement that observer performs often play a central role. For quantum mechanics in the laboratory, where the quantum systems are very small, such a description may be sufficient. However, when considering a theory of quantum gravity or quantum cosmology, systems can be as large as the universe, and it makes no sense to talk about measuring such systems. Thus, if one hopes to develop a justifiable theory of quantum gravity or quantum cosmology, it is reasonable to seek an alternate formalism of quantum physics, one which doesn't depend on measurements in a fundamental way for its interpretation. This is what the topos approach seeks to do, and one central aspect is a generalized state space formulation, in analogy to classical physics. Of course, one must take into account the obstacle provided by the Kochen-Specker theorem.

1.3 The topos approach and the spectral presheaf

The topos approach considers a generalized state space of a quantum system that is not a set, but, being an object in a topos, still has some desirable set-like properties. This generalized state space is not subject to the Kochen-Specker theorem, but at the same time can still serve the same purpose as a state space. We now give a brief overview of this generalized state space, called the *spectral presheaf of a von Neumann algebra*, and the intuition behind it, and mention some of its important properties.

This discussion is not mathematically rigorous, but is intended to introduce and explain the intuition behind the spectral presheaf.

The spectral presheaf was originally defined in [19], and subsequently explored in [5], [16], and [6]. A comprehensive overview of the topos approach can be found in [9], and a more complete description of the spectral presheaf and its role in quantum physics can be found in [10].

As in the Hilbert space formalism, observables are represented by self-adjoint operators in a Hilbert space. Two self-adjoint operators commute when the two observables they represent are co-measurable, implying that their values can be known simultaneously. For example, a quantum particle's total spin and its spin in a certain coordinate direction can be known simultaneously. Two self-adjoint operators do not commute when the observables are not co-measurable and their values cannot be known simultaneously; an example is a quantum particle's position and momentum.

The bounded linear operators of a given Hilbert space $\mathcal{H}$, including the self-adjoint ones that correspond to observables, form an algebra $\mathcal{B}(\mathcal{H})$. More generally, one can consider a von Neumann algebra, which is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. The algebra $\mathcal{B}(\mathcal{H})$ itself is a von Neumann algebra and is not abelian unless $\mathcal{H}$ is one-dimensional because not all observables are co-measurable. However, $\mathcal{B}(\mathcal{H})$ has von Neumann subalgebras that are abelian. Within such an abelian subalgebra, all self-adjoint operators commute and thus all observables of the abelian subalgebra are co-measurable. For this reason, such an abelian subalgebra is sometimes called a *classical context*, because observables act as they would in a classical system.

These abelian subalgebras can be ordered by inclusion to form a partially ordered set whose elements are the classical contexts of the quantum system; we call this poset the context category. For each classical context, the associated observables can be assigned values simultaneously because they are co-measurable; in this way, it is possible to get a space of 'local valuations,' also called local states, on classical contexts. In order to obtain some sort of global state space, it only remains to combine these local valuations in some way. This is precisely what the spectral presheaf does.

The spectral presheaf $\underline{\Sigma(\mathcal{N})}$ of von Neumann algebra $\mathcal{N}$ is a contravariant functor with domain the context category of $\mathcal{N}$. The functor maps each abelian von Neumann subalgebra V of $\mathcal{N}$ to its Gelfand spectrum, which is the set of algebra homomorphisms from V to $\mathbb{C}$ with a certain topology that makes it into a compact Hausdorff space. The Gelfand spectrum can be interpreted as the set of *local valuations* on the classical context V which assign values to observables of V, a 'local state space' of sorts. To each inclusion $i_{V'V} : V' \hookrightarrow V$ of abelian subalgebras, the spectral presheaf assigns the canonical restriction map from $\underline{\Sigma(\mathcal{N})}_V$ to $\underline{\Sigma(\mathcal{N})}_{V'}$, which acts as $\lambda \mapsto \lambda|_{V'}$. This map is continuous, surjective, closed, and open. This is how the local state spaces are linked together.

Most importantly, while the spectral presheaf provides a sort of global state space, it is not a set. It gives a way of using local spaces of valuations to construct a sort of global state space for all of $\mathcal{N} = \mathcal{B}(\mathcal{H})$ in a way that doesn't violate the Kochen-Specker theorem. In fact, the Kochen-Specker theorem is equivalent to the fact that the spectral presheaf has no global sections [19, 16], which means that none of the local valuations can be extended to a global one. Instead of a set, the spectral presheaf is a set-valued functor, whose codomain consists of topological spaces linked together by continuous maps. This functor is in fact an object in a topos, specifically the topos of presheaves over the context category of $\mathcal{B}(\mathcal{H})$. A topos is a special kind of category whose objects have several set-like properties.

One can show that if $\mathcal{N}$ and $\mathcal{M}$ are two von Neumann algebras with no type I_2 summand such that there is an isomorphism between their spectral presheaves, then there is a Jordan $*$-isomorphism between $\mathcal{N}$ and $\mathcal{M}$ [11]. This shows how much algebraic information the spectral presheaf of a von Neumann algebra encodes.

In classical physics, propositions about the values of observables are represented by subsets of the state space. For example, consider the proposition "the position observable, denoted Q, has a value between -1 and 5 (in suitable units)", which for short we will denote "$Q\varepsilon[-1, 5]$", and let f_Q be the real-valued function on the state space representing position. Then the subset $f_Q^{(-1)}([-1, 5])$ of the state space represents this proposition. As all objects in a topos have subobjects, one can also talk about the subobjects of the spectral presheaf. Via a process known as daseinisation, from a proposition such as "$Q\varepsilon[-1, 5]$", one can construct a subobject of the spectral presheaf that is a 'best approximation' to this proposition. In this way the spectral presheaf captures one of the most important properties of a classical state space, the existence of subsets corresponding to propositions.

Beyond its use as a state space in quantum physics, the spectral presheaf is an interesting mathematical structure itself. The spectral presheaf utilizes the well-known mathematical duality between abelian von Neumann algebras and their Gelfand spectra. It also provides a new sort of spectrum and a partial duality result for nonabelian von Neumann algebras, associating with it not a simple Gelfand spectrum (which does not exist for nonabelian algebras), but a collection of Gelfand spectra linked together by continuous maps. The notion of a spectrum of a nonabelian operator algebra and duality results, even partial ones, are of interest in noncommutative geometry and potentially in other areas of mathematics.

The use of abelian subalgebras, called contexts, and a classical duality for each context, suggest that the idea of a spectral presheaf may have broader applications beyond representing a generalized state space of a quantum system. This dissertation begins to explore such a notion of generalized spectral presheaves, beginning with the spectral presheaf of an orthomodular lattice.

1.4 Orthomodular lattices and quantum logic

As we saw in the previous section, in classical physics propositions of the form "observable A has a value in the Borel set Δ" are represented by (Borel) subsets of the state space. The Borel subsets form a complete Boolean algebra, so one can take the conjunction $\wedge$, disjunction $\vee$, and negation $\neg$ of these propositions. In particular, conjunction and disjunction of propositions distribute over each other.

However, in the Hilbert space formalism of quantum theory, a similarly straightforward kind of algebra of propositions does not exist. As originally proposed by Birkhoff and von Neumann in 1936, a proposition about a quantum system corresponds to a closed linear subspace of a Hilbert space [2]. Equivalently, it corresponds to the projection operator that projects onto that subspace. Conjunction, disjunction, and negation can be defined for these propositions as the intersection, closure of span, and orthogonal complement of the corresponding subspaces. However, these operations do not satisfy distributivity, meaning that quantum logic is nondistributive and cannot be modeled by a Boolean algebra as classical logic can. Instead, the closed subspaces of a separable Hilbert space with these operations form an orthomodular lattice, a nondistributive analog of a Boolean algebra [27].

A von Neumann algebra can be seen as a mathematical abstraction of the algebra of bounded operators on a Hilbert space, and its lattice of projections is an abstraction of the lattice of projections of a Hilbert space [10]. This lattice is in fact an orthomodular lattice, and is closely connected to the interpretation of quantum logic given above where projections correspond to propositions about the quantum system [22]. In considerations of the spectral presheaf of a von Neumann algebra, its lattice of projections plays an essential role.

The spectral presheaf of an orthomodular lattice is in this sense a natural generalization of the spectral presheaf of a von Neumann algebra, as we will consider arbitrary orthomodular lattices instead of only the orthomodular lattices of projections of a von Neumann algebra. Exploring the spectral presheaf of an orthomodular lattice can yield an alternate interpretation of quantum logic that is similar to the spectral presheaf interpretation of the state space of a quantum system given by the topos approach.

We will prove in Section 5 that orthomodular lattices are isomorphic if and only if their spectral presheaves are (Theorem 5.7.3). This shows that the alternative 'state space picture' of quantum logic is at least as rich as the traditional orthomodular lattice representation. Thus, when considering spectral presheaves instead of orthomodular lattices, no information is lost. These results lend further support to one of the central claims of the topos approach, that the notion of a spectral presheaf is of critical value for an alternate formalism of quantum physics.

1.5 Overview

Chapter 2 provides the necessary background in category theory. Of particular relevance is Section 2.2, which introduces morphisms between functors with different domains but a common codomain. These definitions and results will play a central role in our arguments about morphisms between spectral presheaves, also seen in Sections 5.2–5.4. Chapter 3 presents some basic results on lattice theory, while Chapter 4 presents some non-standard lattice constructions such as the context category of an orthomodular lattice (Section 4.1) and partial orthomodular lattices (Section 4.2).

In Chapter 5, we define the spectral presheaf of an orthomodular lattice, and after exploring some categorical machinery, prove that two orthomodular lattices are isomorphic if and only if their spectral presheaves are isomorphic (Theorem 5.7.3). This result is stronger than the corresponding result for von Neumann algebras, where an isomorphism between spectral presheaves gives only a Jordan $*$-isomorphism between the algebras. In Chapter 6, we restate the results of Chapter 5 for complete orthomodular lattices; the relevant classical duality is that between complete Boolean algebras and Stonean spaces.

In Chapter 7, we consider clopen subobjects of the spectral presheaf of a complete orthomodular lattice, defined in analogy to the clopen subsets of a Stonean space. We prove that the clopen subobjects form a bi-Heyting algebra and present a map $\underline{\delta^o} : L \to Sub_{cl}\underline{\Omega(L)}$ called daseinisation from a complete orthomodular lattice L to the bi-Heyting algebra $Sub_{cl}\underline{\Omega(L)}$ of clopen subobjects of its spectral presheaf. This map is injective and preserves joins. Using the right adjoint $\epsilon : Sub_{cl}\underline{\Omega(L)} \rightarrow L$ of daseinisation, we define an equivalence relation on $Sub_{cl}\underline{\Omega(L)}$, equip the set E of equivalence classes with a meet operation, and show that E becomes a complete lattice that is isomorphic to the orthomodular lattice L. This generalization of Stone's representation theorem to complete orthomodular lattices, presented as Theorem 7.5.4, is the second main result of this dissertation.

Chapter 8 concludes by discussing some other possible generalizations of the spectral presheaf and states some goals of future work.

Chapter 2

Category Theory Background

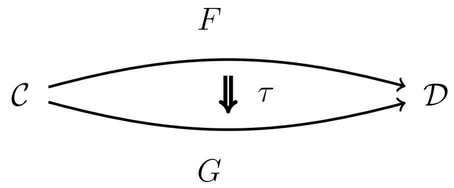
2.1 Categories, functors, and natural transformations

Basic familiarity with categories, functors, and natural transformations will be assumed. For those not familiar with these concepts and others discussed in this subsection, [24] and [26] provide good references. We now briefly discuss the notation that will be used throughout.

The collection of objects of a category $\mathcal{C}$ will be denoted $Ob(\mathcal{C})$, while its collection of morphisms will be $Morph(\mathcal{C})$; the terms 'morphism' and 'arrow' will be used interchangeably. The collection of morphisms in $\mathcal{C}$ from object A to object B will be written $\mathcal{C}(A, B)$. Familiarity with the notions of monic arrow, opposite category $\mathcal{C}^{op}$, small category, subcategory, faithful subcategory, and full subcategory will also be assumed.

The action of a functor $F : \mathcal{C} \to \mathcal{D}$ on an object A will be denoted F_A, and occasionally as $F(A)$ when the latter notation provides a much greater degree of clarity. The action of (covariant) functor F on an arrow $a : A \to B$ will be denoted $F(a) : F_A \to F_B$. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}^{op}$, or, equivalently, a functor from $\mathcal{C}^{op}$ to $\mathcal{D}$. These two definitions of a contravariant functor will be used interchangeably. Familiarity with full functors, faithful functors, and inclusion functors will be assumed.

A natural transformation from functor $F : \mathcal{C} \to \mathcal{D}$ to functor $G : \mathcal{C} \to \mathcal{D}$ will be written as $\tau : F \Rightarrow G$, as will often be depicted graphically as



The component of this natural transformation at object $A \in \mathcal{C}$ will be denoted $\tau_A : F_A \to G_A$. Every functor has an associated identity natural transformation, of which each component is simply an identity arrow in the codomain category.

We now proceed to define some additional categorical concepts that will play an important role in our investigations.

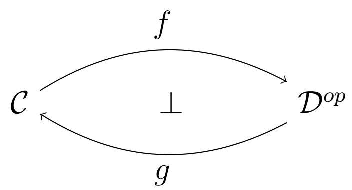
Definition 2.1.1. A *presheaf* is a contravariant functor with codomain **Set**, the category whose objects are all small sets and whose morphisms are all functions between those sets.

Dual to the notion of a presheaf is that of a copresheaf.

Definition 2.1.2. A *copresheaf* is a covariant **Set**-valued functor.

Though the traditional definitions of a presheaf and a copresheaf, above, require that they be **Set**-valued, we often instead consider presheaves and copresheaves where the codomain is not quite **Set**, but rather some collection of sets with additional structure (i.e., a certain topology) and structure-preserving maps between them (i.e., continuous maps). There is always a forgetful functor from these more structured categories into **Set**, however, and postcomposing any presheaf that is not **Set**-valued with this forgetful functor yields a presheaf in the strict sense.

Definition 2.1.3. Given categories $\mathcal{C}$ and $\mathcal{D}$, an *equivalence of categories* between $\mathcal{C}$ and $\mathcal{D}$ consists of covariant functors $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms $\epsilon_{\mathcal{C}} : Id_{\mathcal{C}} \Rightarrow g \circ f$ and $\epsilon_{\mathcal{D}} : Id_{\mathcal{D}} \Rightarrow f \circ g$, where $Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ denote the identity functors on categories $\mathcal{C}$ and $\mathcal{D}$, respectively. There is a *dual equivalence of categories* if f and g are both contravariant; dual equivalences will be written



2.2 Functor categories

In an additional layer of abstraction, the collection of functors between two categories is itself a category.

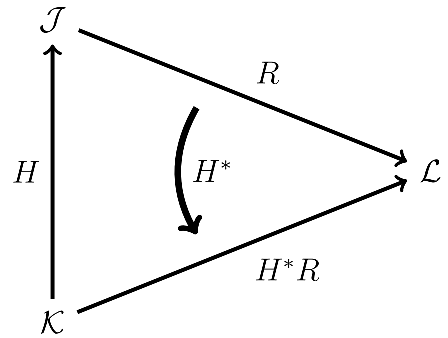
Definition 2.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The functor category $\mathcal{D}^{\mathcal{C}}$ has functors $F : \mathcal{C} \to \mathcal{D}$ as objects and natural transformations $\tau : F \Rightarrow G$ between such functors as morphisms. If $\tau : F \Rightarrow G$ and $\rho : G \Rightarrow H$, then $\rho \circ \tau$ is given by vertical composition, that is, it is the natural transformation with components, for each $A \in Ob(\mathcal{C})$,

$$(\rho \circ \tau)_A = \rho_A \circ \tau_A : F_A \to G_A \to H_A.$$

The following exploration of maps between functor categories with a common codomain, related to content in [11], is not standard will prove useful for later results. Let $H : \mathcal{K} \to \mathcal{J}$ be a functor between small categories. For the sake of simplicity of notation later in this section, the action of H on an object K of $\mathcal{K}$ will be written as $H(K)$ rather than as H_K. Then, for any category $\mathcal{L}$, H induces a map H^* between the functor categories $\mathcal{L}^{\mathcal{J}}$ and $\mathcal{L}^{\mathcal{K}}$ which acts by precomposing by H. On objects, that is, on a functor $R : \mathcal{J} \to \mathcal{L}$,

$$H^*R = R \circ H : \mathcal{K} \to \mathcal{L}.$$

This is captured by the following commutative diagram for each $R \in \mathcal{L}^{\mathcal{J}}$:



It will be useful to note that on objects $K \in \mathcal{K}$,

$$(H^*R)_K = (R \circ H)_K = R_{H(K)}.$$

An arrow in the functor category $\mathcal{L}^{\mathcal{J}}$ is a natural transformation $\tau : R \Rightarrow R'$, for $R, R' : \mathcal{J} \to \mathcal{L}$. The map H^* then yields a natural transformation $H^*\tau : H^*R \Rightarrow H^*R'$ in $\mathcal{L}^{\mathcal{K}}$, where components are defined for each object $K \in \mathcal{K}$ by

$$(H^*\tau)_K = \tau_{H(K)}.$$

Checking the necessary diagram shows that $H^*\tau$ is a valid natural transformation precisely because τ is. It only remains to show that H^* is a functor, that is, to show that it preserves identity arrows and composition.

Proposition 2.2.2. *$H^* : \mathcal{L}^{\mathcal{J}} \to \mathcal{L}^{\mathcal{K}}$ is a functor.*

Proof. First consider identity arrows in functor category $\mathcal{L}^{\mathcal{J}}$. The identity arrow on functor R is the natural transformation $Id_R : R \Rightarrow R$, where each component $(Id_R)_J$ for $J \in \mathcal{J}$ is given by id_{R_J}, the identity arrow of object R_J in $\mathcal{D}$. Under the action of H^* on arrows as given above, $H^*(Id_R)$ is the natural transformation with components for each $K \in \mathcal{K}$ given by

$$(H^*(Id_R))_K = (Id_R)_{H(K)} = id_{R_{H(K)}} = id_{(H^*R)_K}.$$

Thus $H^*(Id_R)$ is in fact the identity natural transformation $Id_{H^*R} : H^*R \Rightarrow H^*R$, meaning that H^* preserves identities and thus satisfies the first requirement necessary to be a functor.

Next, consider composition of natural transformations in functor category $\mathcal{L}^{\mathcal{J}}$. By the definition of vertical composition, for natural transformations $\eta : R \Rightarrow R'$ and $\tilde{\eta} : R' \Rightarrow R''$, $\tilde{\eta} \circ \eta$ is defined to be the natural transformation with components, for all $J \in \mathcal{J}$,

$$(\tilde{\eta} \circ \eta)_J = \tilde{\eta}_J \circ \eta_J : R_J \to R''_J.$$

Then, it follows that for all $K \in \mathcal{K}$,

$$(H^*(\tilde{\eta} \circ \eta))_K = (\tilde{\eta} \circ \eta)_{H(K)} = \tilde{\eta}_{H(K)} \circ \eta_{H(K)} = (H^*\tilde{\eta})_K \circ (H^*\eta)_K$$

Thus, $H^*(\tilde{\eta} \circ \eta) = (H^*\tilde{\eta}) \circ (H^*\eta)$, meaning H^* preserves composition and thus is a functor. □

The following elementary facts about H^* will be useful in later proofs.

Fact 2.2.3. *For functor $H : \mathcal{J}' \to \mathcal{J}$, functor category map $H^* : \mathcal{L}^{\mathcal{J}} \to \mathcal{L}^{\mathcal{J}'}$, functor $\tilde{H} : \mathcal{J}'' \to \mathcal{J}'$, and functor category map $\tilde{H}^* : \mathcal{L}^{\mathcal{J}'} \to \mathcal{L}^{\mathcal{J}''}$,*

$$(H \circ \tilde{H})^* = \tilde{H}^* \circ H^*.$$

Proof. First, consider the actions of these two functors on presheaf $R : \mathcal{J} \to \mathcal{L}$.

$$(H \circ \tilde{H})^*R = R \circ (H \circ \tilde{H}) = (R \circ H) \circ \tilde{H} = (H^*R) \circ \tilde{H} = \tilde{H}^*(H^*R) = (\tilde{H}^* \circ H^*)R.$$

Thus, the two functors act the same on objects R of $\mathcal{L}^{\mathcal{J}}$. It only remains to show that they act the same on morphisms, that is, on natural transformations in $\mathcal{L}^{\mathcal{J}}$.

Let $\tau : R \Rightarrow R'$ be a natural transformation in $\mathcal{L}^{\mathcal{J}}$. For an object $J \in \mathcal{J}''$,

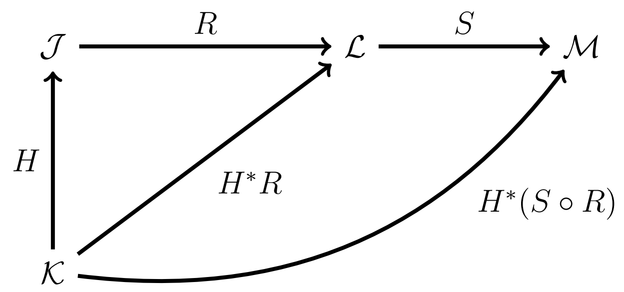
$$\left[(\tilde{H}^* \circ H^*)\tau\right]_J = \left[\tilde{H}^*(H^*\tau)\right]_J = (H^*\tau)_{\tilde{H}(J)} = \tau_{H(\tilde{H}(J))} = \tau_{(H\circ\tilde{H})(J)} = \left[(H \circ \tilde{H})^*\tau\right]_J .$$

As $(\tilde{H}^* \circ H^*)\tau$ and $(H \circ \tilde{H})^*\tau$ have the same component at each $J \in \mathcal{J}''$, the two natural transformations are equal. ☐

Fact 2.2.4. *Suppose $H : \mathcal{K} \to \mathcal{J}$, $R : \mathcal{J} \to \mathcal{L}$, and $S : \mathcal{L} \to \mathcal{M}$. Then*

$$H^*(S \circ R) = S \circ (H^*R).$$

That is, the right triangle of the following diagram commutes:



Proof. This fact follows immediately from the associativity of functor composition and the definition of H^*.

$$S \circ (H^*R) = S \circ (R \circ H) = (S \circ R) \circ H = H^*(S \circ R)$$

☐

Fact 2.2.5. *Let $Id : J \to J$ be the identity functor on category $\mathcal{J}$. Let $R, R' \in \mathcal{L}^{\mathcal{J}}$, and let $\eta : R \Rightarrow R'$ be a natural transformation. Then $Id^*R = R$, $Id^*R' = R'$, and $Id^*\eta = \eta : R \Rightarrow R'$.*

Proof. First, note that $Id^*R = R \circ Id = R$, and similarly for R'. Components of natural transformation $Id^*\eta$ are given by

$$(Id^*\eta)_J = \eta_{Id(J)} = \eta_J$$

As $Id^*\eta$ and η have the same components for each $J \in \mathcal{J}$, they are the same natural transformation. ☐

2.3 Topoi

A topos is a category whose objects have many set-like properties. For example, every object has 'subobjects' that form a distributive lattice, just as every set has subsets that also form a distributive lattice. Though this dissertation is motivated by the topos approach to quantum computer science, very little knowledge of topoi will be required. We briefly define a topos and consider an important example; more about topoi can be found in [15].

Definition 2.3.1. Let $\mathcal{E}$ be a category. A *subobject classifier* for $\mathcal{E}$ is an object Ω and an arrow $true : 1 \to \Omega$ such that for each monic arrow $f : A \rightarrowtail D$, there is a unique arrow $\chi_f : D \to \Omega$ such that the following is a pullback square:

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & D \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

Definition 2.3.2. A *topos* is a category $\mathcal{E}$ that is finitely complete, finitely cocomplete, has exponentiation, and has a subobject classifier.

The most important fact about topoi for our purposes is the existence of subobjects. Specifically, in a topos, a *subobject* of object A is a monic morphism with codomian A. Every object has a collection of subobjects which can be ordered to form a Heyting algebra, just as every set has a collection of subsets that form a Boolean algebra. This illuminates why an object in a topos is a good choice for a generalized state space that is set-like but not a set.

One important example to consider is the functor category $\mathbf{Set}^{\mathcal{C}}$. For any small category $\mathcal{C}$, this functor category is in fact a topos. One can thus consider the subobjects of any functor from $\mathcal{C}$ to **Set**. As the spectral presheaf, of both a von Neumann algebra and an orthomodular lattice, is a functor from a small category to **Set**, this allows us to consider subobjects of the spectral presheaf in the topos $\mathbf{Set}^{\mathcal{C}}$.

Of note, if $H : \mathcal{K} \to \mathcal{J}$ is a functor between small categories, then $H^* : \mathbf{Set}^{\mathcal{J}} \to \mathbf{Set}^{\mathcal{K}}$ is in fact a functor between topoi. Further investigation shows that H^* is the inverse image part of the essential geometric morphism induced by $H : \mathcal{K} \to \mathcal{J}$ [21].

Definition 2.3.3. Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ consists of a pair of functors $f^* : \mathcal{E} \to \mathcal{F}$ and $f_* : \mathcal{F} \to \mathcal{E}$ such that f^* preserves all finite limits and is left adjoint to f_*. The functor f^* is called the *inverse image* of f and f_* is called the *direct image* of f.

Definition 2.3.4. A geometric morphism is *essential* if its inverse image part f^* has a left adjoint, that is, it is right adjoint to some other functor $g : \mathcal{F} \to \mathcal{E}$.

More details on this can be found in [21], Section A4.1. While we will use the functoriality of H^* in later arguments, we will not need to know that H^* is an essential geometric morphism.

Chapter 3

Lattice Theory Background

For background in general lattice theory that will be useful for our purposes, [8] is a good reference. For orthomodular lattices specifically, consider [22].

3.1 Posets and lattices

Definition 3.1.1. A *partially ordered set*, or *poset*, is a set with a reflexive, transitive, antisymmetric binary relation $\leq$.

A poset is also a category, where the elements of the poset are objects and there is a single morphism with domain a and codomain b whenever $a \leq b$. It then makes sense to consider functors between categories, that is, structure-preserving maps between posets. In this case, the structure to be preserved is the relation $\leq$, and such functors are called monotone maps.

Definition 3.1.2. A *monotone map* between two posets P and Q is a function $f : P \to Q$ such that whenever $a \leq b$ in P, then $f(a) \leq f(b)$ in Q.

In an additional layer of abstraction, posets and monotone maps form a category **Pos**.

Definition 3.1.3. A *lattice* is a poset P in which any two elements a and b have a greatest lower bound $a \wedge b$, called the *meet* of a and b, as well as a least upper bound $a \vee b$, called the *join* of a and b. A lattice is *bounded* if it has a least element 0 and a greatest element 1.

The definitions above of meets and joins have a close relationship to the relation $\leq$ of P.

Fact 3.1.4. *For any lattice P and $a, b \in P$, the following are equivalent:*

$$a \leq b \qquad a \wedge b = a \qquad a \vee b = b$$

Some common examples of lattices include the power set of a set X, with subsets partially ordered by inclusion; open sets of a topological space X, again ordered by inclusion; and factors of a given natural number x, ordered by divisibility. Any total orders are also lattices.

Definition 3.1.5. A *sublattice* S of a lattice L is a subset of L such that whenever $a, b \in S$, then $a \wedge b$ and $a \vee b$ are also in S. That is, S is closed under meets and joins.

All sublattices discussed throughout will be assumed to be nonempty.

Just as monotone maps are structure-preserving maps between posets, it is possible to define structure-preserving maps between lattices. In this case, the structure that must be preserved is meets and joins.

Definition 3.1.6. A *lattice homomorphism* $\varphi : P \to Q$ is a function from P to Q such that for all $a, b \in P$,

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) \text{ and } \varphi(a \vee b) = \varphi(a) \vee \varphi(b)$$

By Fact 3.1.4, lattice homomorphisms are also monotone maps. Just as with posets, all lattices and lattice homomorphisms between them form a category **Lat**, which is a faithful subcategory of **Pos**.

3.2 Ortholattices and orthomodular lattices

We now consider a specific subcategory of **Lat**, that of orthomodular lattices.

Definition 3.2.1. An *orthocomplementation function* on a bounded lattice L is a map $(-)' : L \to L$, acting on each $a \in L$ as $a \mapsto a'$, satisfying

1. $a' \vee a = 1$, $a' \wedge a = 0$ (Complement Law)
2. $a'' = a$ (Involution Law)
3. If $a \leq b$, then $b' \leq a'$ (Order-Reversing)

Definition 3.2.2. An *orthocomplemented lattice*, also called an *ortholattice*, is a bounded lattice with an orthocomplementation function.

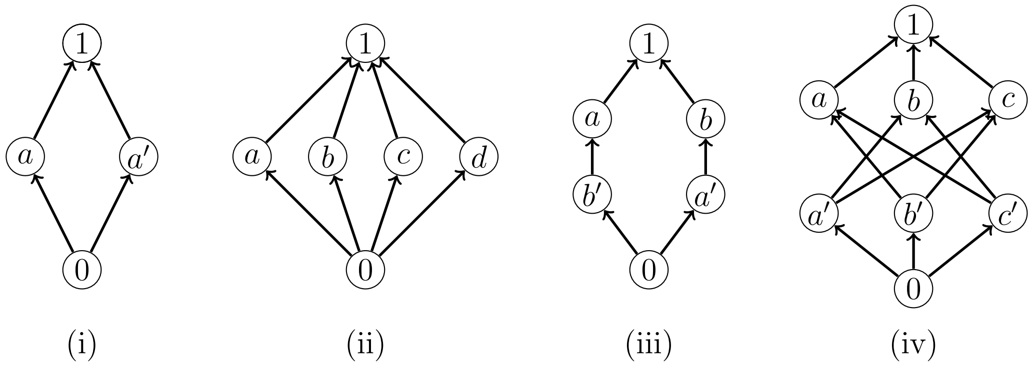


Figure 3.1: Four ortholattices. An arrow $a \to b$ means $a \leq b$.

Definition 3.2.3. An *orthomodular lattice* L is an ortholattice such that for any $x, y \in L$ with $x \leq y$, it holds that $x \vee (x' \wedge y) = y$. This is the *orthomodularity property.*

Figure 3.1 depicts four small ortholattices. Ortholattices (i), (iii), and (iv) have a unique orthocomplementation function, as shown. The second has three valid orthocomplementation functions; a's orthocomplement could be any of b, c, or d, and each of these three options determines a different orthocomplementation function. In general, we assume that an ortholattice comes with a specified orthocomplementation function to avoid such ambiguity. Of these ortholattices, (i), (ii), and (iv) are orthomodular lattices. In (iii), elements b' and a satisfy $b' \leq a$, but

$$b' \vee (b \wedge a) = b' \vee 0 = b' \neq a.$$

Another example of an ortholattice that is also orthomodular is the lattice of subspaces of any inner product space, with the orthogonal complement operation on these subspaces as the orthocomplementation function. Additionally, the closed subspaces of a separable Hilbert space form an orthomodular lattice; as discussed in Chapter **??**, such lattices are useful for representing quantum logic, where the closed subspaces represent quantum propositions [22]. Boolean algebras are also orthomodular lattices, specifically orthomodular lattices that are distributive, and will be considered in the next subsection. It will be useful to note that de Morgan's Laws, which are an important property of Boolean algebras, hold in the more general case for all ortholattices (and thus all orthomodular lattices).

Fact 3.2.4 ([22])**.** *For any ortholattice L and $a, b \in L$,*

$$(a \wedge b)' = a' \vee b'$$
$$(a \vee b)' = a' \wedge b'$$

Definition 3.2.5. An *ortholattice homomorphism* $\varphi : L \to M$ is a lattice homomorphism preserving orthocomplementation; for all $a \in L$,

$$\varphi(a') = \varphi(a)'.$$

Just as in the previous subsections, ortholattices and structure-preserving morphisms between them form a faithful subcategory **OLat** of **Lat**.

Definition 3.2.6. An *orthomodular lattice homomorphism* $\varphi : L \to M$ is an ortholattice homomorphism in which both the domain and codomain are orthomodular lattices.

Note that there are no additional constraints placed on an orthomodular lattice homomorphism when compared to an ortholattice homomorphism. This is because the orthomodularity property can be expressed solely in terms of meets, joins, and orthocomplements, which are already preserved by all ortholattice homomorphisms. As above, there is a category **OML** of orthomodular lattices and orthomodular lattice homomorphisms between them. **OML** is a full and faithful subcategory of **OLat**.

3.3 Boolean algebras

Definition 3.3.1. A *Boolean lattice*, also called a *Boolean algebra*, is a bounded complemented distributive lattice L. That is, every element $a \in L$ has a complement a' such that $a \wedge a' = 0$ and $a \vee a' = 1$, and for any $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{3.1}$$

The number of elements in any finite Boolean algebra must be 2^n for some n, and any two Boolean algebras with the same number of elements are isomorphic [30]. Thus, we can talk about the two-element Boolean algebra, the four-element Boolean algebra, the eight-element Boolean algebra, etc. The two-element Boolean algebra consists of only a top element and a bottom element, and will be denoted by its set of elements $\{0, 1\}$ or as B_0. The four-element Boolean algebra is pictured in (i) of Figure 3.1, and such a Boolean algebra will be denoted B_a. The eight-element Boolean algebra is pictured in (iv) of the same figure, and such a Boolean algebra will be denoted $B_{a,b,c}$.

All distributive ortholattices are Boolean algebras, with complements given by the orthocomplementation function. In Figure 3.1, (i) and (iv) are distributive. For (ii), elements a, b, and c do not satisfy (3.1), while for (iii) elements a, b', and a' do not satisfy (3.1). Additionally, both the lattice of subspaces on an inner product space and the lattice of closed subspaces of a separable Hilbert space are nondistributive and thus are not Boolean algebras.

The definition of a Boolean algebra only requires the existence of a complement for each element and not the existence of an orthocomplementation function, which might suggest that there are some Boolean algebras which are not ortholattices. However, one can show that the distributivity property of a Boolean algebra ensures that complementation in any Boolean algebra is in accordance with a valid orthocomplementation function, meaning every Boolean algebra is an ortholattice. Further investigation shows that every Boolean algebra is an orthomodular lattice as well, with distributivity ensuring that the orthomodularity property always holds.

Definition 3.3.2. A *Boolean algebra homomorphism* $\varphi : B \to B'$ is a lattice homomorphism which preserves complements.

That is, a Boolean algebra homomorphism is an ortholattice homomorphism where both the domain and codomain are Boolean algebras. As distributivity is expressed only in terms of meets and joins, it is not necessary to place an additional constraint on Boolean algebra homomorphisms to ensure distributivity is preserved. There is a category **BA** of Boolean algebras and Boolean algebra homomorphisms, which is a full and faithful subcategory of **OML**.

Because Boolean algebras are distributive, it is generally simpler to work with Boolean algebras instead of ortholattices or orthomodular lattices. In particular, there is a useful dual equivalence between the category **BA** and the topological category **Stone**, which we now discuss.

3.4 Stone duality

There is a well-known duality between Boolean algebras and Stone spaces. The existence of this duality is a main motivation for considering Boolean substructures of an orthomodular lattice, as no such duality exists in the nondistributive setting. The details of this subsection are phrased in terms of Boolean algebra homomorphisms, adapted from [4] where the equivalent definitions are stated in terms of ultrafilters.

Definition 3.4.1. A *Stone space* is a compact totally disconnected Hausdorff space.

There is a category **Stone** whose objects are Stones spaces and whose arrows are continuous functions between these topological spaces. As we will soon see, this category is closely related to the category **BA** of Boolean algebras and Boolean algebra homomorphisms.

There is a specific Stone space associated with each Boolean algebra, defined as follows. Let $\{0,1\}$ denote the two-element Boolean algebra consisting of only a bottom element 0 and a top element 1.

Definition 3.4.2. The *Stone space* of Boolean algebra B is the compact totally disconnected Hausdorff space with set of elements

$$\Omega_B = \{\lambda : B \to \{0,1\} \mid \lambda \text{ is a Boolean algebra homomorphism,} \\ \text{also called a state or an ultrafilter in } B\}$$

and topology generated by a basis of all sets of the form

$$U_b = \{\lambda \in \Omega_B : \lambda(b) = 1\},$$

where $b \in B$.

In a slight abuse of notation, Ω_B will denote both the set of elements of the Stone space of B as well as the Stone space itself with its additional topological structure.

This definition of the Stone space of a Boolean algebra gives rise to a contravariant functor Ω from the category **BA** of Boolean algebras and the category **Stone** of Stone spaces. On objects (Boolean algebras B), Ω is simply the Stone space of B,

$$\Omega_B = \Omega(B).$$

On a Boolean algebra homomorphisms $\varphi : B' \to B$,

$$\begin{aligned} \Omega(\varphi) : \Omega_B \to \;& \Omega_{B'} \\ \lambda \;\mapsto\;& \lambda \circ \varphi \end{aligned}$$

The functoriality of this map Ω is easy to verify. Note that if B' is a subalgebra of B and φ is the inclusion homomorphism $inc_{B',B} : B' \hookrightarrow B$, then

$$\Omega(inc_{B',B})(\lambda) = \lambda \circ inc_{B',B} = \lambda|_{B'} = r_{B,B'}(\lambda),$$

where $r_{B,B'}$ is defined to be the map that restricts functions with domain B to the smaller domain B'. That is, $\Omega(inc_{B',B}) = r_{B,B'}$.

One can also define a contravariant functor cl from **Stone** to **BA**. The subsets of a Stone space that are both closed and open form a Boolean algebra under inclusion, with meets defined as intersections and joins defined as unions of subspaces. On a Stone space X, cl_X is this Boolean algebra of clopen subsets of X, by convention written $cl(X)$, where a clopen subset is one that is both closed and open. On a continuous map of Stone spaces $h : X' \to X$, $cl(h)$ is a Boolean algebra homomorphsism from $cl(X)$ to $cl(X')$ that acts on each clopen subset S as

$$[cl(h)]\,(S) = h^{(-1)}(S) := \{x \in X' \mid h(x) \in S\} \in cl(X').$$

Because h is a continuous function, $h^{(-1)}(S)$ is open. Additionally, because S is closed the complement S' of S is open; $h^{(-1)}(S')$ is then open and its complement $h^{(-1)}(S)$ is closed, meaning $h^{(-1)}(S)$ is clopen. Note that in this context, the exponent (-1) denotes the inverse image rather than the inverse. Throughout, inverse image functions will be denoted $h^{(-1)}$ to differentiate from inverse morphisms, which will be written as h^{-1}.

The two functors Ω and cl give rise to a dual equivalence between the categories **BA** and **Stone**:

$$\mathbf{BA} \underset{cl}{\overset{\Omega}{\rightleftarrows}} \mathbf{Stone} \quad (\perp)$$

That is, there are natural isomorphisms $Bo : Id_{\mathbf{BA}} \Rightarrow cl \circ \Omega$ in **BA** and $St : Id_{\mathbf{Stone}} \Rightarrow \Omega \circ cl$ in **Stone**. In particular, the components of these isomorphisms are given as follows:

$$\begin{aligned} Bo_B : B &\to cl(\Omega_B) \\ b &\mapsto \{\lambda \in \Omega_B \mid \lambda(b) = 1\} \\ St_X : X &\to \Omega(cl(X)) \\ x &\mapsto \lambda_x \end{aligned}$$

where $\lambda_x : cl(X) \to \{0, 1\}$ is given by

$$\lambda_x(S) = \begin{cases} 1 & : x \in S \\ 0 & : x \notin S \end{cases}$$

Later, it will be of use to know the explicit components of $Bo^{-1} : cl \circ \Omega \Rightarrow Id_{\mathbf{BA}}$. Each component Bo_B^{-1} is a map from $cl(\Omega_B)$ to B. Let S be any clopen subset in

$cl(\Omega_B)$. As S is closed and it is a subspace of compact space Ω_B, then S is compact. As S is open, it can be written as a union of basic open sets. Compactness then implies this open cover of S has a finite subcover consisting of open sets in the basis of Ω_B, which are of the form $U_b = \{\lambda \in \Omega_B : \lambda(b) = 1\}$. That is, for some finite index set $J \subseteq B$,

$$S = \bigcup_{b \in J} U_b.$$

Let

$$s_* := \bigvee_{b \in J} b \in B$$

This meet is defined because J is finite. Then, the action of Bo_B^{-1} is as follows.

$$\begin{aligned} Bo_B^{-1} : cl(\Omega_B) &\Rightarrow B \\ S &\mapsto s_* \end{aligned}$$

3.4.1 Examples

Consider the Boolean algebra B_a consisting of four elements $\{0, a, a', 1\}$ and shown in Figure 3.1 (i). Any Boolean algebra homomorphism from B_a to $B_0 = \{0, 1\}$ must map 0 to 0 and must map 1 to 1. It must also map a' to the orthocomplement of the value that a is assigned, meaning such a homomorphism is completely determined by its action on element a. Thus there are two Boolean algebra homomorphisms from B_a to $\{0, 1\}$. The first, which we will call λ_a, has $\lambda_a(a) = 1$, while the second, which we will call $\lambda_{a'}$, has $\lambda_{a'}(a) = 0$. Thus the stone space Ω_{B_a} has two elements, λ_a and $\lambda_{a'}$, and has a basis given by the open sets:

$$\begin{aligned} U_0 &= \{\lambda \in \Omega_{B_a} \mid \lambda(0) = 1\} = \emptyset \\ U_a &= \{\lambda \in \Omega_{B_a} \mid \lambda(a) = 1\} = \{\lambda_a\} \\ U_{a'} &= \{\lambda \in \Omega_{B_a} \mid \lambda(a') = 1\} = \{\lambda_{a'}\} \\ U_1 &= \{\lambda \in \Omega_{B_a} \mid \lambda(1) = 1\} = \{\lambda_a, \lambda_{a'}\} \end{aligned}$$

These four subsets are precisely the clopen subsets of Ω_{B_a}, and they form a Boolean algebra under inclusion that is isomorphic to B_a, shown in Figure 3.2 (i).

For a more involved example, consider the Boolean algebra $B_{a,b,c}$ shown in Figure 3.1 (iv). Careful examination shows that any Boolean algebra homomorphism from $B_{a,b,c}$ to $\{0, 1\}$ must map exactly two of a, b, and c to 1 and the other to 0, with actions on a', b', and c' determined actions on a, b, and c. We will refer the homomorphism

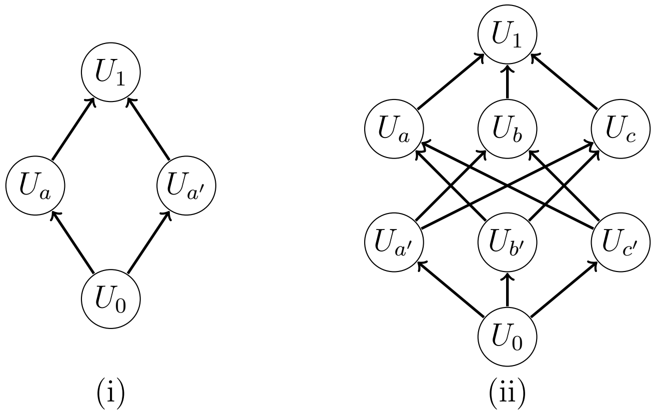


Figure 3.2: The algebra of clopen subsets of the Stone space of (i) four-element Boolean algebra B_a and (ii) eight-element Boolean algebra $B_{a,b,c}$.

from $B_{a,b,c}$ to $\{0,1\}$ that maps a and b to 1 and maps c to 0 as $\lambda_{a,b}$. We can define $\lambda_{a,c}$ and $\lambda_{b,c}$ similarly. Thus the Stone space of B_a has three elements, and a basis is given by the open sets:

$$\begin{aligned}
U_0 &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(0) = 1\} = \emptyset \\
U_{a'} &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(a') = 1\} = \{\lambda_{b,c}\} \\
U_{b'} &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(b') = 1\} = \{\lambda_{a,c}\} \\
U_{c'} &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(c') = 1\} = \{\lambda_{a,b}\} \\
U_a &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(a) = 1\} = \{\lambda_{a,b}, \lambda_{a,c}\} \\
U_b &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(b) = 1\} = \{\lambda_{a,b}, \lambda_{b,c}\} \\
U_c &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(c) = 1\} = \{\lambda_{a,c}, \lambda_{b,c}\} \\
U_1 &= \{\lambda \in \Omega_{B_{a,b,c}} \mid \lambda(1) = 1\} = \{\lambda_{a,b}, \lambda_{a,c}, \lambda_{b,c}\}
\end{aligned}$$

These subsets are all clopen, and comprise all clopen subsets of $\Omega_{B_{a,b,c}}$. When ordered by inclusion, as in Figure 3.2 (ii), they form a Boolean algebra that is isomorphic to $B_{a,b,c}$. It is easy to see that in both of these examples the clopen subspaces of the Stone space of B form a Boolean algebra that is isomorphic to B.

Chapter 4

Distributive substructure of an orthomodular lattice

Having completed the above review of lattice theory and Stone duality, we now move on to less standard explorations of the Boolean substructure of an orthomodular lattice. This material will play an important role in our considerations of the spectral presheaf of an orthomodular lattice. In particular, we will be concerned with the distributive substructures of orthomodular lattice L given in the following definition.

Definition 4.0.3. A *Boolean sublattice*, also called a *Boolean subalgebra*, of an orthomodular lattice L is a complemented distributive sublattice with complements given by the orthocomplementation function of L.

It is important to note that for our purposes, we consider only those subalgebras which are Boolean algebras with complementation inherited from L. Consequently, Boolean subalgebras must be closed under L's orthocomplementation. This means that for any Boolean subalgebra B of L containing some element a, also $a' \in B$ and thus $a \wedge a' = 0$ and $a \vee a' = 1$ are both in B. The following propositions illustrate two important properties of Boolean subalgebras.

Proposition 4.0.4. *Every element a of an orthomodular lattice L is in some Boolean subalgebra of L.*

For $a \neq 0, 1$, one Boolean subalgebra of L that contains a is the four-element Boolean subalgebra B_a, shown in Figure 4.1 (i).

Proposition 4.0.5 ([22])**.** *In an orthomodular lattice L, for any elements $a, b \in L$ satisfying $a \leq b$ there is Boolean subalgebra of L containing both a and b.*

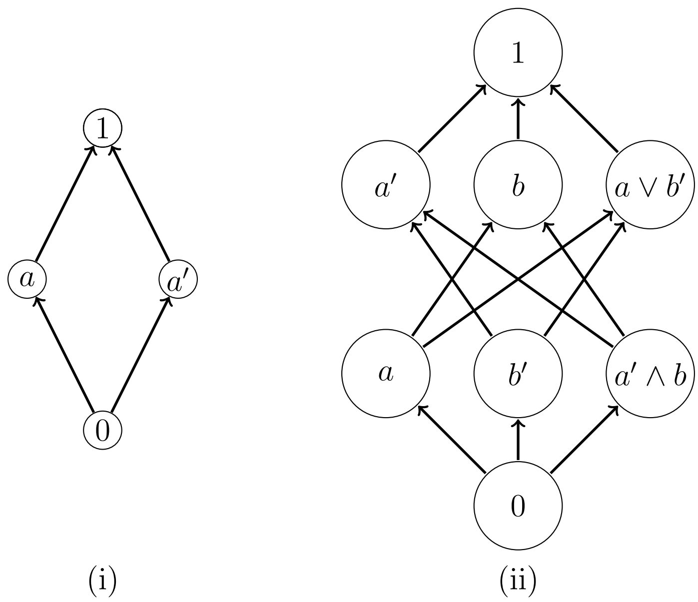


Figure 4.1: (i) A Boolean subalgebra of L that contains element a, for $a \neq 0, 1$, and (ii) a Boolean subalgebra of L that contains elements a and b with $a \leq b$, for distinct $a, b \neq 0, 1$.

When a and b with $a \leq b$ are distinct and not equal to 0 or 1, the eight-element Boolean algebra $B_{a',b,a \vee b'}$, shown in Figure 4.1 (ii), contains both a and b.

Proposition 4.0.5 does not hold for all ortholattices; ortholattice (iii) of Figure 3.1 is a counterexample. In fact, this proposition is true if and only if L is orthomodular.

Proposition 4.0.6 ([22]). *Let L be an ortholattice. L is orthomodular if and only if for all elements $a, b \in L$ with $a \leq b$ there is a Boolean subalgebra of L containing both a and b.*

Proof. The forward implication is Proposition 4.0.5. We also prove the converse to get bidirectional implication; assume that for all $a \leq b$ in L there is some Boolean subalgebra of L containing both a and b. Then elements a and b and their complements satisfy distributivity, meaning

$$a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b) = 1 \wedge b = b.$$

This is precisely the orthomodularity condition, so as it holds for all $a \leq b$ then L is orthomodular. □

This is precisely the reason why we consider orthomodular lattices instead of ortholattices. Proposition 4.0.5 plays a key role in the proofs of Lemma 4.2.3 and subsequently

Lemma 4.2.4 and Theorem 5.7.2, which says that an isomorphism between the spectral presheaves of two orthomodular lattices induces an isomorphism between the orthomodular lattices, a main result of Chapter 5.

Beyond the standard definitions and facts above, one can consider two different structures based on the Boolean subalgebras of an orthomodular lattice L. These will be described in the next two sections.

4.1 The context category $\mathcal{B}(L)$

Definition 4.1.1. For any orthomodular lattice L, $\mathcal{B}(L)$ denotes the *poset of Boolean subalgebras of* L, partially ordered by inclusion. $\mathcal{B}(L)$ is also called the *context category* of L.

The poset category $\mathcal{B}(L)$ has a unique arrow from Boolean subalgebra B' to Boolean subalgebra B whenever $B' \subseteq B$. This arrow will be denoted $i_{B',B}$, and simply means that $B' \subseteq B$.

Additionally, whenever $B' \subseteq B$, one can define an inclusion map between Boolean subalgebras $inc_{B',B} : B' \to B$ given by $inc_{B',B}(b) = b$ for all $b \in B'$. As B' is closed under meets, joins, and orthocomplements, it follows that $inc_{B',B}$ is a Boolean algebra homomorphism. That is, for any b_1, b_2 in B',

$$inc_{B',B}(b_1 \wedge b_2) = b_1 \wedge b_2 = inc_{B',B}(b_1) \wedge inc_{B',B}(b_2),$$

and similarly for joins and orthocomplements. Thus, it is also possible to consider $\mathcal{B}(L)$ as a subcategory of **BA**, where objects are Boolean algebras that are Boolean subalgebras of L with inclusion Boolean algebra homomorphisms $inc_{B',B}$ between them. To clarify when $\mathcal{B}(L)$ is being considered as a poset and when it is being considered as a subcategory of **BA**, $i_{B',B}$ will denote an arrow in the poset and $inc_{B',B}$ will denote an arrow in the subcategory. For the majority of the subsequent sections, it is only necessary to consider $\mathcal{B}(L)$ as a poset.

It may be informative to note that as the bottom element 0 and the top element 1 of L are both in every nonempty Boolean subalgebra of L, then the two-element Boolean subalgebra $B_0 = \{0, 1\}$ is contained in every Boolean subalgebra of L. Thus, the context category $\mathcal{B}(L)$ has B_0 as a bottom element.

Proposition 4.1.2. *For any orthomodular lattice L, $\mathcal{B}(L)$ is a poset in which any two elements have a well-defined unique meet, where the meet $\wedge$ is defined as follows for $B, B' \in \mathcal{B}(L)$:*

$$B \wedge B' := B \cap B'$$

Note that $\cap$ simply denotes set intersection.

Proof. First, it is necessary to show that the meet as defined above is in fact a Boolean algebra. Note that as meets, joins and orthocomplementation in both B and B' are inherited from L, then these operations coincide on $B \cap B'$. Clearly $B \cap B'$ is closed under these three operations, as both B and B' are. All of the axioms that define a Boolean algebra hold in both B and B'; because $B \cap B'$ inherits the same meet, join, and orthocomplementation operations as well as the same top and bottom elements as both B and B', then $B \cap B'$ satisfies all Boolean algebra axioms as well. Thus, $B \wedge B'$ is a Boolean subalgebra of L, so $B \wedge B' \in \mathcal{B}(L)$.

Next, in order to see that $B \wedge B'$ is the greatest lower bound of B and B' in the poset $\mathcal{B}(L)$ ordered by inclusion, first note that clearly $B \cap B' \subseteq B$ and $B \cap B' \subseteq B'$, so it is a lower bound. Additionally, any other Boolean algebra which is a lower bound for both B and B' must be contained in both B and B', and thus is contained in $B \cap B'$. Thus, $B \cap B'$ is the greatest lower bound of B and B' in the poset $\mathcal{B}(L)$. □

The proposition above proves that $\mathcal{B}(L)$ is in fact a meet-semilattice, that is, a poset in which the meet of any two elements exists. It is possible to show (with little modification) that $\mathcal{B}(L)$ is a complete meet-semilattice, meaning arbitrary (infinite) meets exist, which we discuss in Chapter 6 as it is not necessary for our purposes at this point. However, meets are not necessarily preserved by the homomorphisms $\tilde{\varphi}$ that will be considered; see Note 4.1.5. We will instead consider $\mathcal{B}(L)$ more generally as a poset, though the fact that every two Boolean algebras have a meet in $\mathcal{B}(L)$ will prove useful in the proof of Theorem 5.7.2.

Of interest, any orthomodular lattice homomorphism $\varphi : L \to M$ induces a map $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$, where on each Boolean subalgebra B of L,

$$\tilde{\varphi}(B) := \{\varphi(b) : b \in B\}.$$

As φ is an orthomodular lattice homomorphism, meets, joins, and orthocomplementation are preserved. It follows from this that $\tilde{\varphi}(B)$ is in fact a Boolean subalgebra of M, meaning that $\tilde{\varphi}$ is well-defined.

Proposition 4.1.3. *Let $\varphi : L \to M$ be an orthomodular lattice homomorphism. For every $B \in \mathcal{B}(L)$, $\tilde{\varphi}(B) \in \mathcal{B}(M)$ and $\varphi|_B : B \to \tilde{\varphi}(B)$ is a Boolean algebra homomorphism. If φ is an orthomodular lattice isomorphism, then $\varphi|_B$ is a Boolean algebra isomorphism.*

Proof. As φ is an orthomodular lattice homomorphism and B is closed under meets, joins, and orthocomplementation, then $\varphi|_B : B \to \tilde{\varphi}(B)$ also preserves these meets, joins, and orthocomplements. This means $\varphi|_B$ is an orthomodular lattice homomorphism, and its codomain $\tilde{\varphi}(B) \subseteq M$ is also closed under these operations. As B is distributive, a condition expressed solely in terms of meets and joins, then as $\varphi|_B$ is surjective and preserves meets and joins, $\tilde{\varphi}(B)$ is also distributive and thus a Boolean subalgebra in $\mathcal{B}(M)$. As $\varphi|_B$ is an orthomodular lattice homomorphism with Boolean algebras as both domain and codomain, $\varphi|_B$ is a Boolean algebra homomorphism.

If $\varphi : L \to M$ is an isomorphism of orthomodular lattices, it has an inverse $\psi : M \to L$. In this case it clearly follows that $\psi|_{\tilde{\varphi}(B)} : \tilde{\varphi}(B) \to B$ is an inverse of $\varphi|_B$, meaning $\varphi|_B$ is an isomorphism of Boolean algebras. □

In fact, the following results holds.

Lemma 4.1.4. *$\tilde{\varphi}$ is a monotone map between posets.*

Proof. First, it must be demonstrated that $\tilde{\varphi}$ is order-preserving. Suppose that $B' \subseteq B$ holds in $\mathcal{B}(L)$. Then

$$\tilde{\varphi}(B') = \{\varphi(b) \mid b \in B'\}$$
$$\tilde{\varphi}(B) = \{\varphi(b) \mid b \in B\}$$

Clearly, $B' \subseteq B$ implies $\tilde{\varphi}(B') \subseteq \tilde{\varphi}(B)$, meaning $\tilde{\varphi}$ is a monotone map. □

Note 4.1.5. If φ is injective, then $\tilde{\varphi}$ preserves meets and is in fact a meet-semilattice homomorphism. In this case,

$$\begin{aligned}
\tilde{\varphi}(B' \wedge B) &= \{\varphi(b) \in L \mid b \in B' \wedge B\} \\
&= \{\varphi(b) \in L \mid b \in B' \cap B\} \\
&= \{\varphi(b) \mid b \in B'\} \cap \{\varphi(b) \mid b \in B\} \\
&= \tilde{\varphi}(B') \cap \tilde{\varphi}(B) \\
&= \tilde{\varphi}(B') \wedge \tilde{\varphi}(B)
\end{aligned}$$

In the general case, however, the third line above does not necessarily hold; it could be that $b' \in B' \setminus B$ and $b \in B \setminus B'$ such that $\varphi(b') = \varphi(b) \notin \{\varphi(b) \in L \mid b \in B' \cap B\}$. Such an example is orthomodular lattices L and M and homomorphism φ of Section 5.2.1, which we discuss in the next chapter.

Lemma 4.1.4 implies that $\tilde{\varphi}$ is in fact a functor between two posets $\mathcal{B}(L)$ and $\mathcal{B}(M)$. It can even be considered more generally as an arrow in the category **Pos**, consisting of posets and monotone maps between them.

There is a functor from **OML** to **Pos**, sending each orthomodular lattice to its context category each each homomorphism φ to $\tilde{\varphi}$. Call this functor $\mathcal{B} : \mathbf{OML} \to \mathbf{Pos}$, where for orthomodular lattice L and orthomodular lattice homomorphism $\varphi : L \to M$,

$$\mathcal{B}_L = \mathcal{B}(L)$$
$$\mathcal{B}(\varphi) = \tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$$

Proposition 4.1.6. $\mathcal{B} : \mathbf{OML} \to \mathbf{Pos}$ *is a functor.*

Proof. In order to show that $\mathcal{B}$ is a functor, it is necessary to show that it preserves identities and composition. First, consider the identity orthomodular lattice homomorphism $i : L \to L$. Then $\tilde{i} : \mathcal{B}(L) \to \mathcal{B}(L)$ is a monotone map in **Pos**. For each $\mathcal{B} \in \mathcal{B}(L)$,

$$\tilde{i}(B) = \{i(b) \mid b \in B\} = \{b \mid b \in B\} = B.$$

Thus $\tilde{i}$ is the identity arrow on $\mathcal{B}(L)$, meaning that $\mathcal{B}$ preserves identities.

Suppose $\varphi : L \to M$ and $\xi : M \to N$ are orthomodular lattice homomorphisms. Then, for each $B \in \mathcal{B}(L)$,

$$(\tilde{\xi} \circ \tilde{\varphi})(B) = \tilde{\xi}(\{\varphi(b) \mid b \in B\}) = \{\xi(\varphi(b)) \mid b \in B\} = \widetilde{(\xi \circ \varphi)}(B)$$

Thus, $\mathcal{B}$ also preserves composition and thus is a valid functor. □

Proposition 4.1.7. *If* $\varphi : L \to M$ *is an isomorphism of orthomodular lattices, then* $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$ *is an order isomorphism in* **Pos**.

Proof. Functors preserve isomorphisms, and $\mathcal{B}$ is a functor. □

Reference [17] also considers the context category $\mathcal{B}(L)$ of an orthomodular lattice L. They prove the converse of Proposition 4.1.7, that any isomorphism $f : \mathcal{B}(L) \to \mathcal{B}(M)$ determines an isomorphism of orthomodular lattices $f^* : L \to M$. However, their isomorphism f^* is unique if and only if L has no maximal four-element Boolean subalgebras. Instead of considering simply isomorphisms between the context categories of orthomodular lattices, instead in Chapter 5 we consider isomorphisms between the spectral presheaves of orthomodular lattices, which consist of an an isomorphism of context categories as well as some additional data. This additional data allows us to construct a bijection between orthomodular lattices isomorphisms and spectral presheaf isomorphisms, a stronger result.

4.2 The partial orthomodular lattice L_{part}

The Boolean subalgebras of an orthomodular lattice can also be used to generate a second structure, called the partial orthomodular lattice associated with L.

Definition 4.2.1. Let L be an orthomodular lattice. The *partial orthomodular lattice* L_{part} *associated with* L has the same elements and orthocomplements as L and lattice operations $\vee$ and $\wedge$ inherited from L, but only defined for families of elements $(a_i)_{i\in I}$ in L such that there is a Boolean subalgebra $B \in \mathcal{B}(L)$ that contains a_i for all $i \in I$. Such families of elements are called *compatible elements*.

Definition 4.2.2. A *morphism of partial orthomodular lattices* is a function $p : L_{part} \to M_{part}$ that preserves orthocomplements and existing meets and joins.

The following lemma illustrates the necessity of the orthomodularity condition to our endeavors.

Lemma 4.2.3. *If $a \leq b$ in orthomodular lattice L and $p : L_{part} \to M_{part}$ is a partial orthomodular lattice homomorphism, then $p(a) \leq p(b)$.*

Proof. Suppose $a, b \in L$ and $a \leq b$. By Proposition 4.0.5, there is some Boolean subalgebra of L that contains both a and b. This means that the meet $a \wedge b = a$ is defined in L_{part}, and thus is preserved by any partial orthomodular lattice homomorphism p:

$$p(a) = p(a \wedge b) = p(a) \wedge p(b).$$

From this it follows that $p(a) \leq p(b)$. □

Just as in the previous sections, partial orthomodular lattices associated with orthomodular lattices and partial orthomodular lattice homomorphisms form a category **POML**. The motivation for considering partial ortholattices comes from the Bohrification of an orthomodular lattice, to be defined in Chapter 5.

Lemma 4.2.4. *Let L and M be orthomodular lattices, and L_{part} and M_{part} their associated partial orthomodular lattices.There is a bijective correspondence between isomorphisms $L \to M$ in* **OML** *and isomorphisms $L_{part} \to M_{part}$ in* **POML**.

Proof. Let $\varphi : L \to M$ be an isomorphism in **OML**. As a homomorphism between orthomodular lattices, it preserves orthocomplements and finite meets and joins. In particular, it preserves all meets and joins that are defined in L_{part}, meaning that it induces a homomorphism $\varphi : L_{part} \to M_{part}$. As $\varphi : L \to M$ is an isomorphism, so is $\varphi : L_{part} \to M_{part}$.

Conversely, let $p : L_{part} \to M_{part}$ be an isomorphism of partial ortholattices in **POML**. Let $(a_i)_{i \in I}$ be any finite family of elements in L; our goal is to show that

$$p\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} p(a_i),$$

which implies that p preserves all joins, not just those joins that are defined in L_{part}. The same result for meets then follows by taking orthocomplements and using de Morgan's Law (Fact 3.2.4).

First, suppose that there is some Boolean subalgebra B of L such that $a_i \in B$ for all $i \in I$. Thus $\bigvee_{i \in I} a_i$ is defined in L_{part}, and as partial orthomodular lattice homomorphism p preserves all joins that are defined in L_{part},

$$p\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} p(a_i),$$

as desired.

Now, assume that there is no $B \in \mathcal{B}(L)$ such that $a_i \in B$ for all $i \in I$. Consider the element $\bigvee_{i \in I} a_i$ of L. Note that for each i, $a_i \leq \bigvee_{i \in I} a_i$, meaning that by Lemma 4.2.3,

$$p(a_i) \leq p\left(\bigvee_{i \in I} a_i\right).$$

As this is true for all i, it follows that

$$\bigvee_{i \in I} p(a_i) \leq p\left(\bigvee_{i \in I} a_i\right), \tag{4.1}$$

where the join on the left hand side is taken in M.

Let $p^{-1} : M_{part} \to L_{part}$ be the inverse of partial orthomodular lattice isomorphism p, which it is easy to see is also a partial orthomodular lattice isomorphism. Note that for all i,

$$p(a_i) \leq \bigvee_{i \in I} p(a_i).$$

Again by Lemma 4.2.3, p^{-1} preserves inequalities, so this equation becomes

$$a_i = p^{-1}(p(a_i)) \leq p^{-1}\left(\bigvee_{i \in I} p(a_i)\right).$$

As this is true for all $i \in I$, it follows that

$$\bigvee_{i \in I} a_i \leq p^{-1}\left(\bigvee_{i \in I} p(a_i)\right),$$

where the join on the left hand side is taken in L. Applying p to the above equation and invoking Lemma 4.2.3 one last time, the above equation becomes

$$p\left(\bigvee_{i \in I} a_i\right) \leq p\left(p^{-1}\left(\bigvee_{i \in I} p(a_i)\right)\right) = \bigvee_{i \in I} p(a_i) \tag{4.2}$$

Equations 4.1 and 4.2 together imply

$$p\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} p(a_i),$$

showing that p preserves all joins in L, not only those joins which are defined in L_{part}, as desired.

Showing that p preserves all meets in L follows easily. Let $(a_i)_{i \in I}$ be any family of elements in L. Then $(a_i')_{i \in I}$ is also a family of elements in L, and we know

$$p\left(\bigvee_{i \in I} a_i'\right) = \bigvee_{i \in I} p(a_i').$$

Recall that orthocomplementation is preserved by p and satisfies de Morgan's laws. Then,

$$p\left(\bigwedge_{i \in I} a_i\right) = p\left(\left[\bigvee_{i \in I} a_i'\right]'\right) = \left[p\left(\bigvee_{i \in I} a_i'\right)\right]' = \left[\bigvee_{i \in I} p(a_i')\right]' = \bigwedge_{i \in I} [p(a_i')]' = \bigwedge_{i \in I} p(a_i'') = \bigwedge_{i \in I} p(a_i).$$

Thus, as p preserves all meets and joins in L, as well as all orthocomplements, p is in fact an isomorphism of orthomodular lattices, $p : L \to M$.

Note that $p : L_{part} \to M_{part}$ and $p : L \to M$ are the same on every element of L and $\varphi : L \to M$ and the induced $\varphi : L_{part} \to M_{part}$ are the same on every element of L. Thus there is a bijective correspondence between isomorphisms $\varphi : L \to M$ and isomorphisms $p : L_{part} \to M_{part}$. □

Note that it is in the construction of an isomorphism of orthomodular lattices from an isomorphism of partial orthomodular lattices that the orthomodularity condition (in the form of Lemma 4.2.3) is essential. This result does not hold for arbitrary ortholattices, and is the reason we consider orthomodular lattices instead of the more general class of ortholattices.

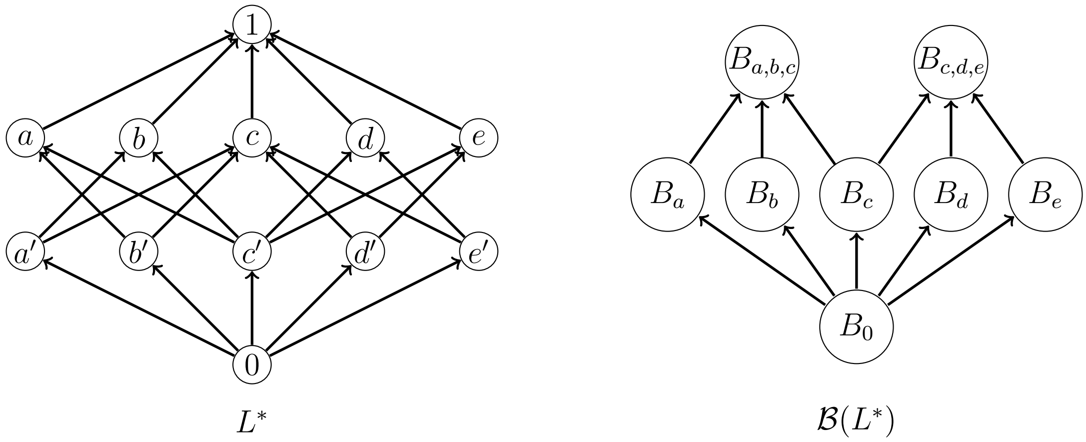


Figure 4.2: An orthomodular lattice L^* with twelve elements and its context category $\mathcal{B}(L^*)$.

a, d	a', d	b, d	b', d
a, d'	a', d'	b, d'	b', d'
a, e	a', e	b, e	b', e
a, e'	a', e'	b, e'	b', e'

Table 4.1: Pairs of elements that are not compatible in L^*; that is, pairwise meets and joins between these elements are not defined in L^*_{part}.

4.3 Example

We now consider a small orthomodular lattice L^*, and examine $\mathcal{B}(L^*)$ and L^*_{part}. Let L^* be as in Figure 4.2. Consider the Boolean subalgebras of L^*. The two-element Boolean algebra $B_0 = \{0, 1\}$ is a sublattice of L^*, as it is for all orthomodular lattices. The four-element Boolean algebra B_a that is (i) of Figure 3.1 appears as a sublattice of L five times, as B_a, B_b, B_c, B_d, and B_e. The eight-element Boolean algebra $B_{a,b,c}$ that is (iv) of Figure 3.1 appears twice, as $B_{a,b,c}$ and $B_{c,d,e}$. This yields the context category shown shown in Figure 4.2.

The partial ortholattice L^*_{part} has the same elements as L^* but meets and joins only defined for compatible elements. Table 4.1 lists all pairs of elements in L that do not have a well-defined meet or join. For L^*, larger families of elements are compatible precisely when they contain none of the pairs in Table 4.1, though this is not the case in general. To see this, consider L^* with additional elements f and f' such that $B_{e,f,a}$ is the eight-element Boolean algebra. Then, for elements a, c, and e, all pairwise meets and joins are defined but not the meet or join of all three elements.

Chapter 5

The Spectral Presheaf of an Orthomodular Lattice

We now proceed to define the spectral presheaf of an orthomodular lattice in an analogous way to the spectral presheaf of a von Neumann algebra.

5.1 Defining the spectral presheaf

Given any orthomodular lattice L, the *spectral presheaf* $\underline{\Omega(L)}$ is a contravariant functor from poset category $\mathcal{B}(L)$ to **Set**. This functor maps an object (Boolean subalgebra) B of $\mathcal{B}(L)$ to the set of elements of Stone space Ω_B of B, without its topological structure:

$$\underline{\Omega(L)}_B = \Omega_B = \{\lambda : B \to \{0,1\} \mid \lambda \text{ is a Boolean algebra homomorphism}\}.$$

On inclusion arrows $i_{B',B} : B' \to B$ in poset $\mathcal{B}(L)$, the contravariant action of the spectral presheaf $\underline{\Omega(L)}$ is defined by restriction:

$$\begin{aligned} \underline{\Omega(L)}(i_{B',B}) : \underline{\Omega(L)}_B &\to \underline{\Omega(L)}_{B'} \\ \lambda &\mapsto \lambda|_{B'} \end{aligned}$$

Let $r_{B,B'} : \Omega_B \to \Omega_{B'}$ be the function that restricts homomorphisms in Ω_B which have domain B to the smaller domain B'. Then,

$$\underline{\Omega(L)}(i_{B',B}) = r_{B,B'}.$$

Note the contravariance here. One can easily check that $\underline{\Omega(L)}$ is a valid contravariant functor. Thus the presheaf $\underline{\Omega(L)}$ lives in the topos $\mathbf{Set}^{\mathcal{B}(L)^{op}}$ of contravariant functors from $\mathcal{B}(L)$ to **Set**, and its image is a subcategory of **Set**.

Of note, we will also at times consider the spectral presheaf alternately as a functor with codomain **Stone**; $\underline{\Omega(L)}_B$ in this case is the Stone space of B instead of its set of elements, and it can be shown that with the Stone topology the $r_{B,B'}$ as defined above are continuous maps. This interpretation will useful for generalizing Stone duality to spectral presheaves in Section 5.5. However, the functor category $\mathbf{Stone}^{\mathcal{B}(L)^{op}}$ is not a topos, so in order to talk about topos structure such as subobjects it is necessary to consider the spectral presheaf as a functor with codomain **Set**.

Just as the spectral presheaf of a von Neumann algebra can be interpreted as a generalized Gelfand spectrum, the spectral presheaf of an orthomodular lattice can be interpreted as a generalized Stone space.

Recall that the elements of an orthomodular lattice can correspond to quantum propositions. A Boolean subalgebra B of an orthomodular lattice is distributive, meaning it behaves similarly to logical structures of classical physics with respect to conjunction, disjunction, and complementation of propositions. Consider a Boolean algebra homomorphism from B to $\{0,1\}$. Interpreting those propositions mapped to 1 as 'true' and those propositions mapped to 0 as 'false', one can interpret the elements of the Stone space of B as possible local valuation functions for the propositions (elements) that comprise B. These local valuation functions are 'classical' because logic in B is distributive (in fact, Boolean), as it is in classical settings. The spectral presheaf of an orthomodular lattice can then be interpreted as taking these spaces of local valuations for classical contexts and linking them together with restriction maps.

5.1.1 Example

Consider the orthomodular lattice L^* from Section 4.3. This lattice and its context category are shown in Figure 4.2. The spectral presheaf of L^* is a functor from $\mathcal{B}(L)$ to **Set**. Each Boolean subalgebra B of L^* is mapped to its Stone space Ω_B. The Stone spaces of all the Boolean subalgebras of L can be found in Section 3.4.1.

We now consider the action of the spectral presheaf on an inclusion map in $\mathcal{B}(L)$. We know that $B_a \subseteq B_{a,b,c}$, meaning there is an arrow $i_{B_a,B_{a,b,c}}$ in $\mathcal{B}(L)$. Recall from Section 3.4.1 that the Stone space of B_a has two elements, called λ_a and $\lambda_{a'}$, where $\lambda_a(a) = 1$ and $\lambda_{a'}(a) = 0$. Additionally, the Stone space of $B_{a,b,c}$ has three elements $\lambda_{a,b}$, $\lambda_{a,c}$, and $\lambda_{b,c}$, where the subscripts denote the two elements out of a, b, and c that are mapped to 1. Then, $\underline{\Omega(L^*)}(i_{B_a,B_{a,b,c}})$ is a map r from $\Omega_{B_{a,b,c}}$ to Ω_{B_a} whose action on elements of $\Omega_{B_{a,b,c}}$ simply restricts the domains of the homomorphisms to

B_a:

$$r(\lambda_{a,b}) = \lambda_{a,b}|_{B_a} = \lambda_a$$
$$r(\lambda_{a,c}) = \lambda_{a,c}|_{B_a} = \lambda_a$$
$$r(\lambda_{b,c}) = \lambda_{b,c}|_{B_a} = \lambda_{a'}$$

Note that as the inverse image of any open set of Ω_{B_a} is open in $\Omega_{B_{a,b,c}}$, then this map r is in fact a continuous map when Ω_{B_a} and $\Omega_{B_{a,b,c}}$ are considered as topological spaces rather than simply as sets. The images of other inclusion arrows under the spectral presheaf of L^* can be determined similarly, and are also continuous maps between topological spaces.

5.2 Maps between spectral presheaves

The next obvious step is to consider maps between spectral presheaves of orthomodular lattices. Specifically, if L and M are orthomodular lattices and $\varphi : L \to M$ is an orthomodular lattice homomorphism, then we want to define some map, that is determined by φ, from $\underline{\Omega(M)}$ to $\underline{\Omega(L)}$. This is done in two steps, below. The first step transforms $\underline{\Omega(M)}$ into a contravariant functor from $\mathcal{B}(L)$ to **Set**, while the second step then gives a natural transformation within $\mathbf{Set}^{\mathcal{B}(L)^{op}}$ from this new functor to $\underline{\Omega(L)}$. In particular, such a map will be used to show that $L \cong M$ if and only if $\underline{\Omega(L)} \cong \underline{\Omega(M)}$, the goal of this section. This result will imply that the spectral presheaf determines up to isomorphism the orthomodular lattice it comes from.

Step 1. Recall from Section 4.1 that homomorphism $\varphi : L \to M$ induces a monotone map $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$. This map $\tilde{\varphi}$ then induces a map between functor categories $\tilde{\varphi}^* : \mathbf{Set}^{\mathcal{B}(M)^{op}} \to \mathbf{Set}^{\mathcal{B}(L)^{op}}$ as in Section 2.2. Specifically, $\tilde{\varphi}^*$ is the inverse image part of the essential geometric morphism between presheaf topoi that arises from functor $\tilde{\varphi}$ between the base categories $\mathcal{B}(L)$ and $\mathcal{B}(M)$ [21].

Recall from Section 2.2 that on functors in $\mathbf{Set}^{\mathcal{B}(M)^{op}}$, $\tilde{\varphi}^*$ acts by precomposition by $\tilde{\varphi}$. Specifically, for the spectral presheaf $\underline{\Omega(M)}$ or orthomodular lattice M, $\tilde{\varphi}^*(\underline{\Omega(M)}) = \underline{\Omega(M)} \circ \tilde{\varphi}$ is a functor in the topos $\mathbf{Set}^{\mathcal{B}(L)^{op}}$. The action of this functor on objects $B \in \mathcal{B}(L)$ is given by

$$\tilde{\varphi}^*(\underline{\Omega(M)})_B = (\underline{\Omega(M)} \circ \tilde{\varphi})_B = \underline{\Omega(M)}_{\tilde{\varphi}(B)} = \Omega_{\tilde{\varphi}(B)}.$$

On arrows, this functor acts as

$$\tilde{\varphi}^*(\underline{\Omega(M)})(i_{B',B}) = (\underline{\Omega(M)} \circ \tilde{\varphi})(i_{B',B}) = \underline{\Omega(M)}(i_{\tilde{\varphi}(B'),\tilde{\varphi}(B)}) = r_{\tilde{\varphi}(B),\tilde{\varphi}(B')}.$$

Thus, $\tilde{\varphi}^*$ maps $\underline{\Omega(M)}$ to some functor from $\mathcal{B}(L)$ to **Set**, which is not necessarily $\underline{\Omega(L)}$. However, since a map from $\underline{\Omega(M)}$ to $\underline{\Omega(L)}$ is desired, it is now necessary to define a way to transform $\tilde{\varphi}^*(\underline{\Omega(M)})$ to $\underline{\Omega(L)}$ within the functor category $\mathbf{Set}^{\mathcal{B}(L)^{op}}$. This is done via a natural transformation as follows.

Step 2. By Proposition 4.1.3, for each $B \in \mathcal{B}(L)$, orthomodular lattice homomorphism $\varphi : L \to M$ induces a Boolean algebra homomorphism $\varphi|_B : B \to \tilde{\varphi}(B)$. This in turn gives a morphism as follows:

$$\begin{aligned} \zeta_{\varphi,B} :\ \tilde{\varphi}^*(\underline{\Omega(M)})_B = \underline{\Omega(M)}_{\tilde{\varphi}(B)} &\to \qquad\qquad \underline{\Omega(L)}_B \\ (\lambda : \tilde{\varphi}(B) \to \{0,1\}) \quad &\mapsto (\lambda \circ \varphi|_B : B \to \tilde{\varphi}(B) \to \{0,1\}) \end{aligned}$$

In fact, these maps $\zeta_{\varphi,B}$ are the components of a natural transformation ζ_φ.

Lemma 5.2.1. *The $\zeta_{\varphi,B}$ are the components of a natural transformation between functors in $\mathbf{Set}^{\mathcal{B}(L)^{op}}$:*

$$\zeta_\varphi : \tilde{\varphi}^*(\underline{\Omega(M)}) \Rightarrow \underline{\Omega(L)}.$$

Proof. Recall

$$\begin{aligned} \underline{\Omega(L)}_B &= \Omega_B \\ \underline{\Omega(L)}(i_{B',B}) &= r_{B'B} \\ \tilde{\varphi}^*(\underline{\Omega(M)})_B &= \underline{\Omega(M)}_{\tilde{\varphi}(B)} = \Omega_{\tilde{\varphi}(B)} \\ \tilde{\varphi}^*(\underline{\Omega(M)})(i_{B',B}) &= \underline{\Omega(M)}(i_{\tilde{\varphi}(B'),\tilde{\varphi}(B)}) = r_{\tilde{\varphi}(B),\tilde{\varphi}(B')} \end{aligned}$$

For $B', B \in \mathcal{B}(L)$, where $i_{B',B}$ is an inclusion arrow, to show ζ_φ is a natural transformation it is sufficient to show that the following diagram commutes:

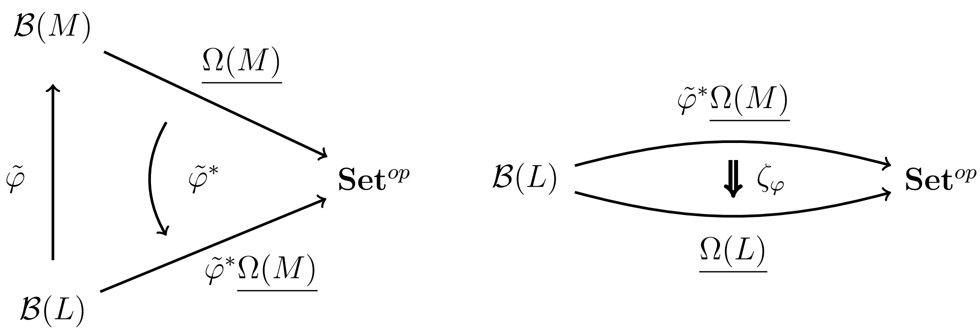
$$\begin{array}{ccc} \Omega_{\tilde{\varphi}(B')} & \xleftarrow{\ r_{\tilde{\varphi}(B),\tilde{\varphi}(B')}\ } & \Omega_{\tilde{\varphi}(B)} \\ \downarrow{\scriptstyle \zeta_{\varphi,B'}} & & \downarrow{\scriptstyle \zeta_{\varphi,B}} \\ \Omega_{B'} & \xleftarrow{\ r_{B,B'}\ } & \Omega_B \end{array}$$

Let $\lambda : \tilde{\varphi}(B) \to \{0,1\}$ be any element of $\Omega_{\tilde{\varphi}(B)}$. Then,

$$\begin{aligned}
\left[\zeta_{\varphi,B'} \circ r_{\tilde{\varphi}(B),\tilde{\varphi}(B')}\right](\lambda) &= \zeta_{\varphi,B'}(\lambda|_{\tilde{\varphi}(B')}) \\
&= \lambda|_{\tilde{\varphi}(B')} \circ \varphi|_{B'} \\
&= (\lambda \circ \varphi)|_{B'} \\
\left[r_{B,B'} \circ \zeta_{\varphi,B}\right](\lambda) &= r_{B,B'}(\lambda \circ \varphi|_B) \\
&= (\lambda \circ \varphi|_B)|_{B'} \\
&= (\lambda \circ \varphi)|_{B'}.
\end{aligned}$$

Thus, the diagram commutes and ζ_φ is a natural transformation. □

The two maps $\tilde{\varphi}^*$ and ζ_φ defined above can be combined to give, for any homomorphism $\varphi : L \to M$, a map from $\underline{\Omega(M)}$ to $\underline{\Omega(L)}$, written $\langle \tilde{\varphi}^*, \zeta_\varphi \rangle$. As $\tilde{\varphi}^*$ is completely determined by $\tilde{\varphi}$, this can also equivalently be written $\langle \tilde{\varphi}, \zeta_\varphi \rangle$, and we will follow this second convention. Note that the process described above is not a standard composition $\zeta_\varphi \circ \tilde{\varphi}^*$, as these two maps are not within the same category; $\tilde{\varphi}^*$ is a map between topoi $\mathbf{Set}^{\mathcal{B}(M)^{op}}$ and $\mathbf{Set}^{\mathcal{B}(L)^{op}}$, while ζ_φ is a natural transformation within $\mathbf{Set}^{\mathcal{B}(L)^{op}}$:



Step 1 **Step 2**

However, a perhaps more intuitive way of understanding the action of $\langle \tilde{\varphi}^*, \zeta_\varphi \rangle$ is to consider it as a relation between the images in **Set** of $\mathcal{B}(M)$ under $\underline{\Omega(M)}$ and of $\mathcal{B}(L)$ under $\underline{\Omega(L)}$. The image in **Set** of $\mathcal{B}(M)$ under $\underline{\Omega(M)}$ will be denoted $\underline{\Omega(M)}_{\mathcal{B}(M)}$, not to be confused with $\underline{\Omega(M)}_B$ for an object B of $\mathcal{B}(M)$. This is a subcategory of **Set**, because it is the image of a functor.

First, $\tilde{\varphi}^*$ acts on $\underline{\Omega(M)}_{\mathcal{B}(M)}$ by restricting it to the image of only those Boolean subalgebras of M that are the image of some Boolean subalgebra of L under $\tilde{\varphi}$. That is, if

$$\tilde{\varphi}(\mathcal{B}(L)) := \{\tilde{\varphi}(B) \mid B \in \mathcal{B}(L)\} \subseteq \mathcal{B}(M),$$

then $\tilde{\varphi}^*$ restricts $\underline{\Omega(M)}_{\mathcal{B}(M)}$ to $\underline{\Omega(M)}_{\tilde{\varphi}(\mathcal{B}(L))}$. Then, natural transformation ζ_φ gives a relation between $\underline{\Omega(M)}_{\tilde{\varphi}(\mathcal{B}(L))}$ and $\underline{\Omega(L)}_{\mathcal{B}(L)}$, both of which are subcategories of

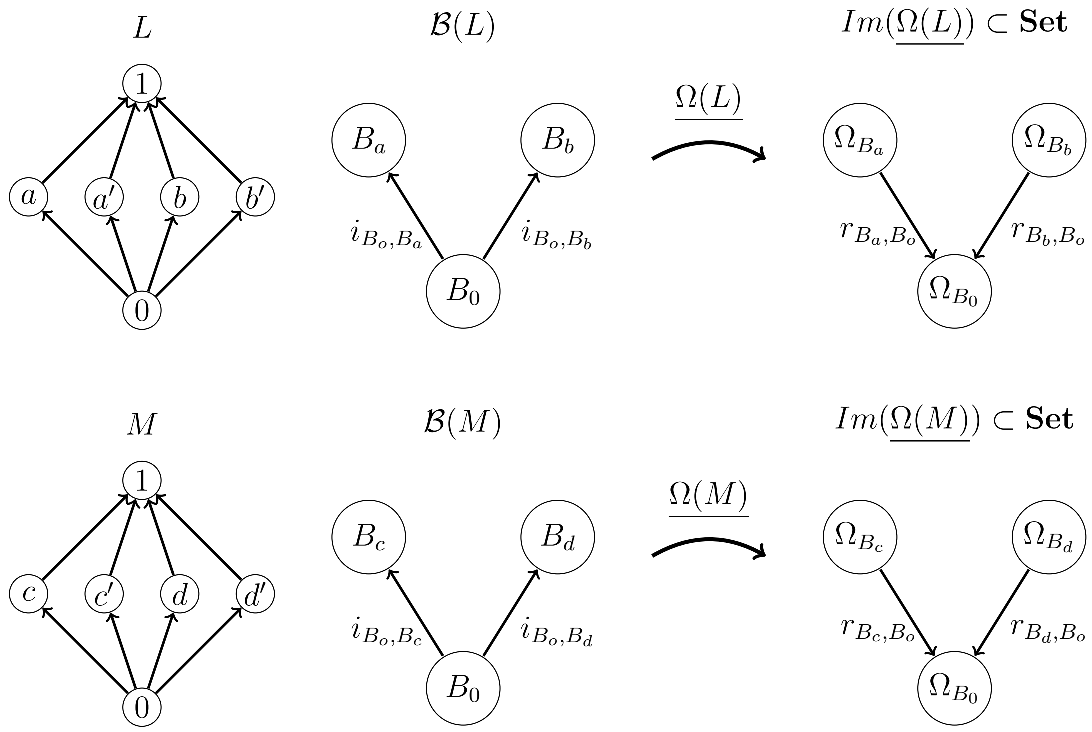


Figure 5.1: Two orthomodular lattices L and M, their context categories, and the images of those context categories under their spectral presheaves.

Set. Thus, $\langle \tilde{\varphi}^*, \zeta_\varphi \rangle$ can be viewed as a relation from $\underline{\Omega(M)}_{\mathcal{B}(M)}$ to $\underline{\Omega(L)}_{\mathcal{B}(L)}$. If $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$ is surjective, then this relation is defined on all of $\underline{\Omega(M)}_{\mathcal{B}(M)}$, while if $\tilde{\varphi}$ is injective, then this relation is functional.

5.2.1 Example

Consider orthomodular lattices L and M as given in Figure 5.1. The same figure also shows their context categories and their spectral prsheaves.

Suppose $\varphi : L \to M$ is an orthomodular lattice homomorphism, characterized by $\varphi(a) = c$ and $\varphi(b) = c'$ (note this is sufficient to determine the action of φ on all elements of L). This homomorphism is neither injective nor surjective, but is clearly a valid orthomodular lattice homomorphism. Then $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$ is given by:

$$\tilde{\varphi}(B_a) = B_c$$
$$\tilde{\varphi}(B_b) = B_{c'} = B_c$$
$$\tilde{\varphi}(B_0) = B_0$$

Applying $\tilde{\varphi}^*$ to $\underline{\Omega(M)}$ involves precomposition by $\tilde{\varphi}$. Thus, $\tilde{\varphi}^*(\underline{\Omega(M)})$ is the map from $\mathcal{B}(L)$ to **Set** given on objects by:

$$
\begin{aligned}
\tilde{\varphi}^*(\underline{\Omega(M)}) : \mathcal{B}(L) \to \mathcal{B}(M) &\to \textbf{Set} \\
B_a \quad \mapsto \quad B_c \quad &\mapsto \Omega_{B_c} \\
B_b \quad \mapsto \quad B_c \quad &\mapsto \Omega_{B_c} \\
B_0 \quad \mapsto \quad B_0 \quad &\mapsto \Omega_{B_0}
\end{aligned}
$$

Note that the image of $\mathcal{B}(L)$ under $\tilde{\varphi}^*(\underline{\Omega(M)})$ consists of only Ω_{B_c}, Ω_{B_0}, and the restriction map between them.

Now we apply natural transformation ζ_φ to this functor $\tilde{\varphi}^*(\underline{\Omega(M)})$. This functor has three components, one each for the three Boolean subalgebras of L, and each component acts by precomposition by φ with an appropriate domain restriction:

$$
\begin{aligned}
\zeta_\varphi &: \tilde{\varphi}^*(\underline{\Omega(M)}) \Rightarrow \underline{\Omega(L)} \\
\zeta_{\varphi,B_a} &: \Omega_{B_c} \to \Omega_{B_a} \\
\zeta_{\varphi,B_b} &: \Omega_{B_c} \to \Omega_{B_b} \\
\zeta_{\varphi,B_0} &: \Omega_{B_0} \to \Omega_{B_0}
\end{aligned}
$$

Recall from Section 3.4.1 that Ω_{B_a} has two elements, called λ_a and $\lambda_{a'}$, where $\lambda_a(a) = 1$ and $\lambda_{a'}(a) = 0$, and similarly for Ω_{B_b} and Ω_{B_c}. Ω_{B_0} simply has one element, λ_0. Using these facts, the action of the three components of ζ_φ can be specified explicitly:

$$
\begin{aligned}
\zeta_{\varphi,B_a} &: \Omega_{B_c} \to \Omega_{B_a} \\
&\quad \lambda_c \mapsto \lambda_c \circ \varphi|_{B_a} = \lambda_a \\
&\quad \lambda_{c'} \mapsto \lambda_{c'} \circ \varphi|_{B_a} = \lambda_{a'} \\
\zeta_{\varphi,B_b} &: \Omega_{B_c} \to \Omega_{B_b} \\
&\quad \lambda_c \mapsto \lambda_c \circ \varphi|_{B_b} = \lambda_{b'} \\
&\quad \lambda_{c'} \mapsto \lambda_{c'} \circ \varphi|_{B_b} = \lambda_b \\
\zeta_{\varphi,B_0} &: \Omega_{B_0} \to \Omega_{B_0} \\
&\quad \lambda_0 \mapsto \lambda_0 \circ \varphi|_{B_0} = \lambda_0
\end{aligned}
$$

Thus, we have shown how to use an orthomodular lattice homomorphism to define a map between spectral presheaves. As every orthomodular lattice has a spectral presheaf and every orthomodular lattice homomorphism determines a map between spectral presheaves, this suggests that it might be possible to define a functor from **OML** to spectral presheaves and maps between them. First, it is necessary to find some category in which the spectral presheaves of orthomodular lattices are objects and maps between spectral presheaves are morphisms. This leads to the definitions in the next section.

5.3 The category of $\mathcal{D}$-valued presheaves

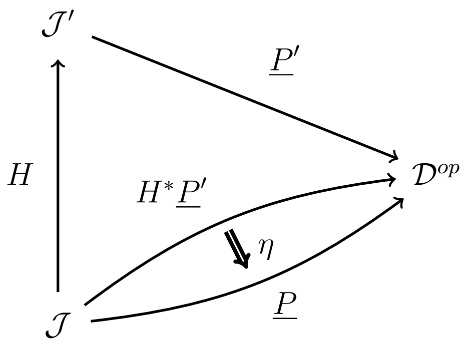
The rather unintuitive definition of a map between spectral presheaves, above, is in fact simply an arrow in a category **Presh(Stone)**. This category is defined and explored here in order to give more intuition about the definition of a spectral presheaf map. Additionally, this more general framework will provide us with the necessary tools to show that the spectral presheaf of an orthomodular lattice is a complete invariant, determining its associated orthomodular lattice up to isomorphism. Definitions and explorations of this category appeared in [11].

Recall from Section 2.2 that for categories $\mathcal{J}$, $\mathcal{J}'$, and $\mathcal{D}$, a functor $H : \mathcal{J} \to \mathcal{J}'$ induces a functor $H^* : (\mathcal{D}^{op})^{\mathcal{J}'} \to (\mathcal{D}^{op})^{\mathcal{J}}$ whose action on objects is precomposition by H. We proceed to define the following.

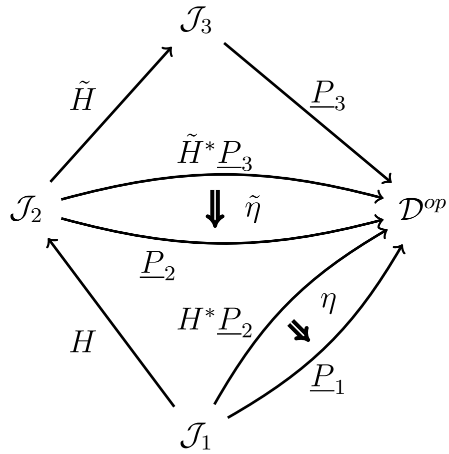
Definition 5.3.1. The category **Presh($\mathcal{D}$)** of $\mathcal{D}$-valued presheaves has as its objects functors (presheaves) of the form $\underline{P} : \mathcal{J} \to \mathcal{D}^{op}$, where $\mathcal{J}$ is a small category. Arrows are pairs

$$\langle H, \eta \rangle : (\underline{P}' : \mathcal{J}' \to \mathcal{D}^{op}) \to (\underline{P} : \mathcal{J} \to \mathcal{D}^{op}),$$

where $H : \mathcal{J} \to \mathcal{J}'$ is a functor and $\eta : H^*\underline{P}' \Rightarrow \underline{P}$ is a natural transformation in $(\mathcal{D}^{op})^{\mathcal{J}}$:



Let $\underline{P}_i : \mathcal{J}_i \to \mathcal{D}^{op}$, for $i = 1, 2, 3$, be functors; let $\langle \tilde{H}, \tilde{\eta} \rangle : \underline{P}_3 \to \underline{P}_2$ and $\langle H, \eta \rangle : \underline{P}_2 \to \underline{P}_1$ be two arrows:



The composition $\langle H, \eta \rangle \circ \langle \tilde{H}, \tilde{\eta} \rangle : \underline{P}_3 \to \underline{P}_1$ is given by

$$\langle H, \eta \rangle \circ \langle \tilde{H}, \tilde{\eta} \rangle = \langle \tilde{H} \circ H, \eta \circ H^* \tilde{\eta} \rangle,$$

where $\eta \circ H^* \tilde{\eta}$ denotes vertical composition of natural transformations.

Lemma 5.3.2. Presh($\mathcal{D}$) *is a category.*

Proof. First, it is necessary to show that composition as given above is well-defined, that is, that $\langle H, \eta \rangle \circ \langle \tilde{H}, \tilde{\eta} \rangle$ is a valid arrow from $\underline{P}_3$ to $\underline{P}_1$. Consider the diagram above. Clearly $\tilde{H} \circ H$ is a functor from $\mathcal{J}_1$ to $\mathcal{J}_3$, as required. Then, the natural transformation $\eta \circ H^* \tilde{\eta}$ is from $H^*(\tilde{H}^* \underline{P}_3)$ to $H^* \underline{P}_2$ to $\underline{P}_1$ in $(\mathcal{D}^{op})^{\mathcal{J}_1}$. As $H^* \circ \tilde{H}^* = (\tilde{H} \circ H)^*$ by Fact 2.2.3, it follows that $\eta \circ H^* \tilde{\eta} : (\tilde{H} \circ H)^* \underline{P}_3 \Rightarrow \underline{P}_1$, as required.

It is also necessary to show that this composition is associative, which will be done algebraically. Suppose $\underline{P}_4 : \mathcal{J}_4 \to \mathcal{D}^{op}$ is a presheaf, $\hat{H} : \mathcal{J}_3 \to \mathcal{J}_4$ is a functor, and $\langle \hat{H}, \hat{\eta} \rangle$ is an arrow from $\underline{P}_4$ to $\underline{P}_3$. Then, by the definition of composition, the functoriality of H^*, the associativity of functors and natural transformations, and Fact 2.2.3,

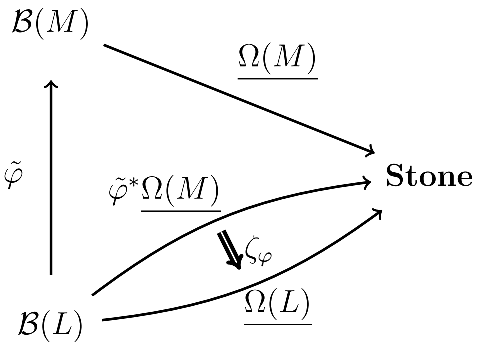
$$\begin{aligned}
\Big(\langle H, \eta \rangle \circ \langle \tilde{H}, \tilde{\eta} \rangle\Big) \circ \langle \hat{H}, \hat{\eta} \rangle &= \langle \tilde{H} \circ H, \eta \circ H^* \tilde{\eta} \rangle \circ \langle \hat{H}, \hat{\eta} \rangle \\
&= \langle \hat{H} \circ \Big(\tilde{H} \circ H\Big), (\eta \circ H^* \tilde{\eta}) \circ (\tilde{H} \circ H)^* \hat{\eta} \rangle \\
&= \langle \hat{H} \circ \Big(\tilde{H} \circ H\Big), \eta \circ \Big(H^* \tilde{\eta} \circ (H^* \circ \tilde{H}^*) \hat{\eta}\Big) \rangle \\
&= \langle \Big(\hat{H} \circ \tilde{H}\Big) \circ H, \eta \circ H^* \Big(\tilde{\eta} \circ \tilde{H}^* \hat{\eta}\Big) \rangle \\
&= \langle H, \eta \rangle \circ \langle \hat{H} \circ \tilde{H}, \tilde{\eta} \circ \tilde{H}^* \hat{\eta} \rangle \\
&= \langle H, \eta \rangle \circ \Big(\langle \tilde{H}, \tilde{\eta} \rangle \circ \langle \hat{H}, \hat{\eta} \rangle\Big)
\end{aligned}$$

Finally, it remains only to show that every object $\underline{P} : \mathcal{J} \to \mathcal{D}^{op}$ of **Presh($\mathcal{D}$)** has an identity arrow. But, if $Id_J : \mathcal{J} \to \mathcal{J}$ is the identity functor on J and $id_{\underline{P}} : \underline{P} \Rightarrow \underline{P}$ is the identity natural transformation on $\underline{P}$, then $\langle Id_J, id_{\underline{P}} \rangle$ is the appropriate identity arrow on $\underline{P}$, which can be easily verified using the definitions above. Thus, **Presh($\mathcal{D}$)** is a valid category. □

It is possible to view spectral presheaves and spectral presheaf maps as defined in the previous subsection as a subcategory of **Presh(Stone)**. Note that we will now assume that the spectral presheaf of an orthomodular lattice has codomain **Stone** rather than codomain **Set**. Specifically, it is the subcategory with objects and arrows determined as follows.

$$\text{Objects: } \{\underline{\Omega(L)} : \mathcal{B}(L) \to \textbf{Set} \mid L \text{ is an orthomodular lattice.}\}$$
$$\text{Morphisms: } \{\langle \tilde{\varphi}, \zeta_\varphi \rangle \mid \varphi \text{ is a orthomodular lattice homomorphism.}\}$$

The latter is an arrow in **Presh(Stone)**, depicted here:



In fact, this subcategory is the image of a functor; there is a contravariant functor SP: **OML** $\to$ **Presh(Stone)**, which acts as follows for all orthomodular lattices L and all orthomodular lattice homomorphisms $\varphi : L \to M$:

$$SP(L) = \underline{\Omega(L)}$$
$$SP(\varphi) = \langle \tilde{\varphi}, \zeta_\varphi \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$$

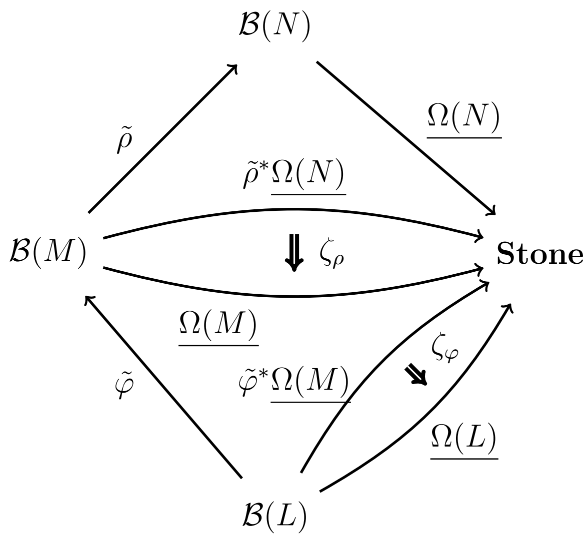
Proposition 5.3.3. *SP is a functor.*

Proof. First, we must check that SP preserves identities. Suppose $i : L \to L$ is the identity orthomodular lattice homomorphism on L. Then, $\tilde{i} : \mathcal{B}(L) \to \mathcal{B}(L)$ is also the identity functor on category $\mathcal{B}(L)$. Further, ζ_i has components given by

$$\zeta_{i,B} : \underline{\Omega(L)}_B \to \underline{\Omega(L)}_B$$
$$\lambda \mapsto \lambda \circ i = \lambda$$

Thus, as each $\zeta_{i,B}$ is just the identity map on $\underline{\Omega(L)}_B$ in **Stone**, it follows that ζ_i is the identity natural transformation on $\underline{\Omega(L)}$. Thus, $\langle\tilde{i}, \zeta_i\rangle$ is the identity arrow of $\underline{\Omega(L)}$ in category **Presh(Stone)**.

Next, it is necessary to show that SP preserves composition. Suppose $\varphi : L \to M$ and $\rho : M \to N$ are orthomodular lattice homomorphisms. Recalling that SP is contravariant, we wish to show that $SP(\rho \circ \varphi) = SP(\varphi) \circ SP(\rho)$. Consider the following diagram, which depicts arrows $SP(\varphi) : \underline{\Omega(M)} \to \underline{\Omega(L)}$ and $SP(\rho) : \underline{\Omega(N)} \to \underline{\Omega(M)}$ in **Presh(Stone)**:



Recall the definition of composition in **Presh(Stone)**:

$$SP(\varphi) \circ SP(\rho) = \langle\tilde{\varphi}, \zeta_\varphi\rangle \circ \langle\tilde{\rho}, \zeta_\rho\rangle = \langle\tilde{\rho} \circ \tilde{\varphi}, \zeta_\varphi \circ \tilde{\varphi}^*\zeta_\rho\rangle$$

Note also that the map from $\mathcal{B}(L)$ to $\mathcal{B}(N)$ induced by the composition $\rho \circ \varphi$ is precisely $\tilde{\rho} \circ \tilde{\varphi}$, which follows from the definition in Section 4.1 of such induced maps. Thus,

$$SP(\rho \circ \varphi) = \langle\tilde{\rho} \circ \tilde{\varphi}, \zeta_{\rho\circ\varphi}\rangle$$

It simply remains to show that the natural transformations $\zeta_\varphi \circ \tilde{\varphi}^*\zeta_\rho$ and $\zeta_{\rho\circ\varphi}$ from presheaf $\tilde{\varphi}^*\tilde{\rho}^*\underline{\Omega(N)}$ to presheaf $\underline{\Omega(L)}$ in $\mathbf{Stone}^{\mathcal{B}(L)^{op}}$ are equal. Consider any element $B \in \mathcal{B}(L)$. Recall, from Fact 2.2.3 and previous definitions, that

$$(\tilde{\varphi}^*\tilde{\rho}^*\underline{\Omega(N)})_B = ((\tilde{\rho} \circ \tilde{\varphi})^*\underline{\Omega(N)})_B = \underline{\Omega(N)}_{(\tilde{\rho}\circ\tilde{\varphi})(B)} = \Omega_{(\tilde{\rho}\circ\tilde{\varphi})(B)}.$$

The action of the component at B of natural transformation $\zeta_{\rho\circ\varphi}$ is, by the definition of ζ,

$$\begin{aligned}\zeta_{\rho\circ\varphi,B} : \Omega_{(\tilde{\rho}\circ\tilde{\varphi})(B)} &\to \Omega_B \\ \lambda &\mapsto \lambda \circ (\rho \circ \varphi)|_B\end{aligned}$$

Now consider natural transformation $\zeta_\varphi \circ \tilde{\varphi}^*\zeta_\rho$.

$$(\zeta_\varphi \circ \tilde{\varphi}^*\zeta_\rho)_B = \zeta_{\varphi,B} \circ (\tilde{\varphi}^*\zeta_\rho)_B = \zeta_{\varphi,B} \circ \zeta_{\rho,\tilde{\varphi}(B)}$$

The action of this composition is given as follows.

$$\begin{aligned}\zeta_{\varphi,B} \circ \zeta_{\rho,\tilde{\varphi}(B)} : \underline{\Omega(N)}_{(\tilde{\rho}\circ\tilde{\varphi})(B)} &\to \underline{\Omega(M)}_{\tilde{\varphi}(B)} \to \underline{\Omega(L)}_B \\ \lambda &\mapsto \lambda \circ \rho|_{\tilde{\varphi}(B)} \mapsto \lambda \circ \rho|_{\tilde{\varphi}(B)} \circ \varphi|_B \\ &= \lambda \circ (\rho \circ \varphi)|_B\end{aligned}$$

As the two natural transformations $\zeta_{\rho\circ\varphi}$ and $\zeta_\varphi \circ \tilde{\varphi}^*\zeta_\rho$ have the same component for every $B \in \mathcal{B}(L)$, then they must be the same natural transformation, implying SP preserves composition and is a functor. □

Thus, the image in **Presh(Stone)** of functor SP, consisting of the spectral presheaves of orthomodular lattices and the spectral presheaf maps between them, is a category. Of note, functor SP is neither full nor faithful.

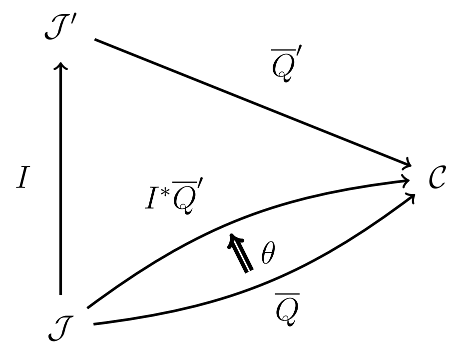
5.4 The category of $\mathcal{C}$-valued copresheaves

Dual to the notion of a presheaf is that of a copresheaf. This definition yields another category **Copresh**$(\mathcal{C})$ as follows.

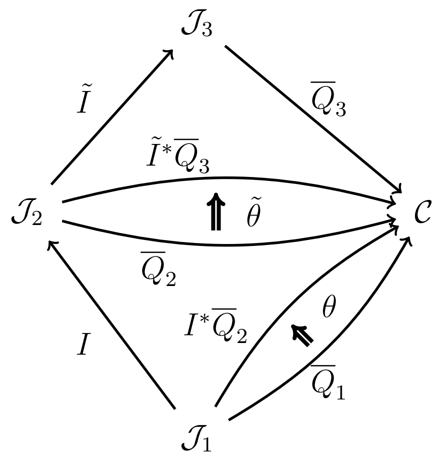
Definition 5.4.1. Let $\mathcal{C}$ be a category. The category **Copresh**$(\mathcal{C})$ of $\mathcal{C}$-valued copresheaves has as its objects functors (copresheaves) of the form $\overline{Q} : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a small category. Arrows are pairs

$$\langle I, \theta\rangle : (\overline{Q} : \mathcal{J} \to \mathcal{C}) \to (\overline{Q}' : \mathcal{J}' \to \mathcal{C}),$$

where $I : \mathcal{J} \to \mathcal{J}'$ is a functor and $\theta : \overline{Q} \Rightarrow I^*\overline{Q}'$ is a natural transformation in $\mathcal{C}^{\mathcal{J}}$:



Let $\overline{Q}_i : \mathcal{J}_i \to \mathcal{C}$, for $i = 1, 2, 3$, be functors. Consider two arrows $\langle I, \theta \rangle : \overline{Q}_1 \to \overline{Q}_2$ and $\langle \tilde{I}, \tilde{\theta} \rangle : \overline{Q}_2 \to \overline{Q}_3$:



The composition $\langle \tilde{I}, \tilde{\theta} \rangle \circ \langle I, \theta \rangle : \overline{Q}_1 \to \overline{Q}_3$ is given by

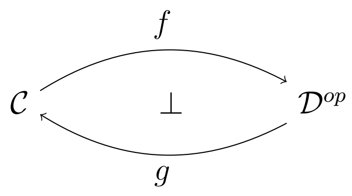
$$\langle \tilde{I}, \tilde{\theta} \rangle \circ \langle I, \theta \rangle = \langle \tilde{I} \circ I, (I^* \tilde{\theta}) \circ \theta \rangle,$$

where $(I^* \tilde{\theta}) \circ \theta$ denotes vertical composition of natural transformations within functor category $\mathcal{C}^{\mathcal{J}_1}$.

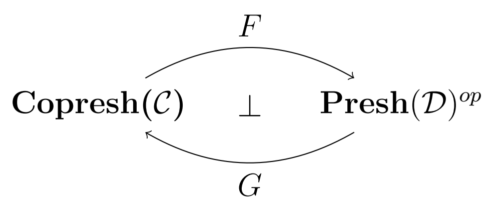
Just as with category **Presh($\mathcal{D}$)**, it follows from the above that **Copresh($\mathcal{C}$)** is a well-defined category, though this proof is omitted due to its similarities to the proof above.

Reference [11] explores the relationship between **Presh($\mathcal{D}$)** and **Copresh($\mathcal{C}$)** and proves the following result.

Lemma 5.4.2 ([11])**.** *Let $\mathcal{C}$, $\mathcal{D}$ be two categories that are dually equivalent,*



Then there is a dual equivalence



The action of F and G is defined in the proof of the above theorem in the following way. First, consider $G : Presh(\mathcal{D})^{op} \to Copresh(\mathcal{C})$. If $\underline{P} : \mathcal{J} \to \mathcal{D}^{op}$ is an object of $Presh(\mathcal{D})^{op}$, then $G(\underline{P}) : \mathcal{J} \to \mathcal{C}$ is the (covariant) functor $g \circ \underline{P}$. That is, for all objects J and arrows $a : J' \to J$ in $\mathcal{J}$,

$$G(\underline{P})_J = (g \circ \underline{P})_J = g(\underline{P}_J) \in Ob(\mathcal{C})$$
$$G(\underline{P})(a) = (g \circ \underline{P})(a) \in Morph(\mathcal{C})$$

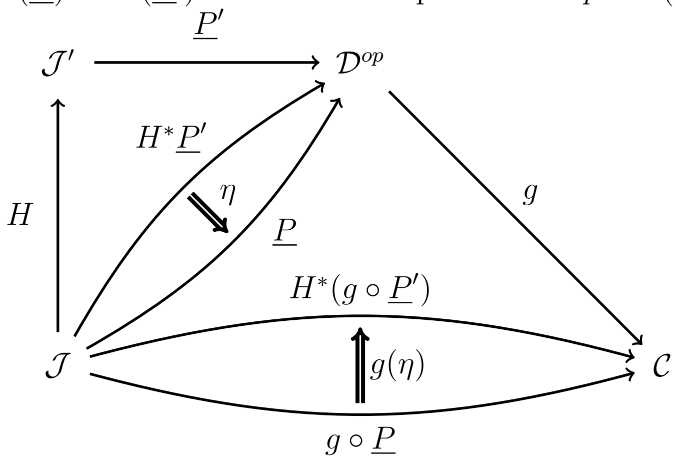
It is now time to consider the action of G on morphisms on $Presh(\mathcal{D})$. Let

$$\langle H, \eta \rangle : (\underline{P}' : \mathcal{J}' \to \mathcal{D}^{op}) \to (\underline{P} : \mathcal{J} \to \mathcal{D}^{op})$$

be an arrow in $Presh(\mathcal{D})$. Then, as G is contravariant, $G(\langle H, \eta \rangle)$ is an arrow in $Copresh(\mathcal{C})$ from $G(\underline{P}) = g \circ \underline{P}$ to $G(\underline{P}') = g \circ \underline{P}'$. Specifically, $G(\langle H, \eta \rangle) = \langle H, g(\eta) \rangle$, where $g(\eta) : g \circ \underline{P} \Rightarrow H^*(g \circ \underline{P}')$ is a natural transformation with components

$$g(\eta)_J = g(\eta_J) : (g \circ \underline{P})_J \to (g \circ H^* \underline{P}')_J.$$

Recall from Fact 2.2.4 that $g \circ H^* \underline{P}' = H^*(g \circ \underline{P}')$, so the above definition of $g(\eta)_J$ does in fact have the correct codomain. It may be useful to note that because g is a contravariant functor, components $g(\eta)_J$ are arrows in the opposite direction of components η_J. The following diagram is not a commutative diagram, but is intended to give some visual intuition behind the definitions above and the reason why $\langle H, g(\eta) \rangle : G(\underline{P}) \to G(\underline{P}')$ is in fact a morphism in $Copresh(\mathcal{C})$.



In [11], the action of functor $F : Copresh(\mathcal{C}) \to Presh(\mathcal{D})^{op}$ is defined as follows. On an object $\overline{Q} : \mathcal{J} \to \mathcal{C}$ of $Copresh(\mathcal{C})$, F acts as postcomposition by $f : \mathcal{C} \to \mathcal{D}^{op}$. That is,

$$F(\overline{Q}) = f \circ \overline{Q} : \mathcal{J} \to \mathcal{C} \to \mathcal{D}^{op}.$$

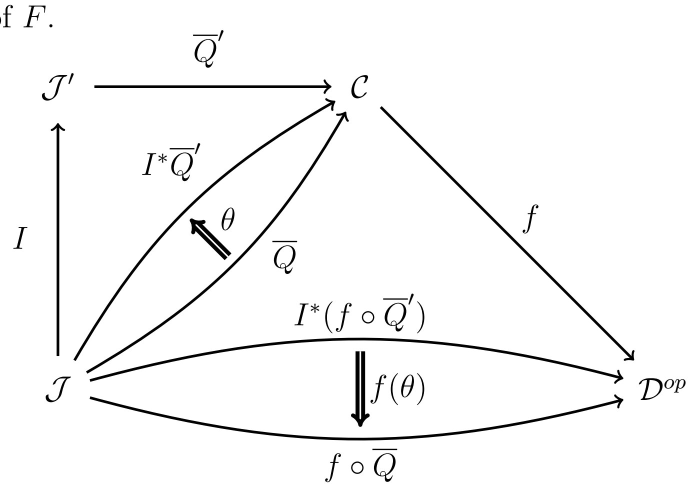
On morphisms $\langle I, \theta \rangle : (\overline{Q} : \mathcal{J} \to \mathcal{C}) \to (\overline{Q}' : \mathcal{J}' \to \mathcal{C})$ in $Copresh(\mathcal{C})$, contravariant functor F acts as follows.

$$F(\langle I, \theta \rangle) = \langle I, f(\theta) \rangle,$$

where $f(\theta) : I^*(F(\overline{Q}')) \Rightarrow F(\overline{Q})$ is a natural transformation with components, for each $J \in \mathcal{J}$, given by

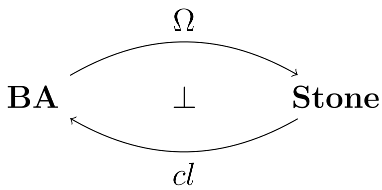
$$f(\theta)_J = f(\theta_J) : (f \circ I^* \overline{Q}')_J \to (f \circ \overline{Q})_J$$

By Fact 2.2.4 and the definition of F on objects, these components $f(\theta)_J$ have the appropriate domain and codomain. It may be useful to note that as functor $f : \mathcal{C} \to \mathcal{D}^{op}$ is contravariant, natural transformations $f(\theta)$ and θ are in opposite directions. The following is again not a commutative diagram, but captures the intuition behind this definition of F.



5.5 Dual equivalences and Stone duality

The above construction of F and G from f and g holds for any dual equivalence of categories. In particular, recall from Section 3.4 that there is a dual equivalence between the category **BA** of Boolean algebras and the category **Stone** of Stone spaces:



This duality is witnessed by natural isomorphisms:

$$St : Id_{Stone} \Rightarrow \Omega \circ cl$$
$$Bo : Id_{BA} \Rightarrow cl \circ \Omega$$

The components of these natural isomorphisms are given in Section 3.4. By Lemma 5.4.2, there is then a duality

$$\mathbf{Copresh(BA)} \underset{CL}{\overset{\Sigma}{\rightleftarrows}} \mathbf{Presh(Stone)} \quad (\perp)$$

The definitions of functors F and G that appear in Lemma 5.4.2 will now be used to describe the actions of CL and Σ on Bohrifications in **Copresh(BA)** and spectral presheaves in **Presh(Stone)**, respectively. The Bohrification of an orthomodular lattice is the tautological inclusion copresheaf, defined as follows in analogy to [18], where the term Bohrification was first used.

Definition 5.5.1. For an orthomodular lattice L, the *Bohrification* $\overline{\mathcal{L}}$ of L is the copresheaf from $\mathcal{B}(L)$ to **BA** given by:

$$\begin{aligned} \text{On objects: } & \overline{\mathcal{L}}_B = B \\ \text{On morphisms: } & \overline{\mathcal{L}}(i_{B',B}) = inc_{B',B}, \text{ the inclusion homomorphism} \end{aligned}$$

Recall that $i_{B',B}$ denotes the arrow in poset $\mathcal{B}(L)$ from B' to B which signifies that $B' \subseteq B$, while $inc_{B',B}$ denotes the Boolean algebra homomorphism $B' \hookrightarrow B$ that maps each element in B' to the same element of B. We now proceed to describe functors CL and Σ of the above dual equivalence of **Copresh(BA)** and **Presh(Stone)**.

Functor Σ: First consider the action of functor Σ on the Bohrification $\overline{\mathcal{L}}$ of orthomodular lattice L, which is an object in **Copresh(BA)**. Σ acts by postcomposition with Ω, that is,

$$\Sigma(\overline{\mathcal{L}}) = \Omega \circ \overline{\mathcal{L}} : \mathcal{B}(L) \to \mathbf{BA} \to \mathbf{Stone}$$

Specifically, on objects B of $\mathcal{B}(L)$, the resulting functor acts as follows:

$$\Sigma(\overline{\mathcal{L}})_B = (\Omega \circ \overline{\mathcal{L}})_B = \Omega(\overline{\mathcal{L}}_B) = \Omega_B.$$

On arrows $i_{B',B}$ in $\mathcal{B}(L)$, this functor has the following action:

$$\Sigma(\overline{\mathcal{L}})(i_{B',B}) = (\Omega \circ \overline{\mathcal{L}})(i_{B',B}) = \Omega(inc_{B',B}) = r_{B,B'},$$

where r denotes the restriction map, that is, precomposition with the inclusion map. Note that as $\Omega \circ \overline{\mathcal{L}}$ is a presheaf from $\mathcal{B}(L)$ to **Stone** with the exact same action on both objects and arrows of $\mathcal{B}(L)$ as $\underline{\Omega(L)}$, then in fact $\Omega \circ \overline{\mathcal{L}} = \underline{\Omega(L)}$. That is,

$$\Sigma(\overline{\mathcal{L}}) = \underline{\Omega(L)}. \tag{5.1}$$

Now consider the action of functor Σ on morphisms between Bohrifications, that is, on arrows $\langle I, \theta\rangle : \overline{\mathcal{L}} \to \overline{\mathcal{M}}$. From above,

$$\Sigma(\langle I, \theta\rangle) = \langle I, \Omega(\theta)\rangle,$$

where $\Omega(\theta)$ is the natural transformation with components $\Omega(\theta)_B = \Omega(\theta_B)$ for all $B \in \mathcal{B}(L)$.

Functor CL: We now describe the action of functor CL on a spectral presheaf $\underline{\Omega(L)} \in$ **Presh**(**Stone**), for some orthomodular lattice L. This action is the composition $cl \circ \underline{\Omega(L)}$, which is now a functor with domain $\mathcal{B}(L)$ in **Copresh(BA)**. The action of this functor on objects $B \in \mathcal{B}(L)$ is given by

$$(cl \circ \underline{\Omega(L)})_B = cl(\underline{\Omega(L)}_B) = cl(\Omega_B),$$

where $cl(\Omega_B)$ is the Boolean algebra of clopen subsets of Ω_B, the Stone space of B. On inclusion arrows $i_{B,B'} : B' \to B$ in $\mathcal{B}(L)$, the action of this functor is given by

$$(cl \circ \underline{\Omega(L)})(i_{B',B}) = cl(r_{B,B'}) : cl(\Omega_{B'}) \to cl(\Omega_B).$$

Recall that the action of functor cl on morphisms is given by the inverse image map, denoted by exponent (-1). For any clopen subset S of $\Omega_{B'}$,

$$cl(r_{B,B'})(S) = r_{B,B'}^{(-1)}(S) = \{\lambda \in \Omega_B : \lambda|_{B'} \in S\},$$

which is a clopen subset of Ω_B.

Now, consider how map CL acts on spectral presheaf morphisms $\langle \tilde{\varphi}, \zeta_\varphi\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in **Presh(Stone)**. From the definition of the action of functor G on arrows that is given above,

$$CL(\langle \tilde{\varphi}, \zeta_\varphi\rangle) = \langle \tilde{\varphi}, cl(\zeta_\varphi)\rangle$$

where $cl(\zeta_\varphi)$ is a natural transformation from functor $cl \circ \underline{\Omega(L)}$ to functor $cl \circ \tilde{\varphi}^*(\underline{\Omega(M)})$. Map $cl(\zeta_\varphi)$ has components for each $B \in \mathcal{B}(L)$ that map from $cl(\underline{\Omega(L)}_B)$, which is equal to $cl(\Omega_B)$, to $cl((\tilde{\varphi}^*\underline{\Omega(M)})_B)$, which is equal to $cl(\Omega_{\tilde{\varphi}(B)})$, given by:

$$cl(\zeta_\varphi)_B = cl(\zeta_{\varphi,B}) = \zeta_{\varphi,B}^{(-1)} : cl(\Omega_B) \to cl(\Omega_{\tilde{\varphi}(B)}).$$

Again, here the exponent denotes the inverse image function, rather than an inverse function. Specifically, the action of $cl(\zeta_\varphi)_B$ on a clopen subset S of $\underline{\Omega(L)}_B$ is given by

$$cl(\zeta_\varphi)_B(S) = \zeta_{\varphi,B}^{(-1)}(S) = \{\lambda \in \Omega_{\tilde{\varphi}(B)} : \zeta_{\varphi,B}(\lambda) \in S\} = \{\lambda \in \Omega_{\tilde{\varphi}(B)} : \lambda \circ \varphi|_B \in S\}.$$

5.6 Presheaf and copresheaf isomorphisms

Now that the action of functors Σ and CL has been defined, this duality can be used to explore the relationship between spectral presheaves in **Presh(Stone)** and Bohrifications in **Copresh(BA)**. We first proceed to show that $\overline{\mathcal{L}}$ and $CL(\underline{\Omega(L)}) = cl \circ \underline{\Omega(L)}$ are isomorphic in the functor category $\mathbf{BA}^{\mathcal{B}(L)}$. That is, there is a natural isomorphism between them, which will be defined below. For each $B \in \mathcal{B}(L)$, note that the component of this natural transformation must be an isomorphism from $\overline{\mathcal{L}}_B$ to $(cl \circ \underline{\Omega(L)})_B$. Recall that

$$\overline{\mathcal{L}}_B = B \text{ and } (cl \circ \underline{\Omega(L)})_B = cl(\underline{\Omega(L)}_B) = cl(\Omega_B).$$

The duality between **BA** and **Stone** from Section 3.4 yields a natural isomorphism with components $Bo_B : B \to cl(\Omega_B)$. Using these maps as components gives a map $\{Bo_B\}_{B \in \mathcal{B}(L)} : \overline{\mathcal{L}} \Rightarrow cl \circ \underline{\Omega(L)}$, which we now show is a natural isomorphism as desired.

Lemma 5.6.1. *The map* $\{Bo_B\}_{B \in \mathcal{B}(L)} : \overline{\mathcal{L}} \Rightarrow cl \circ \underline{\Omega(L)}$ *is a natural isomorphism. That is, these two functors are isomorphic in the functor category* $\mathbf{BA}^{\mathcal{B}(L)}$.

Proof. First it is necessary to show that this map is a natural transformation, that is, that the following diagram commutes for every $B', B \in \mathcal{B}(L)$ such that $B' \subseteq B$ and thus $i_{B',B}$ exists:

$$\begin{array}{ccc}
\overline{\mathcal{L}}_{B'} & \xrightarrow{\overline{\mathcal{L}}(i_{B',B})} & \overline{\mathcal{L}}_B \\
{\scriptstyle Bo_{B'}}\big\downarrow & & \big\downarrow{\scriptstyle Bo_B} \\
(cl \circ \underline{\Omega(L)})_{B'} & \xrightarrow{(cl \circ \underline{\Omega(L)})(i_{B',B})} & (cl \circ \underline{\Omega(L)})_B
\end{array}$$

Recall that

$$(cl \circ \underline{\Omega(L)})_B = cl(\underline{\Omega(L)}_B) = cl(\Omega_B) = (cl \circ \Omega)_B.$$

Additionally, note that

$$(cl \circ \underline{\Omega(L)})(i_{B',B}) = cl(\underline{\Omega(L)}(i_{B',B})) = cl(r_{B,B'}) = cl(\Omega(inc_{B',B})) = (cl \circ \Omega)(inc_{B',B}).$$

Thus, also applying the definition of $\overline{\mathcal{L}}$, the above diagram can be rewritten as

$$
\begin{array}{ccc}
B' & \xrightarrow{\quad inc_{B',B} \quad} & B \\
\Big\downarrow{\scriptstyle Bo_{B'}} & & \Big\downarrow{\scriptstyle Bo_B} \\
(cl \circ \Omega)_{B'} & \xrightarrow{\quad (cl \circ \Omega)(inc_{B',B}) \quad} & (cl \circ \Omega)_B
\end{array}
$$

The above diagram commutes because $inc_{B',B} : B' \to B$ is a morphism in category **BA** and because $Bo : Id_{\mathbf{BA}} \to cl \circ \Omega$ is a natural transformation. Thus, the collection $\{Bo_B\}_{B \in \mathcal{B}(L)} : \overline{\mathcal{L}} \Rightarrow cl \circ \underline{\Omega(L)}$ is a valid natural transformation. It follows that as each arrow Bo_B is an isomorphism then it is in fact a natural isomorphism. □

Natural isomorphism $\{Bo_B\}_{B \in \mathcal{B}(L)}$ will now simply be written in a slight abuse of notation as Bo, and we will remember this natural isomorphism only has components for all $B \in \mathcal{B}(L)$. This will greatly simplify subsequent proofs.

While the above lemma presents an interesting result, it will be more useful to know that the functors $\overline{\mathcal{L}}$ and $cl \circ \underline{\Omega(L)}$ are isomorphic in category **Copresh(BA)**, as this would imply some results about isomorphisms in **Presh(Stone)**, our ultimate goal.

Lemma 5.6.2. *The morphism* $\langle Id_{\mathcal{B}(L)}, Bo \rangle : \overline{\mathcal{L}} \to cl \circ \underline{\Omega(L)}$ *is an isomorphism in* **Copresh(BA)**.

Proof. Natural isomorphism $Bo = \{Bo_B\}_{B \in \mathcal{B}(L)}$ has an inverse natural isomorphism $\{Bo_B^{-1}\}_{B \in \mathcal{B}(L)} : cl \circ \Omega \Rightarrow \overline{\mathcal{L}}$, which will simply be denoted Bo^{-1}. We now use Fact 2.2.5 to show that morphism $\langle Id_{\mathcal{B}(L)}, Bo^{-1} \rangle : cl \circ \underline{\Omega(L)} \to \overline{\mathcal{L}}$ is an inverse to morphism $\langle Id_{\mathcal{B}(L)}, Bo \rangle$ in **Copresh(BA)**:

$$
\begin{aligned}
\langle Id_{\mathcal{B}(L)}, Bo \rangle \circ \langle Id_{\mathcal{B}(L)}, Bo^{-1} \rangle &= \langle Id_{\mathcal{B}(L)} \circ Id_{\mathcal{B}(L)}, (Id^*_{\mathcal{B}(L)} Bo) \circ Bo^{-1} \rangle \\
&= \langle Id_{\mathcal{B}(L)}, Bo \circ Bo^{-1} \rangle \\
&= \langle Id_{\mathcal{B}(L)}, Id_{cl \circ \underline{\Omega(L)}} \rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle Id_{\mathcal{B}(L)}, Bo^{-1} \rangle \circ \langle Id_{\mathcal{B}(L)}, Bo \rangle &= \langle Id_{\mathcal{B}(L)} \circ Id_{\mathcal{B}(L)}, (Id^*_{\mathcal{B}(L)} Bo^{-1}) \circ Bo \rangle \\
&= \langle Id_{\mathcal{B}(L)}, Bo^{-1} \circ Bo \rangle \\
&= \langle Id_{\mathcal{B}(L)}, Id_{\overline{\mathcal{L}}} \rangle
\end{aligned}
$$

Thus, $\langle Id_{\mathcal{B}(L)}, Bo \rangle$ is an isomorphism in **Copresh(BA)**, meaning $\overline{\mathcal{L}}$ and $cl \circ \underline{\Omega(L)}$ are isomorphic in this category of copresheaves. □

The main result of this section is the following theorem, which results from the previous lemma using the dual equivalence between **Copresh(BA)** and **Presh(Stone)**.

Theorem 5.6.3. *Let L and M be orthomodular lattices, $\underline{\Omega(L)}$ and $\underline{\Omega(M)}$ their spectral presheaves, and $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ their Bohrifications. Then there is an isomorphism $\underline{\Omega(M)} \to \underline{\Omega(L)}$ in* **Presh(Stone)** *if and only if there is an isomorphism $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ in* **Copresh(BA)***, and these isomorphisms can be explicitly constructed from each other.*

Proof. Suppose there is an isomorphism $\langle H, \eta \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in **Presh(Stone)**. Then, as functors preserve isomorphisms, there is an isomorphism

$$CL(\langle H, \eta \rangle) : CL(\underline{\Omega(L)}) \to CL(\underline{\Omega(M)})$$

in **Copresh(BA)**. Specifically, recall that $CL(\underline{\Omega(L)}) = cl \circ \underline{\Omega(L)}$, similarly for M, and that

$$CL(\langle H, \eta \rangle) = \langle H, cl(\eta) \rangle,$$

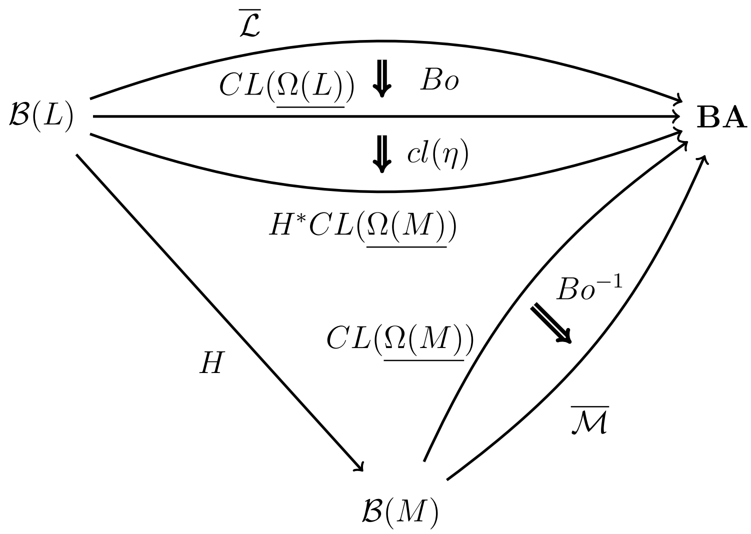
where $cl(\eta)$ is the natural transformation with components $cl(\eta)_J = cl(\eta_J) = \eta_J^{(-1)}$, where the exponent (-1) denotes the inverse image function. By the previous lemma, there are isomorphisms in **Copresh(BA)**

$$\langle Id_{\mathcal{B}(L)}, Bo \rangle : \overline{\mathcal{L}} \to cl \circ \underline{\Omega(L)}$$
$$\langle Id_{\mathcal{B}(M)}, Bo^{-1} \rangle : cl \circ \underline{\Omega(M)} \to \overline{\mathcal{M}}$$

Composing these two isomorphisms on either side of $\langle H, cl(\eta) \rangle$ gives an isomorphism from $\overline{\mathcal{L}}$ to $\overline{\mathcal{M}}$, as desired. Specifically, this composition evaluates as follows:

$$\begin{aligned}
&\langle Id_{\mathcal{B}(M)}, Bo^{-1} \rangle \circ \langle H, cl(\eta) \rangle \circ \langle Id_{\mathcal{B}(L)}, Bo \rangle \\
&\qquad = \langle Id_{\mathcal{B}(M)}, Bo^{-1} \rangle \circ \langle H \circ Id_{\mathcal{B}(L)}, (Id^*_{\mathcal{B}(L)} cl(\eta)) \circ Bo \rangle \\
&\qquad = \langle Id_{\mathcal{B}(M)} \circ H \circ Id_{\mathcal{B}(L)}, (H^* Bo^{-1}) \circ (Id^*_{\mathcal{B}(L)} cl(\eta)) \circ Bo \rangle \\
&\qquad = \langle H, (H^* Bo^{-1}) \circ cl(\eta) \circ Bo \rangle
\end{aligned}$$

By construction, each component of natural transformation $(H^* Bo^{-1}) \circ cl(\eta) \circ Bo$ is an isomorphism in category **BA**, and thus is a Boolean algebra isomorphism. To be even more explicit about this composition, the actions of natural transformations Bo and Bo^{-1} are given in Section 3.4. Some visual intuition for this composition is provided below:



Thus, whenever there is an isomorphism $\langle H, \eta \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in **Presh(Stone)**, then $\langle H, (H^* Bo^{-1}) \circ cl(\eta) \circ Bo \rangle : \overline{\mathcal{L}} \to \overline{\mathcal{M}}$ is an isomorphism in **Copresh(BA)**, completing the first half of this proof.

Now, suppose that there is an isomorphism $\langle I, \theta \rangle : \overline{\mathcal{L}} \to \overline{\mathcal{M}}$ in **Copresh(BA)**. Recall there is a functor Σ : **Copresh**(**BA**) $\to$ **Presh**(**Stone**) that is dual to CL. Then, as functors preserve isomorphisms, there is an isomorphism in **Presh(Stone)** from $\Sigma(\overline{\mathcal{M}})$ to $\Sigma(\overline{\mathcal{L}})$, given by

$$\Sigma(\langle I, \theta \rangle) = \langle I, \Omega(\theta) \rangle,$$

where $\Omega(\theta)$ is the natural transformation with components $\Omega(\theta)_B = \Omega(\theta_B)$ for all B in $\mathcal{B}(L)$. Recalling from (5.1) that

$$\Sigma(\overline{\mathcal{M}}) = \Omega \circ \overline{\mathcal{M}} = \underline{\Omega(M)} \text{ and } \Sigma(\overline{\mathcal{L}}) = \Omega \circ \overline{\mathcal{L}} = \underline{\Omega(L)},$$

it follows that $\langle I, \Omega(\theta) \rangle$ is an isomorphism in **Presh(Stone)** from $\underline{\Omega(M)}$ to $\underline{\Omega(L)}$, meaning the two spectral presheaves are isomorphic, as desired. □

The above theorem represents a major step towards the goal of this section, which is showing that two orthomodular lattices are isomorphic if and only if their spectral presheaves are.

5.7 Spectral presheaf isomorphisms

The main result of this subsection is that two orthomodular lattices are isomorphic if and only if their spectral presheaves are. Equipped with Theorem 5.6.3, we are now able to prove this main result, which is separated into the following two theorems.

Theorem 5.7.1. *Let L and M be orthomodular lattices. If $\varphi : L \to M$ is an isomorphism in* **OML***, then there is an isomorphism $\langle \tilde{\varphi}, \zeta_\varphi \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in* **Presh(Stone)***, where natural transformation ζ_φ has components $\zeta_{\varphi,B} = \Omega(\varphi|_B)$ for all B in $\mathcal{B}(L)$.*

Proof. Suppose $\varphi : L \to M$ is an isomorphism of orthomodular lattices, with inverse $\psi = \varphi^{-1}$. Then, by Proposition 4.1.7, $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$ is an order isomorphism of posets, with inverse $\tilde{\psi}$. Additionally, for each $B \in \mathcal{B}(L)$, $\varphi|_B : B \to \varphi(B)$ is an isomorphism of Boolean algebras by Proposition 4.1.3, with inverse $\psi|_{\tilde{\varphi}(B)}$.

By Stone duality, applying functor Ω to Boolean algebra isomorphism $\varphi|_B : B \to \tilde{\varphi}(B)$ yields a continuous isomorphism $\Omega(\varphi|_B) : \Omega_{\tilde{\varphi}(B)} \to \Omega_B$ in category **Stone**, where objects are Stone spaces and morphisms are continuous maps between them. Note that as $\tilde{\varphi}(B) \in \mathcal{B}(M)$, then

$$\Omega_{\tilde{\varphi}(B)} = \underline{\Omega(M)}_{\tilde{\varphi}(B)} = (\underline{\Omega(M)} \circ \tilde{\varphi})_B = (\tilde{\varphi}^* \underline{\Omega(M)})_B.$$

Additionally, as $B \in \mathcal{B}(L)$, then $\Omega_B = \underline{\Omega(L)}_B$. Thus, $\Omega(\varphi|_B)$ is in fact a Stone space morphism from $(\tilde{\varphi}^* \underline{\Omega(M)})_B$ to $\underline{\Omega(L)}_B$. Let isomorphism $\Omega(\varphi|_B)$ be denoted

$$\Omega(\varphi|_B) := \zeta_{\varphi,B} : (\tilde{\varphi}^* \underline{\Omega(M)})_B \to \underline{\Omega(L)}_B.$$

Note that this coincides exactly with the definition of $\zeta_{\varphi,B}$ given in Step 2 of Section 5.2, where the action of $\zeta_{\varphi,B}$ on a homomorphism $\lambda : \tilde{\varphi}(B) \to \{0,1\}$ is given by precomposition with $\varphi|_B$.

We first check that the components $(\zeta_{\varphi,B})_{B \in \mathcal{B}(L)}$ form a natural isomorphism from $\tilde{\varphi}^* \underline{\Omega(M)}$ to $\underline{\Omega(L)}$. For every $B' \subseteq B$ in $\mathcal{B}(L)$, that is, for every arrow $i_{B',B}$ in poset $\mathcal{B}(L)$, the following diagram must commute:

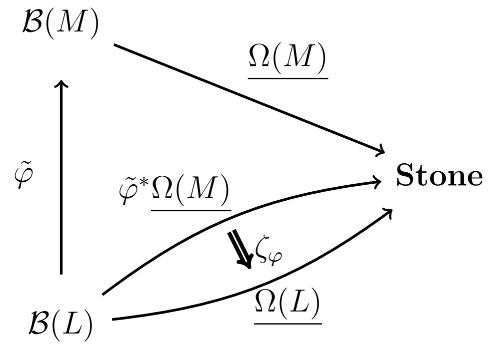
$$\begin{array}{ccc}
\underline{\Omega(M)}_{\tilde{\varphi}(B')} & \xleftarrow[= r_{\tilde{\varphi}(B),\tilde{\varphi}(B')}]{\underline{\Omega(M)}(i_{\tilde{\varphi}(B'),\tilde{\varphi}(B)})} & \underline{\Omega(M)}_{\tilde{\varphi}(B)} \\
\Big\downarrow \zeta_{\varphi,B'} & & \Big\downarrow \zeta_{\varphi,B} \\
\underline{\Omega(L)}_{B'} & \xleftarrow[= r_{B',B}]{\underline{\Omega(L)}(i_{B',B})} & \underline{\Omega(L)}_B
\end{array}$$

This is precisely the same diagram that was shown to commute in Step 2 of Section 5.2. Thus, the $\zeta_{\varphi,B}$ are the components of a natural isomorphism ζ_φ from presheaf $\tilde{\varphi}^* \underline{\Omega(M)}$ to presheaf $\underline{\Omega(L)}$.

Since $\tilde{\varphi} : \mathcal{B}(L) \to \mathcal{B}(M)$ is an isomorphism and $\zeta_\varphi : \tilde{\varphi}^*\underline{\Omega(M)} \to \underline{\Omega(L)}$ is a natural isomorphism, then the composite

$$\langle \tilde{\varphi}, \zeta_\varphi \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$$

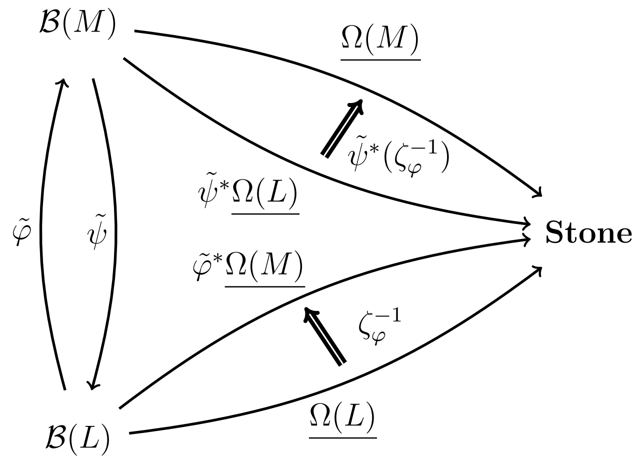
is an arrow in **Presh(Stone)**, depicted here:



It only remains to show that this arrow has an inverse, that is, that it is an isomorphism in **Presh(Stone)**. Recall that $\tilde{\psi} : \mathcal{B}(M) \to \mathcal{B}(L)$ is the inverse of $\tilde{\varphi}$, and consider the arrow

$$\langle \tilde{\psi}, \tilde{\psi}^*(\zeta_\varphi^{-1}) \rangle : \underline{\Omega(L)} \to \underline{\Omega(M)}.$$

This arrow is depicted in the following diagram:



That both compositions of arrow $\langle \tilde{\varphi}, \zeta_\varphi \rangle$ with its inverse give the identity morphism is now checked algebraically, using the functoriality of $\tilde{\varphi}^*$ and $\tilde{\psi}^*$, Fact 2.2.3, and Fact 2.2.5:

$$\begin{aligned}\langle \tilde{\psi}, \tilde{\psi}^*(\zeta_\varphi^{-1})\rangle \circ \langle \tilde{\varphi}, \zeta_\varphi\rangle &= \langle \tilde{\varphi} \circ \tilde{\psi}, \tilde{\psi}^*(\zeta_\varphi^{-1}) \circ \tilde{\psi}^*\zeta_\varphi\rangle \\ &= \langle Id_{\mathcal{B}(M)}, \tilde{\psi}^*(Id_{\tilde{\varphi}^*\underline{\Omega(M)}})\rangle \\ &= \langle Id_{\mathcal{B}(M)}, Id_{\tilde{\psi}^*\tilde{\varphi}^*\underline{\Omega(M)}}\rangle \\ &= \langle Id_{\mathcal{B}(M)}, Id_{\underline{\Omega(M)}}\rangle\end{aligned}$$

$$\begin{aligned}\langle \tilde{\varphi}, \zeta_\varphi\rangle \circ \langle \tilde{\psi}, \tilde{\psi}^*(\zeta_\varphi^{-1})\rangle &= \langle \tilde{\psi} \circ \tilde{\varphi}, \zeta_\varphi \circ \tilde{\varphi}^*(\tilde{\psi}^*(\zeta_\varphi^{-1}))\rangle \\ &= \langle Id_{\mathcal{B}(L)}, \zeta_\varphi \circ (\tilde{\psi} \circ \tilde{\varphi})^*(\zeta_\varphi^{-1}))\rangle \\ &= \langle Id_{\mathcal{B}(L)}, \zeta_\varphi \circ (Id_{\mathcal{B}(L)})^*(\zeta_\varphi^{-1})\rangle \\ &= \langle Id_{\mathcal{B}(L)}, \zeta_\varphi \circ \zeta_\varphi^{-1}\rangle \\ &= \langle Id_{\mathcal{B}(L)}, Id_{\underline{\Omega(L)}}\rangle\end{aligned}$$

Thus, $\langle \tilde{\varphi}, \zeta_\varphi\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ is an isomorphism in **Presh(Stone)**, as desired. □

In order to prove the next result, recall from Section 4.2 the definition of a partial orthomodular lattice. Note that a partial orthomodular lattice captures all aspects of lattice structure within each boolean subalgebra of L, as well as capturing inclusion relations between Boolean subalgebras. This is precisely the same data about L that can be recovered from the Bohrification of L, which is an object in a topos. This makes L_{part} a topos-external description of the Bohrification $\overline{\mathcal{L}}$ of orthomodular lattice L, useful for switching our focus from presheaves and copresheaves to back to lattices.

Theorem 5.7.2. *Let L and M be orthomodular lattices. If there is an isomorphism $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in* **Presh(Stone)***, then there is an isomorphism from L to M in* **OML** *that can be explicitly constructed from $\langle H, \eta\rangle$.*

Proof. Let $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ be an isomorphism between spectral presheaves of orthomodular lattices. By Theorem 5.6.3, there exists a isomorphism from $\overline{\mathcal{L}}$ to $\overline{\mathcal{M}}$ in **Copresh(BA)**, specifically,

$$\langle H, (H^*Bo^{-1}) \circ cl(\eta) \circ Bo\rangle : \overline{\mathcal{L}} \to \overline{\mathcal{M}}.$$

For simplicity, define

$$\rho := (H^*Bo^{-1}) \circ cl(\eta) \circ Bo : \overline{\mathcal{L}} \Rightarrow H^*\overline{\mathcal{M}}.$$

This natural transformation ρ has components for each $B \in \mathcal{B}(L)$ that map from $\overline{\mathcal{L}}_B = B$ to $(H^*\overline{\mathcal{M}})_B = \overline{\mathcal{M}}_{H(B)} = H(B)$, where $H(B)$ is an element of $\mathcal{B}(M)$, that is, a Boolean subalgebra of M:

$$\rho_B : B \to H(B).$$

By the construction of ρ in the proof of Theorem 5.6.3, each component ρ_B is a Boolean algebra isomorphism.

Let $a \in L$, and suppose that $a \in B'$ and $a \in B$, where $B', B \in \mathcal{B}(L)$ with $B' \subseteq B$. That is, $i_{B',B}$ is an arrow in $\mathcal{B}(L)$. Recall that $\overline{\mathcal{L}}(i_{B',B}) = inc_{B',B}$, the inclusion Boolean algebra homomorphism from B' to B. Additionally, $H(i_{B',B})$ is an arrow in $\mathcal{B}(M)$ from $H(B')$ to $H(B)$; as poset categories have at most one arrow with a given domain and codomain, it must be that $H(i_{B',B}) = i_{H(B'),H(B)}$. Then,

$$(\overline{\mathcal{M}} \circ H)(i_{B',B}) = inc_{H(B'),H(B)}.$$

The naturality of ρ then means that the following diagram commutes:

$$\begin{array}{ccc} B' & \xrightarrow{\quad\rho_{B'}\quad} & H(B') \\ \Big\downarrow{\scriptstyle inc_{B',B}} & & \Big\downarrow{\scriptstyle inc_{H(B'),H(B)}} \\ B & \xrightarrow{\quad\rho_{B}\quad} & H(B) \end{array}$$

That is,

$$\rho_B(a) = (\rho_B \circ inc_{B',B})(a) = (inc_{H(B'),H(B)} \circ \rho_{B'})(a) = \rho_{B'}(a).$$

From this, it follows that if element a is in any two Boolean subalgebras B_1, B_2 of L (not necessarily related by containment), then

$$\rho_{B_1}(a) = \rho_{B_1 \cap B_2}(a) = \rho_{B_2}(a).$$

Note that as all Boolean subalgebras of L contain both 0 and 1, the Boolean subalgebra $B_1 \cap B_2$ is nonempty. This yields a well-defined map as follows:

$$\begin{aligned} \varphi : L_{part} &\to M_{part} \\ a &\mapsto \rho_B(a), \text{ where } B \in \mathcal{B}(L) \text{ is any Boolean subalgebra containing } a \end{aligned}$$

This is well defined because $\rho_B(a)$ is the same regardless of which Boolean subalgebra containing a is chosen, and because every element of L is in at least one Boolean subalgebra. This map φ is a partial orthomodular lattice homomorphism because on each Boolean subalgebra of L, ρ_B is a Boolean algebra homomorphism and thus preserves all meets, joins, and orthocomplements within that sublattice. As each Boolean subalgebra is closed under orthocomplementation, this means that φ preserves orthocomplements on all of L as well. It remains to check that φ is an isomorphism of partial orthomodular lattices.

As ρ is a natural isomorphism, each component ρ_B is an isomorphism of Boolean algebras and has an inverse $\rho_B^{-1} : H(B) \to B$. Just as above, for any $m \in M$, it can be shown that $\rho_{B_1}^{-1}(m) = \rho_{B_2}^{-1}(m)$ for any $B_1, B_2 \in \mathcal{B}(M)$ that contain m. Thus, it is possible to define a partial orthomodular lattice homomorphism

$$\begin{aligned} \psi : M_{part} &\to L_{part} \\ m &\mapsto \rho_B^{-1}(m), \text{ where } B \in \mathcal{B}(M) \text{ is any Boolean subalgebra containing } m \end{aligned}$$

One can now verify that ψ is an inverse to φ. Let $a \in L$, and let $B \in \mathcal{B}(L)$ contain a. Then,

$$(\psi \circ \varphi)(a) = (\rho_B^{-1} \circ \rho_B)(a) = Id_B(a) = a$$

Similarly, for any $m \in M$ contained in some Boolean algebra $B \in \mathcal{B}(M)$,

$$(\varphi \circ \psi)(m) = (\rho_{H^{-1}(B)} \circ \rho_{H^{-1}(B)}^{-1})(m) = Id_B(m) = m$$

Clearly ψ is an inverse to φ, meaning that φ is a partial orthomodular lattice isomorphism. By Lemma 4.2.4, φ preserves all meets and joins, not just those within Boolean subalgebras, and as it also already preserves orthocomplementation this means that $\varphi : L \to M$ is an isomorphism of orthomodular lattices. □

Specifically, for any element $a \in L$, the action of φ on a as constructed in the proof above is given as follows. Let $B \in \mathcal{B}(L)$ be any Boolean subalgebra containing a. Then,

$$\begin{aligned} \varphi(a) = \rho_B(a) = ((H^*Bo^{-1}) \circ cl(\eta) \circ Bo)_B(a) &= ((H^*Bo_B^{-1}) \circ cl(\eta)_B \circ Bo_B)(a) \\ &= (Bo_{H(B)}^{-1} \circ cl(\eta_B) \circ Bo_B)(a). \end{aligned}$$

Recall that $Bo_B : B \to cl(\Omega(B))$ is the component at B of the natural transformation that witnesses Stone duality; $Bo_{H(B)}^{-1}$ is the component at $H(B) \in \mathcal{B}(M)$ of the inverse of this same natural transformation; and cl : **Stone** $\to$ **BA** is one functor the dual equivalence of Stone duality. Specific actions of these maps are given in Section 3.4. In practice, to calculate $\varphi(a)$ it is simplest to choose $B = B_a = \{0, a, a', 1\}$, the Boolean algebra with four elements, as we will do in the later proofs of Theorems 5.7.4 and 5.7.5.

Theorem 5.7.3. *Two orthomodular lattices L and M are isomorphic in* **OML** *if and only if their spectral preserves $\underline{\Omega(L)}$ and $\underline{\Omega(M)}$ are isomorphic in* **Presh(Stone)**.

Proof. Theorems 5.7.1 and 5.7.2. □

The above is the main result of this section; the impact of this theorem is discussed in Section 5.8.

In fact, it is possible to prove an even stronger result about isomorphisms of orthomodular lattices and their spectral presheaves. For orthomodular lattice isomorphism $\varphi : L \to M$, denote the spectral presheaf isomorphism constructed in the proof of Theorem 5.7.1 by $SP(\varphi) : \underline{\Omega(M)} \to \underline{\Omega(L)}$. For spectral presheaf isomorphism $\langle H, \eta \rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$, denote the orthomodular lattice isomorphism constructed in the proof of Theorem 5.7.2 by $OML(\langle H, \eta \rangle) : L \to M$.

Theorem 5.7.4. *For all orthomodular lattice isomorphisms $\varphi : L \to M$,*

$$OML(SP(\varphi)) = \varphi.$$

Proof. Consider orthomodular lattice isomorphism $\varphi : L \to M$. Then, $\langle \tilde{\varphi}, \zeta_\varphi \rangle$ is an isomorphism in **Presh(Stone)**, where ζ_φ is a natural isomorphism with components given by

$$\begin{aligned} \zeta_{\varphi,B} : \Omega_{\tilde{\varphi}(B)} &\to \Omega_B \\ \lambda &\mapsto \lambda \circ \varphi|_B \end{aligned}$$

Each component $\zeta_{\varphi,B}$ is an isomorphism of Stone spaces.

To construct $OML(\langle \tilde{\varphi}, \zeta_\varphi \rangle)$, consider natural isomorphism in **Copresh(BA)**:

$$\rho = (\tilde{\varphi}^* Bo^{-1}) \circ cl(\zeta_\varphi) \circ Bo : \overline{\mathcal{L}} \Rightarrow \tilde{\varphi}^* \overline{\mathcal{M}}.$$

Each component of this natural isomorphism is a Boolean algebra isomorphism from B to $\tilde{\varphi}(B)$ given by

$$\begin{aligned} \rho_B &= \left((\tilde{\varphi}^* Bo^{-1}) \circ cl(\zeta_\varphi) \circ Bo \right)_B \\ &= (\tilde{\varphi}^* Bo^{-1})_B \circ cl(\zeta_\varphi)_B \circ Bo_B \\ &= Bo^{-1}_{\tilde{\varphi}(B)} \circ \zeta^{(-1)}_{\varphi,B} \circ Bo_B \end{aligned}$$

Now, let $a \in L$. The four-element Boolean algebra B_a with elements $\{0, a, a', 1\}$ is a Boolean subalgebra of L that contains a. $OML(\langle \tilde{\varphi}, \zeta_\varphi \rangle)$ is the homomorphism from L to M whose action on element a is $\rho_{B_a}(a)$, which we will now calculate. First, recall from Section 3.4.1 that the Stone space of B_a has two elements, called λ_a and $\lambda_{a'}$, where $\lambda_a(a) = 1$ and $\lambda_{a'}(a) = 0$. Thus,

$$Bo_{B_a}(a) = \{\sigma \in \Omega_{B_a} \mid \sigma(a) = 1\} = \{\lambda_a\}.$$

Note that as $\varphi|_B$ is a Boolean algebra isomorphism, then $\tilde{\varphi}(B)$ is the four-element Boolean algebra with elements $\{0, \varphi(a), \varphi(a)', 1\}$, which we will denote $B_{\varphi(a)}$. The Stone space of $B_{\varphi(a)}$ has two elements, which we will denote as $\lambda_{\varphi(a)}$ and $\lambda_{\varphi(a)'}$, where $\lambda_{\varphi(a)}(\varphi(a)) = 1$ and $\lambda_{\varphi(a)'}(\varphi(a)) = 0$. Then,

$$\begin{aligned}\zeta^{(-1)}_{\varphi,B}(Bo_B(a)) &= \zeta^{(-1)}_{\varphi,B}(\{\lambda_a\})\\ &= \{\sigma \in \Omega_{\tilde{\varphi}(B_a)} \mid (\sigma \circ \varphi|_{B_a})(a) = 1\}\\ &= \{\lambda_{\varphi(a)}\}\end{aligned}$$

In order to calculate $\rho_{B_a}(a) = Bo^{-1}_{\tilde{\varphi}(B)}(\{\lambda_{\varphi(a)}\}$, recall the definition for the components of Bo^{-1} given in Section 3.4. We must write $\{\lambda_{\varphi(a)}\}$ as a finite union of basic open sets of $\Omega_{\tilde{\varphi}(B)}$. As $\{\lambda_{\varphi(a)}\} = U_{\varphi(a)}$ is itself a basic open set, then $Bo^{-1}_{\tilde{\varphi}(B)}(\{\lambda_{\varphi(a)}\}) = \varphi(a)$. Thus,

$$\rho_{B_a}(a) = \left(Bo^{-1}_{\tilde{\varphi}(B)} \circ \zeta^{(-1)}_{\varphi,B} \circ Bo_B\right)(a) = Bo^{-1}_{\tilde{\varphi}(B)}(\{\lambda_{\varphi(a)}\}) = \varphi(a)$$

Thus, $OML(\langle\tilde{\varphi}, \zeta_\varphi\rangle)$ is the orthomodular lattice homomorphism from L to M mapping a to $\rho_{B_a}(a) = \varphi(a)$, meaning that $\varphi = OML(\langle\tilde{\varphi}, \zeta_\varphi\rangle) = (OML \circ SP)(\varphi)$. □

Theorem 5.7.5. *Let* $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ *be an isomorphism in* **Presh(Stone)** *between the spectral presheaves of two orthomodular lattices* M *and* L*. Then*

$$SP(OML(\langle H, \eta\rangle)) = \langle H, \eta\rangle.$$

Proof. Consider an isomorphism $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ in **Presh(Stone)**. To construct $OML(\langle H, \eta\rangle)$, consider natural isomorphism in **Copresh(BA)**:

$$\rho = (H^* Bo^{-1}) \circ cl(\eta) \circ Bo : \overline{\mathcal{L}} \Rightarrow H^* \overline{\mathcal{M}}.$$

Each component of this natural isomorphism is a Boolean algebra isomorphism from B to $H(B)$ given by

$$\rho_B = Bo^{-1}_{H(B)} \circ \eta^{(-1)}_B \circ Bo_B.$$

Let $a \in L$. Then the four-element Boolean B_a contains a. We first want to calculate

$$OML(\langle H, \eta\rangle)(a) = \rho_{B_a}(a).$$

Note that $H(B_a)$ is also a four-element Boolean algebra because H is an order isomorphism of posets; as there is no $B \in \mathcal{B}(L)$ such that $\{0,1\} \subset B \subset B_a$, then this also holds true for $H(B_a)$ in $\mathcal{B}(M)$, meaning $H(B_a)$ is also a four-element Boolean subalgebra. We will name its elements $\{0, h(a), h(a)', 1\}$. Note that we are not defining some function $h : L \to M$, but rather simply using function notation to indicate that the elements of $H(B_a)$ depend on the chosen element a. This will minimize the use of subscripts in the following argument. We now calculate

$$\begin{aligned}(\eta_{B_a}^{(-1)} \circ Bo_{B_a})(a) &= \eta_{B_a}^{(-1)}(\{\lambda_a\}) \\ &= \{\sigma \in \Omega_{H(B_a)} \mid \eta_{B_a}(\sigma) = \lambda_a\} \\ &= \{\sigma \in \Omega_{H(B_a)} \mid \eta_{B_a}(\sigma)(a) = 1\}\end{aligned}$$

As η_{B_a} is an isomorphism of Stone spaces, it must be that exactly one of the two elements $\lambda_{h(a)}$ and $\lambda_{h(a)'}$ of $\Omega_{H(B_a)}$ satisfies $\eta_{B_a}(\sigma)(a) = 1$, which is equivalent to $\eta_{B_a}(\sigma) = \lambda_a$. If $(\eta_{B_a}^{(-1)} \circ Bo_{B_a})(a) = \{\lambda_{h(a)}\} = U_{h(a)}$, then applying $Bo_{H(B_a)}^{-1}$ yields $h(a)$, while the other case yields $h(a)'$. Thus,

$$\rho_{B_a}(a) = \begin{cases} h(a) & : \quad \eta_{B_a}(\lambda_{h(a)}) = \lambda_a \\ h'(a) & : \quad \eta_{B_a}(\lambda_{h(a)'}) = \lambda_a \end{cases}$$

Thus, $OML(\langle H, \eta\rangle)$ is a homomorphism $\varphi : L \to M$ given by $\varphi(a) = \rho_{B_a}(a)$ as above.

We now want to show that $SP(\varphi) = \langle H, \eta\rangle$. First, consider $\tilde{\varphi}$, and let B be any element of $\mathcal{B}(L)$. We want to show that $\tilde{\varphi}(B) = H(B)$. First, let $a \in L$ and consider the four-element Boolean subalgebra $B_a = \{0, a, a', 1\}$. Recall that $H(B_a)$ has four elements which we call $\{0, h(a), h'(a), 1\}$, and note that either $\varphi(a) = h(a)$ and $\varphi(a') = h(a)'$, or $\varphi(a) = h(a)'$ and $\varphi(a') = h(a)$. In either case,

$$\tilde{\varphi}(B_a) = \{\varphi(x) \mid x \in B_a\} = \{0, h(a), h(a)', 1\} = H(B_a).$$

Now, let B be an arbitrary Boolean subalgebra of L. Let $\varphi(a)$ be any element in $\tilde{\varphi}(B)$, where a is some element of B. Then, $\varphi(a) \in \tilde{\varphi}(B_a) = H(B_a)$. As $B_a \subseteq B$, then $H(B_a) \subseteq H(B)$, meaning $\varphi(a) \in H(B)$ and thus $\tilde{\varphi}(B) \subseteq H(B)$.

Conversely, let $h \in H(B)$. Then $B_h = \{0, h, h', 1\} \subseteq H(B)$, implying that

$$H^{-1}(B_h) \subseteq H^{-1}(H(B)) = B.$$

As $H^{-1}(B_h)$ is a four-element Boolean subalgebra of B because H is an order isomorphism, then

$$\tilde{\varphi}(H^{-1}(B_h)) = H(H^{-1}(B_h)) = B_h$$

because $\tilde{\varphi}$ and H are the same on four-element Boolean subalgebras. Thus $h \in B_h$ is equal to some element $\varphi(a)$ in

$$\tilde{\varphi}(H^{-1}(B_h)) = \{\varphi(a) \mid a \in H^{-1}(B_h) \subseteq B\}.$$

As a is thus also an element of B, then $h \in \tilde{\varphi}(B)$ as desired. So, $\tilde{\varphi}(B) = H(B)$ for all $B \in \mathcal{B}(L)$, meaning $\tilde{\varphi} = H$.

It only remains to show that $\zeta_\varphi = \eta$, and this will follow from the fact that each component of these natural isomorphisms is the same. Fix some $B \in \mathcal{B}(L)$. Recall that $\zeta_{\varphi,B}$ and η_B are both isomorphisms from $\Omega_{\tilde{\varphi}(B)} = \Omega_{H(B)}$ to Ω_B. Fix $\sigma \in \Omega_{\tilde{\varphi}(B)} = \Omega_{H(B)}$ and fix $a \in B$; we want to show that $\zeta_{\varphi,B}(\sigma)(a) = \eta_B(\sigma)(a)$.

As described in the proof of Theorem 5.7.1, component $\zeta_{\varphi,B}$ acts on an element $\sigma \in \Omega_B$ by precomposing by $\varphi|_B$:

$$\begin{aligned} \zeta_{\varphi,B} : \Omega_{\tilde{\varphi}(B)} &\to \Omega_B \\ \sigma &\mapsto \sigma \circ \varphi|_B \end{aligned}$$

Thus,

$$\zeta_{\varphi,B}(\sigma)(a) = \sigma(\varphi(a)). \tag{5.2}$$

As η is a natural transformation, then as B_a is a Boolean algebra contained in B, it follows that the following diagram commutes:

$$\begin{array}{ccc} \Omega_{\tilde{\varphi}(B_a)} & \xleftarrow{r_{\tilde{\varphi}(B),\tilde{\varphi}(B_a)}} & \Omega_{\tilde{\varphi}(B)} \\ \downarrow \eta_{B_a} & & \downarrow \eta_B \\ \Omega_{B_a} & \xleftarrow{r_{B,B_a}} & \Omega_B \end{array}$$

In particular, this implies that

$$\eta_B(\sigma)(a) = \eta_B(\sigma)|_{B_a}(a) = \eta_{B_a}(\sigma|_{\tilde{\varphi}(B_a)})(a). \tag{5.3}$$

Recall the definition of map φ, where the elements of $H(B_a)$ are denoted $\{0, h(a), h(a)', 1\}$ and the two elements of $\Omega_{H(B_a)}$ are denoted $\lambda_{h(a)}$ and $\lambda_{h(a)'}$:

$$\varphi(a) = \begin{cases} h(a) & : \ \eta_{B_a}(\lambda_{h(a)}) = \lambda_a \Leftrightarrow \eta_{B_a}(\lambda_{h(a)})(a) = 1 \\ h(a)' & : \ \eta_{B_a}(\lambda_{h(a)'}) = \lambda_a \Leftrightarrow \eta_{B_a}(\lambda_{h(a)})(a) = 0 \end{cases}$$

Specifically, for any $\sigma|_{\tilde{\varphi}(B)} \in \Omega_{H(B)} = \{\lambda_{h(a)}, \lambda_{h(a)'}\}$, whether $\sigma|_{\tilde{\varphi}(B)} = \lambda_{h(a)}$ or $\sigma|_{\tilde{\varphi}(B)} = \lambda_{h(a)'}$, an exhaustive check shows that

$$\sigma(\varphi(a)) = \sigma|_{\tilde{\varphi}(B_a)}(\varphi(a)) = \eta_{B_a}(\sigma|_{\tilde{\varphi}(B)})(a).$$

Combining this with Equations 5.2 and 5.3,

$$\zeta_{\varphi,B}(\sigma)(a) = \sigma(\varphi(a)) = \eta_{B_a}(\sigma|_{\tilde{\varphi}(B)})(a) = \eta_B(\sigma)(a)$$

Thus, $\zeta_\varphi = \eta$, meaning

$$SP(OML(\langle H, \eta\rangle)) = \langle H, \eta\rangle.$$

□

Theorem 5.7.6. *There are bijections SP and OML between orthomodular lattice isomorphisms $\varphi : L \to M$ and spectral presheaf isomorphisms $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$.*

Proof. Theorems 5.7.4 and 5.7.5. □

5.8 Impact of Theorem 5.7.3

Theorem 5.7.3 is of considerable mathematical interest. While the duality between Stone spaces and Boolean algebras has been well-known in the mathematical community for many years, we are not familiar with any attempts to generalize this duality to orthomodular lattices. The spectral presheaf of an orthomodular lattices provides a new notion of duality between orthomodular lattices and a functor whose image is, rather than a single Stone space, a collection of Stone spaces linked together by continuous restriction maps. Theorem 5.7.3 implies that this duality preserves all structure of an orthomodular lattice, as we would want such a duality result to do. In the case where orthomodular lattice L is in fact a Boolean algebra, this notion of duality does not quite restrict to Stone duality, as the spectral presheaf considers all Boolean subalgebras of L while Stone duality does not. This is necessary to avoid certain no-go theorems about extending classical dualities [31].

Theorem 5.7.3 demonstrates that the spectral presheaf of an orthomodular lattice is a complete invariant, determining the orthomodular lattice up to isomorphism and vice versa. This is stronger than the corresponding result for von Neumann algebras, where a spectral presheaf determines a von Neumann algebra only up to Jordan $*$-isomorphism rather than up to isomorphism [11].

The fact that the spectral presheaf of an orthomodular lattice is a complete invariant means that instead of modeling quantum logic with an orthomodular lattice, one can model quantum logic with the spectral presheaf of an orthomodular lattice without losing any information. This shows that there is a 'state space' picture of quantum logic that is of a similar form to the generalized state space of a quantum system given by the spectral presheaf of a von Neumann algebra. This further validates the topos approach to quantum physics by demonstrating that its main principles apply to other aspects of quantum theory as well.

Reference [17] proves that an isomorphism of context categories yields an isomorphism of orthomodular lattices, though this isomorphism is only unique when the orthomodular lattices have no maximal four-element Boolean subalgebras. We considered not just the context category but rather a functor on the context category; an isomorphism between spectral presheaves $\langle H, \eta \rangle$ consists of not only an isomorphism H between context categories but also a natural isomorphism η. The additional data of η enables the proof of Theorem 5.7.6, that there is a bijection between orthomodular lattice isomorphisms and spectral presheaf isomorphisms. This result is not true of orthomodular lattice isomorphisms and context category isomorphisms as considered in [17]. Additionally, Theorem 5.7.2 provides a way to construct an isomorphism of orthomodular lattices from an isomorphism of their spectral presheaves by only considering four-element Boolean subalgebras; it is precisely when considering maximal four-element Boolean subalgebras that the process employed by [17] fails to construct a unique isomorphism.

Chapter 6

The Spectral Presheaf of a Complete Orthomodular Lattice

The spectral presheaf of an orthomodular lattice is an object in the topos $\mathbf{Set}^{\mathcal{B}(\mathbf{L})^{\mathbf{op}}}$. As an object in a topos, one can talk about its subobjects as a next logical step. Though this should be possible to do for any orthomodular lattice, some of our constructions currently only apply to the case of complete orthomodular lattices. For this reason, we will focus on complete orthomodular lattices from now on, though it is hoped that future research will extend the results of this section to arbitrary orthomodular lattices. Recall that a complete orthomodular lattice is one in which all infinite meets and joins exist.

Of note, the isomorphism result of Theorem 5.7.3 doesn't immediately apply to complete orthomodular lattices. This is because, among other nuances, the isomorphism between orthomodular lattices L and M constructed from an isomorphism from L_{part} to M_{part} in the proof of Theorem 5.7.2 is not necessarily a complete orthomodular lattice homomorphism, that is, it may only preserve finite meets and joins, not arbitrary meets and joins. Related to this, it is not obvious (although true) that the clopen subsets of the Stone space of a complete Boolean algebra form a complete Boolean algebra themselves, which is necessary for any sort of Stone duality result involving complete orthomodular lattices. For these reasons and to provide additional insights into complete orthomodular lattices and their Stone spaces, the results about the spectral presheaf of an orthomodular lattice from Chapter 5 are rephrased below for complete orthomodular lattices. These results are presented largely without proof, as the proofs are nearly identical to those for orthomodular lattices in the previous section. Often the only change needed is inserting the word 'complete' and requiring morphisms to preserve arbitrary meets and joins rather than finite meets and joins. First, we consider some background information about complete orthomodular lattices.

6.1 Complete orthomodular lattices and their Boolean substructure

Definition 6.1.1. A *complete lattice* L is one for which every (possibly infinite) family $(a_i)_{i\in I}$ of elements of L has a well-defined meet and join, that is, a greatest lower bound and a least upper bound.

Definition 6.1.2. A *complete lattice homomorphism* $\varphi : L \to M$ is one which preserves the meet and join of every (possibly infinite) family $(a_i)_{i\in I}$ of elements of L:

$$\varphi\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} \varphi(a_i)$$
$$\varphi\left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I} \varphi(a_i)$$

While homomorphisms preserving arbitrary meets and arbitrary joins will certainly be of interest, monotone maps satisfying just one of these conditions will be useful as well because of the role they play in Galois connections, which we now take a brief moment to explore. All of the following results on Galois connections can be found in [8], Chapter 7.

Definition 6.1.3. Let P and Q be posets. A pair of monotone maps $f : P \to Q$ and $g : Q \to P$ is a *Galois connection between* P *and* Q if, for all $p \in P$ and all $q \in Q$,

$$f(p) \leq q \text{ iff } p \leq g(q).$$

A Galois connection is written (f, g), where f is called the lower adjoint (or left adjoint) of g and g is called the upper adjoint (or right adjoint) of f.

The importance of Galois connections to our explorations comes from the following result, which is also known as the Adjoint Functor Theorem for Posets. Here we state it for complete lattices, however, as it is complete lattices that will be of interest for our investigations.

Proposition 6.1.4. *Let P and Q be complete lattices and $f : P \to Q$ a monotone map. Then,*

1. *f preserves arbitrary joins if and only if f has an upper adjoint g, meaning (f, g) is a Galois connection. For all $q \in Q$, this map g is given by*

$$g(q) = \bigvee \{p \in P \mid f(p) \leq q\}$$

2. *f preserves arbitrary meets if and only if f has a lower adjoint h, meaning (h, f) is a Galois connection. For all $q \in Q$, this map h is given by*

$$h(q) = \bigwedge \{p \in P \mid q \leq f(p)\}$$

Galois connections have several interesting properties that will be of use to us.

Proposition 6.1.5. *Let P and Q be complete lattices and $f : P \to Q$ and $g : Q \to P$ such that (f, q) is a Galois connection. The following hold:*

1. *f preserves arbitrary joins*
2. *g preserves arbitrary meets*
3. *For all $p \in P$, $p \leq (g \circ f)(p)$*
4. *For all $q \in Q$, $(f \circ g)(q) \leq q$*
5. *For all $p \in P$, $(f \circ g \circ f)(p) = f(p)$*
6. *For all $q \in Q$, $(g \circ f \circ g)(q) = g(q)$*

Returning to complete lattices and complete lattice homomorphisms, one can consider both complete orthomodular lattices and complete Boolean lattices, also called complete Boolean algebras. There is a category **cOML** of complete orthomodular lattices and complete orthomodular lattice homomorphisms between them, and a category **cBA** of complete Boolean algebras and complete Boolean algebra homomorphisms between them.

It is straightforward to show that Propositions 4.0.4 and 4.0.5 about Boolean subalgebras of an orthomodular lattices also hold in the complete case, and the Boolean algebras stated to exist are the same as in Figure 4.1.

Proposition 6.1.6. *Every element a of a complete orthomodular lattice L is in some complete Boolean subalgebra of L.*

Proposition 6.1.7. *In a complete orthomodular lattice L, for any elements $a, b \in L$ satisfying $a \leq b$ there is complete Boolean subalgebra of L containing both a and b.*

On can also define complete analogs of $\mathcal{B}(L)$ and L_{part}.

Definition 6.1.8. The *context category* of a complete orthomodular lattice L, denoted $\mathcal{B}_c(L)$, is the poset of complete Boolean subalgebras of L, ordered by inclusion.

Note $\mathcal{B}_c(L)$ can alternately be considered a poset category or as a subcategory of **cBA**, the category of complete Boolean algebras. As before, arrows in poset $\mathcal{B}_c(L)$ will be denoted $i_{B',B}$, while inclusion Boolean algebra homomorphisms that are arrows in the subcategory $\mathcal{B}_c(L)$ of **cBA** will be denoted $inc_{B',B}$.

Proposition 4.1.2 can also be proved in the complete case.

Proposition 6.1.9. *For any complete orthomodular lattice L, $\mathcal{B}_c(L)$ is a poset in which any non-empty family of elements has a well-defined unique meet, where the meet $\bigwedge$ is defined as follows for family $(B_i \in \mathcal{B}_c(L))_{i\in I}$:*

$$\bigwedge_{i\in I} B_i := \bigcap_{i\in I} B_i$$

Note that $\bigcap$ simply denotes set intersection.

As before, any complete orthomodular lattice homomorphism $\varphi : L \to M$ induces a map $\tilde{\varphi} : \mathcal{B}_c(L) \to \mathcal{B}_c(M)$, where on each complete Boolean subalgebra B of L,

$$\tilde{\varphi}(B) := \{\varphi(b) : b \in B\}.$$

Consider the following facts about $\tilde{\varphi}$.

Proposition 6.1.10. *For every $B \in \mathcal{B}_c(L)$, $\tilde{\varphi}(B) \in \mathcal{B}_c(M)$ and $\varphi|_B : B \to \tilde{\varphi}(B)$ is a complete Boolean algebra homomorphism.*

Proposition 6.1.11. *$\tilde{\varphi}$ is a monotone map between posets, and thus a morphism in category* **Pos**.

Proposition 6.1.12. *If $\varphi : L \to M$ is an isomorphism of complete orthomodular lattices, then $\tilde{\varphi} : \mathcal{B}_c(L) \to \mathcal{B}_c(M)$ is an order isomorphism in* **Pos** *and for each $B \in \mathcal{B}_c(L)$,*

$$\varphi|_B : B \to \tilde{\varphi}(B)$$

is an isomorphism of complete Boolean algebras.

There is a functor from **cOML**, the category of complete orthomodular lattices, to **Pos**, sending each complete orthomodular lattice to its context category and each homomorphism φ to $\tilde{\varphi}$. Call this functor $c\mathcal{B} : \mathbf{cOML} \to \mathbf{Pos}$, where for complete orthomodular lattice L and complete orthomodular lattice homomorphism $\varphi : L \to M$,

$$c\mathcal{B}_L = \mathcal{B}_c(L)$$
$$c\mathcal{B}(\varphi) = \tilde{\varphi} : \mathcal{B}_c(L) \to \mathcal{B}_c(M)$$

Proposition 6.1.13. $c\mathcal{B} : \mathbf{cOML} \to \mathbf{Pos}$ *is a functor.*

One can also consider a complete version of the partial orthomodular lattice L_{part} associated with an orthomodular lattice L.

Definition 6.1.14. Let L be a complete orthomodular lattice. The *partial complete orthomodular lattice* L^c_{part} *associated with* L has the same elements and orthocomplements as L and lattice operations $\bigvee$ and $\bigwedge$ inherited from L, but only defined for (possibly infinite) families of elements $(a_i)_{i\in I}$ in L such that there is a complete Boolean subalgebra $B \in \mathcal{B}_c(L)$ that contains a_i for all $i \in I$. Such families of elements are called *compatible elements*.

Definition 6.1.15. A *morphism of partial complete orthomodular lattices* is a function $p : L^c_{part} \to M^c_{part}$ that preserves orthocomplements and existing meets and joins.

There is a category **cPOML** of partial complete orthomodular lattices and partial complete orthomodular lattice homomorphisms between them.

Lemma 6.1.16. *If* $a \leq b$ *in complete orthomodular lattice* L *and* $p : L_{part} \to M_{part}$ *is a partial complete orthomodular lattice homomorphism, then* $p(a) \leq p(b)$.

Lemma 6.1.17. *Let* L *and* M *be complete orthomodular lattices, and* L_{part} *and* M_{part} *their associated partial complete orthomodular lattices. There is a bijective correspondence between isomorphisms* $L \to M$ *in* **cOML** *and isomorphisms* $L_{part} \to M_{part}$ *in* **cPOML**.

Just as before, it is Lemmas 6.1.16 and 6.1.17 that critically depend on the orthomodularity condition and do not hold for arbitrary complete ortholattices.

6.2 Stonean spaces

Just as there is a duality between Boolean algebras and Stone spaces, there is a duality between complete Boolean algebras and Stonean spaces.

Definition 6.2.1. A *Stonean space* is an extremely disconnected compact Hausdorff space.

In an extremely disconnected topological space, the closure of every open subspace is open and the interior of every closed subspace is closed. It is in part this fact, which holds for Stonean spaces but not for Stone spaces, that motivates our investigation of complete orthomodular lattices.

Recall that a Stone space is a totally disconnected compact Hausdorff space. As ‘extremely disconnected’ is a stronger condition than ‘totally disconnected,’ all Stonean spaces are also Stone spaces but not vice versa. The following lemmas characterize the relation between Stonean spaces and complete Boolean algebras.

Proposition 6.2.2 ([20]). *A Boolean algebra is complete if and only if its Stone space is Stonean.*

Proposition 6.2.3 ([14]). *The clopen subsets of a Stonean space form a complete Boolean algebra. Complementation is given by set-theoretic complementation, and meets and joins for a family of clopen subsets $\{S_i \mid i \in I\}$ are given by:*

$$\bigvee_{i\in I} S_i = cls(\bigcup_{i\in I} S_i)$$
$$\bigwedge_{i\in I} S_i = int(\bigcap_{i\in I} S_i)$$

Here cls denotes the closure and int denotes the interior of a subset.

It is simple to verify that this join is precisely the smallest clopen set containing all S_i and this meet is precisely the largest clopen set contained in all of the S_i.

The correspondence between complete Boolean algebras and Stonean spaces can be extended to a dual equivalence of categories. There is a category **Stonean**, whose objects are Stonean spaces and whose morphisms are continuous open maps. Just as with the category **cBA**, where we consider only those Boolean algebra homomorphisms which are complete, for **Stonean** we consider only those continuous maps which are open.

Proposition 6.2.4 ([1])**.** *There is a dual equivalence of categories between* **cBA** *and* **Stonean***:*

$$\mathbf{cBA} \underset{cl}{\overset{\Omega}{\rightleftarrows}} \mathbf{Stonean} \qquad (\perp)$$

This duality is witnessed by the natural isomorphisms $Bo : Id_{\mathbf{cBA}} \Rightarrow cl \circ \Omega$ and $St : Id_{\mathbf{Stonean}} \Rightarrow \Omega \circ cl$, which we will call by the same names as the natural transformations that witness Stone duality. Propositions 6.2.2 and 6.2.3, above, are consequences of this dual equivalence, but the references listed above provide explicit proofs that give more intuition as to why such results are true. As corollaries of Proposition 6.2.4, we also have the following facts that will be essential for extending the isomorphism result of Theorem 5.7.3 to complete orthomodular lattices.

Fact 6.2.5. *For every $B \in Ob(\mathbf{cBA})$, the component Bo_B of natural isomorphism Bo is a complete Boolean algebra isomorphism.*

Proof. $Bo_B : B \to cl(\Omega_B)$ is an arrow in **cBA**. □

Fact 6.2.6. *If $\eta : X \to Y$ is any continuous open map between Stonean spaces, then $cl(\eta)$ is a complete Boolean algebra homomorphism.*

Proof. $cl(\eta) : cl(Y) \to cl(X)$ is an arrow in **cBA**. □

By Lemma 5.4.2, there is then a duality:

$$\mathbf{Copresh(cBA)} \underset{CL}{\overset{\Sigma}{\rightleftarrows}} \mathbf{Presh(Stonean)} \qquad (\perp)$$

In order to consider the actions of Σ and CL on the Bohrifications of complete orthomodular lattices in **Copresh(cBA)** and the spectral presheaves of complete orthomodular lattices in **Presh(Stonean)**, it is first necessary to define the spectral presheaf and the Bohrification of a complete orthomodular lattice.

6.3 Spectral presheaf isomorphisms

The spectral presheaf of a complete orthomodular lattice is defined in an analogous way to the spectral presehaf of an orthomodular lattice. Let L be a complete orthomodular lattice. Then the spectral presheaf of L is a functor $\underline{\Omega(L)}$ from $\mathcal{B}_c(L)$ to **Stonean** which sends each complete Boolean subalgebra of L to its Stone space. As the Stone space of a complete Boolean algebra is always Stonean, then $\underline{\Omega(L)}$ is indeed a functor in **Presh(Stonean)**. Specifically, for B, B' in $\mathcal{B}_c(L)$ with $B' \subseteq B$,

$$\underline{\Omega(L)}_B = \Omega(B)$$
$$\underline{\Omega(L)}(i_{B',B}) = \Omega(inc_{B',B}) = r_{B,B'} : \Omega(B) \to \Omega(B').$$

Note that as Ω is the map of the dual equivalence between **cBA** and **Stonean**, it maps complete Boolean algebra homomorphisms to continuous open maps. In particular, as the inclusion $inc_{B',B}$ is a complete Boolean algebra homomorphism, the map $r_{B,B'}$ is continuous and open.

The Bohrification of a complete orthomodular lattice can also be defined in an analogous way. For complete orthomodular lattice L, the Bohrification $\overline{\mathcal{L}}$ is a functor from $\mathcal{B}_c(L)$ to **cBA** satisfying:

$$\overline{\mathcal{L}}_B = B$$
$$\overline{\mathcal{L}}(i_{B',B}) = inc_{B',B}$$

As $\mathcal{B}_c(L)$ only contains complete Boolean subalgebras of L, the codomain of $\overline{\mathcal{L}}$ is in fact contained in **cBA**, meaning $\overline{\mathcal{L}}$ is a functor in **Copresh(cBA)** as desired.

Now it is possible to consider the action of functors Σ and CL of the dual equivalence between **Copresh(cBA)** and **Presh(Stonean)** on Bohrifications and spectral presheaves, respectively. A brief check shows that they are just as for the general case of arbitrary orthomodular lattices in Section 5.5.

$$\Sigma(\overline{\mathcal{L}}) = \underline{\Omega(L)}$$
$$\Sigma(\langle I, \theta \rangle) = \langle I, \Omega(\theta) \rangle$$
$$CL(\underline{\Omega(L)}) = cl \circ \underline{\Omega(L)}$$
$$CL(\langle \tilde{\varphi}, \zeta_\varphi \rangle) = \langle \tilde{\varphi}, \zeta_\varphi^{(-1)} \rangle$$

The lemmas and theorems of Sections 5.6 and 5.7 also have analogous versions for the complete case.

Lemma 6.3.1. *The map* $\{Bo_B\}_{B\in\mathcal{B}_c(L)} : \overline{\mathcal{L}} \Rightarrow cl \circ \underline{\Omega(L)}$ *is a natural isomorphism. That is, these two functors are isomorphic in the functor category* $\mathbf{cBA}^{\mathcal{B}_c(L)}$.

Natural isomorphism $\{Bo_B\}_{B\in\mathcal{B}_c(L)}$ will now simply be written in a slight abuse of notation as Bo for the sake of simplicity.

Lemma 6.3.2. *Morphism* $\langle Id_{\mathcal{B}_c(L)}, Bo\rangle : \overline{\mathcal{L}} \to cl \circ \underline{\Omega(L)}$ *is an isomorphism in* **Copresh(cBA)**.

Theorem 6.3.3. *Let* L *and* M *be complete orthomodular lattices,* $\underline{\Omega(L)}$ *and* $\underline{\Omega(M)}$ *their spectral presheaves, and* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ *their Bohrifications. Then there is an isomorphism* $\underline{\Omega(M)} \to \underline{\Omega(L)}$ *in* **Presh(Stonean)** *if and only if there is an isomorphism* $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ *in* **Copresh(cBA)***, and these isomorphisms can be explicitly constructed from each other.*

If $\langle H, \eta\rangle$ is an isomorphism between the spectral presheaves of complete orthomodular lattices L and M, then the corresponding isomorphism from $\overline{\mathcal{L}}$ to $\overline{\mathcal{M}}$ in **Copresh(cBA)** is:

$$\rho := \langle Id_{\mathcal{B}(M)}, Bo^{-1}\rangle \circ \langle H, cl(\eta)\rangle \circ \langle Id_{\mathcal{B}(L)}, Bo\rangle = \langle H, (H^*Bo^{-1}) \circ cl(\eta) \circ Bo\rangle.$$

In particular, each component of natural isomorphism $(H^*Bo^{-1}) \circ cl(\eta) \circ B$ is an isomorphism in **cBA**, that is, it is a complete Boolean algebra isomorphism. This follows from Proposition 6.2.4, and in particular, Facts 6.2.5 and 6.2.6. That this isomorphism (renamed ρ in the proof of Theorem 5.7.2 as it is above) preserves arbitrary meets and joins is essential for being able to construct a complete orthomodular lattice isomorphism from an isomorphism of spectral presheaves in Theorem 6.3.5 below.

Theorem 6.3.4. *Let* L *and* M *be complete orthomodular lattices. If* $\varphi : L \to M$ *is an isomorphism in* **cOML***, then there is an isomorphism* $\langle\tilde{\varphi}, \zeta_\varphi\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ *in* **Presh(Stonean)***, where natural transformation* ζ_φ *has components* $\zeta_{\varphi,B} = \Omega(\varphi|_B)$ *for all* B *in* $\mathcal{B}_c(L)$.

Theorem 6.3.5. *Let* L *and* M *be complete orthomodular lattices. If there is an isomorphism* $\langle H, \eta\rangle : \underline{\Omega(M)} \to \underline{\Omega(L)}$ *in* **Presh(Stonean)***, then one can construct an isomorphism from* L *to* M *in* **cOML**.

The proof of this theorem for complete orthomodular lattices is made possible by the fact that ρ is a complete Boolean algebra isomorphism and thus induces a partial complete orthomodular lattice isomorphism, which can be used to construct a complete orthomodular lattice isomorphism.

Theorem 6.3.6. *Two complete orthomodular lattices L and M are isomorphic in* **cOML** *if and only if their spectral preserves $\underline{\Omega(L)}$ and $\underline{\Omega(M)}$ are isomorphic in* **Presh(Stonean)**.

Chapter 7

Representing a Complete Orthomodular Lattice

Now that a spectral presheaf isomorphism result has been presented for complete orthomodular lattices, we can proceed with our exploration of the subobjects of the spectral presheaf of a complete orthomodular lattice. The goal of this section is to find a 'representation' of a complete orthomodular lattice by clopen subobjects of its spectral presheaf, in analogy to the Stone representation of a Boolean algebra by clopen subsets of its Stone space. In Section 7.1, we define and describe the clopen subobjects of the spectral presheaf of an orthomodular lattice and in Section 7.2 show that they form a complete bi-Heyting algebra. In Section 7.3, we define a 'daseinisation' map from a complete orthomodular lattice to the clopen subobjects of its spectral presheaf. If we interpret the elements of the orthomodular lattice as quantum propositions, then this map can be seen as a 'translation' of the quantum propositions into clopen subobjects of a generalized space. In Sections 7.4 and 7.5, we use the adjoint of this daseinisation map to relate the lattice structure of the clopen subobjects of the spectral presheaf to the lattice structure of the original orthomodular lattice.

For the remainder of this chapter, assume that L is a complete orthomodular lattice and $\underline{\Omega(L)}$ is its spectral presheaf.

7.1 Clopen subobjects of the spectral presheaf

Definition 7.1.1. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. A functor $G : \mathcal{C} \to \mathbf{Set}$ is a *subfunctor* of F if for all $C \in Ob(\mathcal{C})$, $G_C \subseteq F_C$ and for all $a : C \to D$ in $Morph(\mathcal{C})$, $G(a)$ is the restriction of $F(a)$ to domain G_C.

Note that this implies that for all $c \in G_C$, $[G(a)](c) = [F(a)](c) \in G_D$.

Definition 7.1.2. A *subobject* of $\underline{\Omega(L)}$ is a subfunctor $\underline{S} : \mathcal{B}_c(L)^{op} \to \mathbf{Set}$ of $\underline{\Omega(L)}$.

This is the same definition of a subobject of $\underline{\Omega(L)}$ as in the topos sense. That is, recalling the definition of a subobject in a topos, subfunctors of $\underline{\Omega(L)}$ correspond precisely to monic arrows with codomain $\underline{\Omega(L)}$ in the functor category $\mathbf{Set}^{\mathcal{B}(L)^{op}}$ [15].

Definition 7.1.3. A subobject $\underline{S}$ of $\underline{\Omega(L)}$ is *clopen* if for all $B \in \mathcal{B}_c(L)$, the component $\underline{S}_B$ is a clopen subset of $\underline{\Omega(L)}_B$. The set of clopen subobjects of $\underline{\Omega(L)}$ will be denoted $Sub_{cl}\underline{\Omega(L)}$.

It is possible to define a partial order of $Sub_{cl}\underline{\Omega(L)}$ in an obvious way. Let $\underline{S}$ and $\underline{T}$ be clopen subobjects in $Sub_{cl}\underline{\Omega(L)}$. Then define $\underline{S} \leq \underline{T}$ if and only if for each B in $\mathcal{B}_c(L)$, $\underline{S}_B \subseteq \underline{T}_B$. Thus, $Sub_{cl}\underline{\Omega(L)}$ is a partially ordered set.

One can also define the join of a (possibly infinite) family of clopen subobjects $(\underline{S}_i)$ for all i in index set I in the following way. This join will be a functor from $\mathcal{B}_c(L)$ to **Stonean**, and its action on objects $B \in \mathcal{B}_c(L)$ is

$$\left(\bigvee_{i\in I} \underline{S}_i\right)_B := cls\left(\bigcup_{i\in I} \underline{S}_{i,B}\right).$$

Note that cls denotes taking the closure of a subset of Stonean space Ω_B in the Stone topology. Each $\underline{S}_{i;B}$ is a clopen subset of Ω_B. The arbitrary union of open sets is open, but the arbitrary union of closed sets is not necessarily closed. To obtain a clopen set, it is then necessary to take the closure of this union; because Ω_B is Stonean, the closure of any open set is clopen, as desired. Note that it is the smallest clopen subobject of $\underline{\Omega(L)}$ that is larger than all of the subobjects $\underline{S}_i$, as desired.

On the morphisms $i_{B',B}$ in $\mathcal{B}_c(L)$,

$$\left(\bigvee_{i\in I} \underline{S}_i\right)(i_{B',B}) = r_{B,B'}\Big|_{\left(\bigvee_{i\in I} \underline{S}_i\right)_B}$$

A brief topological argument shows that any continuous map between topological spaces, such as $r_{B,B'}$, maps the closure of a set X to within the closure of the image of X, meaning that that this restriction of $r_{B,B'}$ above is well defined.

One can also define the meet of a (possibly infinite) family of clopen subobjects of $\underline{\Omega(L)}$ in a similar way. Let $(\underline{S}_i)_{i\in I}$ be a family of subobjects of $Sub_{cl}\underline{\Omega(L)}$. Then the meet of all of the subobjects of this family is a functor from $\mathcal{B}_c(L)$ to **Stonean**, and its action on objects B in $\mathcal{B}_c(L)$ is defined to be:

$$\left(\bigwedge_{i\in I} \underline{S}_i\right)_B := int\left(\bigcap_{i\in I} \underline{S}_{i,B}\right).$$

Here *int* denotes taking the interior in the Stone topology. The intersection above is necessarily open, and because Ω_B is a Stonean space, taking its interior yields a clopen subset of Ω_B, as desired. This meet is the largest clopen subobject that is less than all of the $\underline{S}_i$.

In order to show that the meet defined above is a well-defined subobject of the spectral presheaf, it remains to show that for $B' \subseteq B$, the image of $\left(\bigwedge_{i\in I} \underline{S}_i\right)_B$ under $r_{B,B'}$ is contained in $\left(\bigwedge_{i\in I} \underline{S}_i\right)_{B'}$. For this it is sufficient to know that $r_{B,B'}$ is an open map between topological spaces, which follows from the dual equivalence between **cBA** and **Stonean** as described above because $r_{B,B'}$ is an arrow in **Stonean**. As $int\left(\bigcap_{i\in I} \underline{S}_{i,B}\right) = \left(\bigwedge_{i\in I} \underline{S}_i\right)_B$ is an open set, then its image under $r_{B,B'}$ is an open set that is contained $\left(\bigcap_{i\in I} \underline{S}_{i,B'}\right)$. Thus, this image is contained in $int\left(\bigcap_{i\in I} \underline{S}_{i,B'}\right) = \left(\bigwedge_{i\in I} \underline{S}_i\right)_{B'}$, meaning $\bigwedge_{i\in I} \underline{S}_i$ is a valid subobject of $\underline{\Omega(L)}$.

With the definitions of meet and join above, $Sub_{cl}\underline{\Omega(L)}$ is a complete lattice. It is also possible to show that $Sub_{cl}\underline{\Omega(L)}$ is distributive by considering its components. For each $B \in \mathcal{B}_c(L)$,

$$\begin{aligned}
[\underline{S} \wedge (\underline{T} \vee \underline{R})]_B &= \underline{S}_B \cap (\underline{T} \vee \underline{R})_B \\
&= \underline{S}_B \cap (\underline{T}_B \cup \underline{R}_B) \\
&= (\underline{S}_B \cap \underline{T}_B) \cup (\underline{S}_B \cap \underline{R}_B) \\
&= (\underline{S} \wedge \underline{T})_B \cup (\underline{S} \wedge \underline{R})_B \\
&= [(\underline{S} \wedge \underline{T}) \vee (\underline{S} \wedge \underline{R})]_B
\end{aligned}$$

Thus, the two subfunctors $\underline{S} \wedge (\underline{T} \vee \underline{R})$ and $(\underline{S} \wedge \underline{T}) \vee (\underline{S} \wedge \underline{R})$ are the same on every object $B \in \mathcal{B}_c(L)$. On each arrow $i_{B',B}$ both functors are simply the restriction of $r_{B,B'}$ to the same domain. Thus,

$$\underline{S} \wedge (\underline{T} \vee \underline{R}) = (\underline{S} \wedge \underline{T}) \vee (\underline{S} \wedge \underline{R}),$$

meaning $Sub_{cl}\underline{\Omega(L)}$ is a distributive lattice.

While the above argument shows that binary mets distributive over binary joins, in a complete Boolean algebra it is also true that finite meets distribute over arbitrary joins and finite joins distribute over arbitrary meets. Thus, a similar argument will show that this also holds in $Sub_{cl}\underline{\Omega(L)}$. First, we will need two brief topological lemmas.

Lemma 7.1.4. *Let X be a clopen subset and Y an open subset of some topological space. Then*

$$X \cap cls(Y) = cls(X \cap Y).$$

Proof. We show both directions of containment. First, note that as both X and $cls(Y)$ are closed, then $X \cap cls(Y)$ is closed. As $X \cap cls(Y)$ also clearly contains $X \cap Y$, then

$$cls(X \cap Y) \subseteq X \cap cls(Y).$$

Now, consider any element $x \in X \cap cls(Y)$; we want to show that $x \in cls(X \cap Y)$. Let U be any neighborhood of x. As $x \in X$ and X is open, there is some open set $V \subseteq U$ with $x \in V$ such that $V \subseteq X$. Additionally, as $x \in cls(Y)$ then any open neighborhood of x contains some point that is in Y. In particular, open neighborhood V of x contains some point y such that $y \in Y$. Because $V \subseteq X$ then $y \in X \cap Y$. As $V \subseteq U$, then y is a point in U such that $y \in X \cap Y$. As this holds for an arbitrary neighborhood U of x, then $x \in cls(X \cap Y)$ and

$$X \cap cls(Y) \subseteq cls(X \cap Y).$$

□

Lemma 7.1.5. *Let X be a clopen subset and Y a closed subset of some topological space. Then,*

$$X \cup int(Y) = int(X \cup Y).$$

Proof. We show both directions of containment. First, note that as X and $int(Y)$ are both open then $X \cup int(Y)$ is open. As it is clearly contained in $X \cup Y$, then

$$X \cup int(Y) \subseteq int(X \cup Y).$$

Now, let $t \in int(X \cup Y)$. If $t \in X$, then $t \in X \cup int(Y)$ and we are done. Suppose $t \notin X$. By the definition of $int(X \cup Y)$, there exists an open neighborhood V of t such that $V \subseteq X \cup Y$. Let X' denote the complement of X in this topological space, and note that X' is clopen and by assumption $t \in X'$, so $t \in V \cap X'$, which is open. Note also that

$$V \cap X' \subseteq (X \cup Y) \cap X' = Y \cap X' \subseteq Y$$

As $V \cap X'$ is an open set containing t that is contained in Y, then $t \in int(Y)$ meaning $t \in X \cup int(Y)$ and thus

$$int(X \cup Y) \subseteq X \cup int(Y).$$

□

Armed with the previous two lemmas, we can now prove the following.

Lemma 7.1.6. *In $Sub_{cl}\underline{\Omega(L)}$, finite meets distribute over arbitrary joins and finite joins distribute over arbitrary meets.*

Proof. In any complete Boolean algebra, finite meets distribute over arbitrary joins and finite joins distribute over arbitrary meets [29]. Using this and Lemma 7.1.4, for functors $\underline{S}$ and $(\underline{T}_i)_{i\in I}$ in $Sub_{cl}\underline{\Omega(L)}$ and for all $B \in \mathcal{B}_c(L)$,

$$\begin{aligned}\left[\underline{S} \wedge \left(\bigvee_{i\in I} \underline{T}_i\right)\right]_B &= \underline{S}_B \cap \left(\bigvee_{i\in I} \underline{T}_i\right)_B \\ &= \underline{S}_B \cap cls\left(\bigcup_{i\in I} \underline{T}_{i,B}\right) \\ &= cls\left(\underline{S}_B \cap \left(\bigcup_{i\in I} \underline{T}_{i,B}\right)\right) \\ &= cls\left(\bigcup_{i\in I} (\underline{S}_B \cap \underline{T}_{i,B})\right) \\ &= cls\left(\bigcup_{i\in I} (\underline{S} \wedge \underline{T}_i)_B\right) \\ &= \left[\bigvee_{i\in I} (\underline{S} \wedge \underline{T}_i)\right]_B .\end{aligned}$$

Thus, functors $\underline{S} \wedge (\bigvee_{i\in I} \underline{T}_i)$ and $\bigvee_{i\in I}(\underline{S} \wedge \underline{T}_i)$ are the same on every object B in their domain $\mathcal{B}_c(L)$. On each arrow $i_{B,B}$, both functors are simply the restriction of $r_{B,B'}$ to the same domain. Thus,

$$\underline{S} \wedge \left(\bigvee_{i\in I} \underline{T}_i\right) = \bigvee_{i\in I} (\underline{S} \wedge \underline{T}_i).$$

Similarly, using Lemma 7.1.5, one can easily show that

$$\underline{S} \vee \left(\bigwedge_{i\in I} \underline{T}_i\right) = \bigwedge_{i\in I} (\underline{S} \vee \underline{T}_i).$$

Inductively, these results can be extended from binary meets and joins to all finite meets and joins. □

Before continuing with our explorations, we will take a brief moment to define and discuss bi-Heyting algebras before demonstrating that $Sub_{cl}\underline{\Omega(L)}$ is a bi-Heyting algebra.

7.2 Bi-Heyting algebras

The information on bi-Heyting algebras that is relevant for our purposes can be found in [13] and [29].

A Heyting algebra is an alternate generalization of a Boolean algebra. While an orthomodular lattice generalizes a Boolean algebra by relaxing the distributivity condition, a Heyting algebra generalizes a Boolean algebra by relaxing the requirement that the join of an element and its complement is 1, the top element of the lattice. Formally, a Heyting algebra can be defined as follows.

Definition 7.2.1. A *Heyting algebra* is a bounded lattice H such that for all elements $a, b \in H$, there is a greatest element $x \in H$ such that $a \wedge x \leq b$. Such an element x is called the *relative pseudocomplement of a with respect to b* or the *Heyting implication from a to b* and is denoted $a \Rightarrow b$. The *pseudocomplement* of a, also called the *Heyting negation of a*, is the element $\neg a := a \Rightarrow 0$.

In the above definition, the element $\neg a$ is called a pseudocomplement of a because $a \wedge \neg a = 0$ but it is not necessarily true that $a \vee \neg a = 1$.

Definition 7.2.2. A Heyting algebra is *complete* if it is complete as a lattice.

In a complete Heyting algebra, finite meets distribute over arbitrary joins [29].

One can also define the dual notion of a co-Heyting algebra, also called a Brouwer algebra.

Definition 7.2.3. A *co-Heyting algebra* is a bounded lattice H such that for all elements $a, b \in H$, there is a least element $x \in H$ such that $a \leq b \vee x$. Such an element x is called the *co-Heyting implication from a to b*, and is denoted $a \Leftarrow b$. The *co-Heyting negation of a* is the element $\sim a := 1 \Leftarrow a$.

In the dual to that above, the co-Heyting negation satisfies $a \vee \sim a = 1$ but it might not necessarily be true that $a \wedge \sim a = 0$.

Definition 7.2.4. A co-Heyting algebra is *complete* if it is complete as a lattice.

In a complete co-Heyting algebra, finite joins distribute over arbitrary meets [29].

Definition 7.2.5. A *bi-Heyting algebra* is a bounded lattice that is both a Heyting algebra and a co-Heyting algebra. A bi-Heyting algebra is *complete* if it is complete as a lattice.

A bi-Heyting algebra is distributive, but generalizes a Boolean algebra by splitting up the notion of complementation into two separate notions, Heyting negation and co-Heyting negation. Heyting negation is intuitionistic, satisfying $a \wedge \neg a = 0$ but not necessarily $a \vee \neg a = 1$; logically, this means that the law of excluded middle need not hold and a proposition may be neither true not false. Co-Heyting negation is paraconsistent, satisfying $a \vee \sim a = 1$ but not necessarily $a \wedge \sim a = 0$; logically, this means that the law of noncontradiction need not hold and a proposition may be both true and false.

Now that we have defined a complete bi-Heyting algebra, we proceed to show that $Sub_{cl}\underline{\Omega(L)}$ is a complete bi-Heyting algebra.

Proposition 7.2.6. *$Sub_{cl}\underline{\Omega(L)}$ is a complete bi-Heyting algebra.*

Proof. Consider the following map on complete lattice $Sub_{cl}\underline{\Omega(L)}$:

$$\begin{aligned} \underline{S} \wedge (-) : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{T} &\mapsto \underline{S} \wedge \underline{T} \end{aligned}$$

Simple lattice properties imply that this is a monotone map. By Lemma 7.1.6, this monotone map preserves arbitrary joins. Then, by Proposition 6.1.4, because $Sub_{cl}\underline{\Omega(L)}$ is complete and the functor $\underline{S} \wedge (-)$ preserves arbitrary joins, it has an upper adjoint which we will call $\underline{S} \Rightarrow (-)$ given by:

$$\begin{aligned} \underline{S} \Rightarrow (-) : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{T} &\mapsto (\underline{S} \Rightarrow \underline{T}) := \bigvee \{\underline{R} \in Sub_{cl}\underline{\Omega(L)} \mid \underline{S} \wedge \underline{R} \leq \underline{T}\} \end{aligned}$$

Additionally, by Proposition 6.1.5, this map satisfies

$$\underline{S} \wedge (\underline{S} \Rightarrow \underline{T}) \leq \underline{T}. \tag{7.1}$$

In the construction of this map it is necessary to know that arbitrary joins exist in $Sub_{cl}\underline{\Omega(L)}$ in order to apply Proposition 6.1.4, necessitating the use of Stonean spaces and thus complete orthomodular lattices.

This maps yields a well-defined Heyting implication in distributive lattice $Sub_{cl}\underline{\Omega(L)}$. For each pair of elements $\underline{S}$ and $\underline{T}$ of $Sub_{cl}\underline{\Omega(L)}$, by definition $\underline{S} \Rightarrow \underline{T}$ is larger than any clopen subobject $\underline{R}$ of $\underline{\Omega(L)}$ satisfying $\underline{S} \wedge \underline{R} \leq \underline{T}$, and by Equation 7.1 the object $\underline{S} \Rightarrow \underline{T}$ satisfies this equation as well. Thus, $Sub_{cl}\underline{\Omega(L)}$ is a Heyting algebra. It is complete as $Sub_{cl}\underline{\Omega(L)}$ is a complete lattice. The Heyting negation of this algebra will be denoted $\neg$, and is given by

$$\begin{aligned} \neg : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{S} &\to \neg\underline{S} := (\underline{S} \Rightarrow \underline{0}) \end{aligned}$$

Here $\underline{0}$ is the clopen subobject of $\underline{\Omega(L)}$ with $\underline{0}_B = \emptyset$ for all $B \in \mathcal{B}_c(L)$, the bottom element of $Sub_{cl}\underline{\Omega(L)}$.

Analogously, Lemma 7.1.6 implies that the following monotone map preserves arbitrary meets:

$$\begin{aligned} \underline{S} \vee (-) : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{T} &\mapsto \underline{S} \vee \underline{T} \end{aligned}$$

Thus, by Proposition 6.1.4, it has a lower adjoint which we will call $(-) \Leftarrow \underline{S}$ given by:

$$\begin{aligned} (-) \Leftarrow \underline{S} : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{T} &\mapsto (\underline{T} \Leftarrow \underline{S}) := \bigwedge\{\underline{R} \in Sub_{cl}\underline{\Omega(L)} \mid \underline{T} \leq \underline{S} \vee \underline{R}\} \end{aligned}$$

By Proposition 6.1.5, this map satisfies

$$\underline{T} \leq \underline{S} \vee (\underline{T} \Leftarrow \underline{S}) \tag{7.2}$$

It is clear by the definition of this map and Equation 7.2 that this gives a co-Heyting implication for $Sub_{cl}\underline{\Omega(L)}$, demonstrating that $Sub_{cl}\underline{\Omega(L)}$ is a complete co-Heyting algebra and thus a complete bi-Heyting algebra. The co-Heyting negation is given by

$$\begin{aligned} \sim : Sub_{cl}\underline{\Omega(L)} &\to Sub_{cl}\underline{\Omega(L)} \\ \underline{S} &\to \sim\underline{S} := (\underline{\Omega(L)} \Leftarrow \underline{S}) \end{aligned}$$

□

7.3 Daseinisation

In this subsection we define a map from complete orthomodular lattice L to $Sub_{cl}\underline{\Omega(L)}$, called the daseinisation map. This can be interpreted as an approximation map, which for each element a of L 'brings into existence' an approximation of a as a subspace of each of the Stonean spaces Ω_B for $B \in \mathcal{B}_c(L)$.

Let L be a complete orthomodular lattice, let $a \in L$, and let $B \in \mathcal{B}_c(L)$ be a complete Boolean subalgebra of L, not necessarily containing a. Then, we define

$$\delta^o_B(a) := \bigwedge \{b \in B \mid b \geq a\},$$

the smallest element of B that is greater than or equal to a. If $a \in B$, then $\delta^o_B(a) = a$. Note that the superscript of o denotes that this is outer daseinisation, that is, approximating element a in B from above. It is precisely at this step that completeness of orthomodular lattice L is required to define daseinisation; we need to know that the infinite meet in the definition of $\delta^o_B(a)$ exists.

By Stone duality, we know that there is an isomorphism between complete Boolean algebra B and the clopen subobjects of its Stone space, which is Stonean because B is complete. From Section 3.4, this isomorphism is given by

$$\begin{aligned} Bo_B : B &\to cl(\Omega_B) = cl(\underline{\Omega(L)}_B) \\ b &\mapsto \{\sigma \in \underline{\Omega(L)}_B \mid \sigma(b) = 1\} \end{aligned}$$

Recall cl is the functor which maps a Stone space to its Boolean algebra of clopen subsets; if the Stone space is Stonean, this Boolean algebra is complete. In fact, this Bo_B is a complete Boolean algebra isomorphism. In particular, element $\delta^o_B(a)$ of B corresponds to the clopen subset of $\underline{\Omega(L)}_B$ given by:

$$\underline{\delta^o}_B(a) := Bo_B(\delta^o_B(a)) = \{\sigma \in \underline{\Omega(L)}_B \mid \sigma(\delta^o_B(a)) = 1\}.$$

The reason for naming this clopen subset in such a manner will become clear shortly. For now, simply note that the clopen subsets in $cl(\Omega_B) = cl(\underline{\Omega(L)}_B)$ are precisely the possible images of B under some clopen subfunctor $\underline{S}$ of $\underline{\Omega(L)}$.

Suppose that $B' \subseteq B$ in $\mathcal{B}_c(L)$. Clearly, it holds that $\delta^o_B(a) \leq \delta^o_{B'}(a)$. Then,

$$\begin{aligned} \sigma \in \underline{\delta^o}_B(a) &\Leftrightarrow \sigma(\delta^o_B(a)) = 1 \\ &\Rightarrow \sigma(\delta^o_{B'}(a)) = 1 \\ &\Leftrightarrow \sigma|_{B'}(\delta^o_{B'}(a)) = 1 \\ &\Leftrightarrow \sigma|_{B'} \in \underline{\delta^o}_{B'}(a) \end{aligned}$$

We conjecture that this result can be strengthened to show that $\sigma \in \underline{\delta^o}_B(a)$ if and only if $\sigma|_{B'} \in \underline{\delta^o}_{B'}(a)$, but such a result is not necessary for our purposes so we do not pursue this line of investigation. Note that this result implies that for every inclusion arrow $i_{B',B}$ in $\mathcal{B}_c(L)$, the restriction of $\underline{\Omega(L)}(i_{B',B}) = r_{B,B'}$ to domain $\underline{\delta^o}_B(a) \subseteq \underline{\Omega(L)}_B$ has codomain contained in $\underline{\delta^o}_{B'}(a) \subseteq \underline{\Omega(L)}_{B'}$. This means that the functor from $\mathcal{B}_c(L)$

to **Set** which sends B to $\underline{\delta^o}_B(a)$ is a valid subfunctor of $\underline{\Omega(L)}$; we will call this functor $\underline{\delta^o}(a)$. Clearly this functor

$$\underline{\delta^o}(a) := (\underline{\delta^o}_B(a))_{B \in \mathcal{B}_c(L)}$$

is thus also a subobject of the spectral presheaf. It is a clopen subobject because for each $B \in \mathcal{B}_c(L)$, the subset $\underline{\delta^o}_B(a)$ of Ω_B is clopen.

We are now ready to define the daseinisation map for complete orthomodular lattice L and discuss its properties.

Definition 7.3.1. The map

$$\underline{\delta^o} : L \to Sub_{cl}\underline{\Omega(L)}$$
$$a \mapsto \underline{\delta^o}(a)$$

from the complete orthomodular lattice L to the complete bi-Heyting algebra $Sub_{cl}\underline{\Omega(L)}$ is called *outer daseinisation*, or more simply *daseinisation.*

The daseinisation map can be seen as a process by which an element a in complete orthomodular lattice L is approximated in each classical context B and subsequently each Stone space Ω_B, ultimately yielding a clopen subobject of $\underline{\Omega(L)}$. Returning to the notion of an orthomodular lattice as a quantum logic whose elements are propositions, for each classical context B the daseinisation process first associates to proposition a the strongest proposition within B that must be true if proposition a is true, which above we called $\delta^o_B(a)$. The next step of daseinisation associates to each of these strongest propositions the collection of local valuations (elements of the Stone space of B, i.e., Boolean algebra homomorphisms from B to $\{0,1\}$) for which the proposition holds, which we called $\underline{\delta^o}_B(a)$. These sets of local valuations are clopen and are linked together by restriction maps to create a clopen subobject $\underline{\delta^o}(a)$. This analysis shows that the daseinisation process associates to each quantum proposition a subobject of the spectral presheaf of the complete orthomodular lattice to which it belongs, just as a classical proposition corresponds to a subset of the state space of the classical system.

Lemma 7.3.2. *The daseinisation map* $\underline{\delta^o} : L \to Sub_{cl}\underline{\Omega(L)}$ *has the following properties:*

1. $\underline{\delta^o}(0) = \underline{0}$, $\underline{\delta^o}(1) = \underline{\Omega(L)}$,

2. $\underline{\delta^o}$ *is monotone, that is,* $a \leq b$ *in* L *implies* $\underline{\delta^o}(a) \leq \underline{\delta^o}(b)$ *in* $Sub_{cl}\underline{\Omega(L)}$,

3. $\underline{\delta^o}$ *is injective*

4. $\underline{\delta^o}$ *preserves all joins*

Proof. (1) is obvious form the definition of $\underline{\delta^o}$; for all $B \in \mathcal{B}_c(L)$, $\delta^o_B(0) = 0$ and $\underline{\delta^o}_B(0) = \emptyset$. Similarly, $\delta^o_B(1) = 1$ and $\underline{\delta^o}_B(1) = \Omega_B = \underline{\Omega(L)}_B$.

(2) also follows from the definition of $\underline{\delta^o}$. If $a \leq b$, then $\delta^o_B(a) \leq \delta^o_B(b)$ and $\underline{\delta^o}_B(a) \subseteq \underline{\delta^o}_B(b)$ for all $B \in \mathcal{B}_c(L)$, meaning that $\underline{\delta^o}(a) \leq \underline{\delta^o}(b)$ in $Sub_{cl}\underline{\Omega(L)}$.

For (3), let a and b be distinct elements of L. Then,

$$\bigwedge_{B \in \mathcal{B}_c(L)} \delta^o_B(a) = a \neq b = \bigwedge_{B \in \mathcal{B}_c(L)} \delta^o_B(b)$$

This implies that there must be some $B \in \mathcal{B}_c(L)$ such that $\delta^o_B(a) \neq \delta^o_B(b)$. As Bo_B is a complete Boolean algebra isomorphism, it follows for this B that

$$\underline{\delta^o}_B(a) = Bo_B(\delta^o_B(a)) \neq Bo_B(\delta^o_B(a)) = \underline{\delta^o}_B(b)$$

As $\underline{\delta^o}(a)$ and $\underline{\delta^o}(b)$ differ at this component, then they are not the same subobject of $\underline{\Omega(L)}$. Thus, $\underline{\delta^o}$ is injective.

For (4), let $(a_i)_{i \in I}$ be some (possibly infinite) family of elements of L indexed by set I. For all $i \in I$, from statement (2) of this lemma we know that

$$\underline{\delta^o}(a_i) \leq \underline{\delta^o}\left(\bigvee_{i \in I} a_i\right)$$

Consequently,

$$\bigvee_{i \in I} \underline{\delta^o}(a_i) \leq \underline{\delta^o}\left(\bigvee_{i \in I} a_i\right)$$

Conversely, recall that for all i, $a_i \leq \delta^o_B(a_i)$. Then, for each $B \in \mathcal{B}_c(L)$,

$$\begin{aligned}
\delta^o_B\left(\bigvee_{i \in I} a_i\right) &= \bigwedge\{b \in B \mid b \geq \bigvee_{i \in I} a_i\} \\
&\leq \bigwedge\{b \in B \mid b \geq \bigvee_{i \in I} \delta^o_B(a_i)\} \\
&= \delta^o_B\left(\bigvee_{i \in I} \delta^o_B(a_i)\right) \\
&= \bigvee_{i \in I} \delta^o_B(a_i)
\end{aligned}$$

The last equality above holds because for $a \in B$, $\delta^o_B(a) = a$. As Bo_B is an isomorphism of Boolean algebras, it preserves order and arbitrary joins, so

$$\underline{\delta^o}_B\left(\bigvee_{i\in I} a_i\right) = Bo_B\left(\delta^o_B\left(\bigvee_{i\in I} a_i\right)\right) \leq Bo_B\left(\bigvee_{i\in I} \delta^o_B(a_i)\right) = \bigvee_{i\in I} Bo_B\left(\delta^o_B(a_i)\right) = \bigvee_{i\in I} \underline{\delta^o}_B(a_i).$$

As this is true for all $B \in \mathcal{B}_c(L)$, it follows that

$$\underline{\delta^o}\left(\bigvee_{i\in I} a_i\right) \leq \bigvee_{i\in I} \underline{\delta^o}(a_i)$$

This result and the one above together imply

$$\underline{\delta^o}\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} \underline{\delta^o}(a_i),$$

demonstrating that $\underline{\delta^o}$ is join-preserving, as desired. □

7.4 The adjoint of daseinisation

Now that the daseinisation map gives a relationship between complete orthomodular lattice L and complete bi-Heyting algebra $Sub_{cl}\underline{\Omega(L)}$, we want to explore this correspondence a bit more. In analogy to Stone's representation theorem for Boolean algebras, we want to find some sort of representation of L inside $Sub_{cl}\underline{\Omega(L)}$.

It is clear that L and $Sub_{cl}\underline{\Omega(L)}$ cannot be isomorphic as lattices in general, because L is not necessarily distributive but $Sub_{cl}\underline{\Omega(L)}$ is. Additionally, $Sub_{cl}\underline{\Omega(L)}$ contains significantly more elements in general that L. However, we will show that $Sub_{cl}\underline{\Omega(L)}$ modulo a certain equivalence relation is a complete lattice with a complete lattice isomorphism to L. We first consider the adjoint of the daseinisation map.

As $\underline{\delta^o}$ is a join-preserving map between two complete lattices, we can now apply Proposition 6.1.4. Thus, $\underline{\delta^o}$ has a meet-preserving upper adjoint $\epsilon : Sub_{cl}\underline{\Omega(L)} \to L$. This map ϵ is defined by:

$$\begin{aligned} \epsilon : Sub_{cl}\underline{\Omega(L)} &\to L \\ \underline{S} &\mapsto \bigvee\{a \in L \mid \underline{\delta^o}(a) \leq \underline{S}\} \end{aligned}$$

The following lemma, adapted from an unpublished result by Carmen Constantin, provides more insight into this map ϵ.

Lemma 7.4.1. *Let L be a complete orthomodular lattice, with spectral presheaf $\underline{\Omega(L)}$. The map ϵ generated by Proposition 6.1.4 is given by*

$$\begin{aligned} \epsilon : Sub_{cl}\underline{\Omega(L)} &\to L \\ \underline{S} \quad &\mapsto \bigwedge_{B \in \mathcal{B}_c(L)} Bo_B^{-1}(\underline{S}_B) \end{aligned}$$

Proof. Suppose that a is some lower bound for the set $\{Bo_B^{-1}(\underline{S}_B) \mid B \in \mathcal{B}_c(L)\}$. That is, for each $B \in \mathcal{B}_c(L)$,

$$a \leq Bo_B^{-1}(\underline{S}_B).$$

As $Bo_B^{-1}(\underline{S}_B)$ is an element of B that is greater than a and $\delta_B^o(a)$ is the least element of B that is greater than B, then

$$\begin{aligned} & a \leq Bo_B^{-1}(\underline{S}_B) \\ \Leftrightarrow \quad & \delta_B^o(a) \leq Bo_B^{-1}(\underline{S}_B) \\ \Leftrightarrow \quad & Bo_B(\delta_B^o(a)) \subseteq Bo_B(Bo_B^{-1}(\underline{S}_B)) \\ \Leftrightarrow \quad & \underline{\delta^o}_B(a) \subseteq \underline{S}_B \end{aligned}$$

This exactly characterizes the lower bounds a of the set $\{Bo_B^{-1}(\underline{S}_B) \mid B \in \mathcal{B}_c(L)\}$. That is,

$$\begin{aligned} \{a \in L \mid a \leq Bo_B^{-1}(\underline{S}_B) \ \forall \ B \in \mathcal{B}_c(L)\} &= \{a \in L \mid \underline{\delta^o}_B(a) \subseteq \underline{S}_B \ \forall \ B \in \mathcal{B}_c(L)\} \\ &= \{a \in L \mid \underline{\delta^o}(a) \leq \underline{S}\} \end{aligned}$$

In a complete lattice, joins can be written in terms of meets. That is,

$$\begin{aligned} \bigwedge_{B \in \mathcal{B}_c(L)} Bo_B^{-1}(\underline{S}_B) &= \bigvee \{a \in L \mid a \leq Bo_B^{-1}(\underline{S}_B) \ \forall \ B \in \mathcal{B}_c(L)\} \\ &= \bigvee \{a \in L \mid \underline{\delta^o}(a) \leq \underline{S}\} \\ &= \epsilon(\underline{S}) \end{aligned}$$

□

The previous lemma implies the following result, which is stronger than could be expected for an arbitrary Galois connection:

Lemma 7.4.2. $\epsilon \circ \underline{\delta^o} = id_L$

Proof. Let $a \in L$. Then,

$$\begin{aligned}(\epsilon \circ \underline{\delta^o})(a) = \epsilon(\underline{\delta^o}(a)) &= \bigwedge_{B \in \mathcal{B}_c(L)} Bo_B^{-1}(\underline{\delta^o}_B(a)) \\ &= \bigwedge_{B \in \mathcal{B}_c(L)} Bo_B^{-1}(Bo_B(\delta^o_B(a))) \\ &= \bigwedge_{B \in \mathcal{B}_c(L)} \delta^o_B(a) \\ &= a\end{aligned}$$

□

We proceed to use this map ϵ to define an equivalence relation on $Sub_{cl}\underline{\Omega(L)}$.

Definition 7.4.3. For $\underline{S}$, $\underline{T}$ in $Sub_{cl}\underline{\Omega(L)}$, define $\underline{S} \sim \underline{T}$ if and only if $\epsilon(\underline{S}) = \epsilon(\underline{T})$.

This is clearly a well-defined equivalence relation. Let

$$E = \left\{ [\underline{S}] \mid \underline{S} \in Sub_{cl}\underline{\Omega(L)} \right\}.$$

This particular set E will be shown in the next subsection to be partial representation of ortholattice L inside of $Sub_{cl}\underline{\Omega(L)}$. In fact, we will see that E, when given an appropriate lattice structure, can be used to reconstruct the structure of L as a complete lattice.

7.5 Representing L in the clopen subobjects of its spectral presheaf

This section will show that E is a complete lattice, and that there is a complete lattice isomorphism from E to L. First, we show that there is a set bijection between E and L.

Lemma 7.5.1. *There is a bijective set map from E to the set underlying complete orthomodular lattice L, given by*

$$\begin{aligned} g : E &\to L \\ [\underline{S}] &\mapsto \epsilon(\underline{S}) \end{aligned}$$

Proof. Clearly g is well defined, as if $[\underline{S}] = [\underline{T}]$ then $g([\underline{S}]) = \epsilon(\underline{S}) = \epsilon(\underline{T}) = g([\underline{T}])$ by definition. Consider the function

$$f : L \to E$$
$$a \mapsto [\underline{\delta^o}(a)]$$

I will now show that f is an inverse to g, meaning E and L are isomorphic as sets. First, let $a \in L$. Then, by Lemma 7.4.2,

$$(g \circ f)(a) = g\left([\underline{\delta^o}(a)]\right) = \epsilon\left(\underline{\delta^o}(a)\right) = a.$$

Now, let $\underline{S} \in Sub_{cl}\underline{\Omega(L)}$. Then,

$$(f \circ g)\left([\underline{S}]\right) = f\left(\epsilon(\underline{S})\right) = \left[\underline{\delta^o}\left(\epsilon(\underline{S})\right)\right].$$

By the properties of a Galois connection (Proposition 6.1.5),

$$\epsilon\left(\underline{\delta^o}\left(\epsilon(\underline{S})\right)\right) = \epsilon(\underline{S}),$$

meaning

$$(f \circ g)\left([\underline{S}]\right) = [\underline{\delta^o}(\epsilon(\underline{S}))] = [\underline{S}].$$

Thus, as both compositions of f and g are the identity, then $g : E \to L$ is a set bijection and E and L are isomorphic as sets. □

Note that a consequence of this lemma is that map ϵ is surjective. We now proceed to give E the structure of a complete meet-semilattice.

Definition 7.5.2. For all families $([\underline{S_i}])_{i\in I}$ of elements of E, where all $\underline{S_i} \in Sub_{cl}\underline{\Omega(L)}$,

$$\bigwedge_{i\in I}[\underline{S_i}] := \left[\bigwedge_{i\in I}\underline{S_i}\right],$$

where the meet on the right hand side above is taken in $Sub_{cl}\underline{\Omega(L)}$.

It is simple to check that such meets in E are well defined; suppose that $(\underline{T_i})_{i\in I}$ is such that $[\underline{S_i}] = [\underline{T_i}]$ for all i, that is, that $\epsilon(\underline{S_i}) = \epsilon(\underline{T_i})$. Then, because ϵ preserves meets,

$$\epsilon\left(\bigwedge_{i\in I}\underline{S_i}\right) = \bigwedge_{i\in I}\epsilon(\underline{S_i}) = \bigwedge_{i\in I}\epsilon(\underline{T_i}) = \epsilon\left(\bigwedge_{i\in I}\underline{T_i}\right).$$

Thus $\bigwedge_{i\in I}[\underline{S_i}] = \bigwedge_{i\in I}[\underline{T_i}]$, and meets in E are well-defined. Of note, this definition of meets also yields a poset structure on E; $[\underline{S}] \leq [\underline{T}]$ if and only if $[\underline{S}] \wedge [\underline{T}] = [\underline{S}]$. As

E has a top element $[\underline{\Omega(L)}]$, that is, an empty meet, then this makes E a complete meet-semilattice, that is, a poset with a top element in which arbitrary meets are well-defined.

A complete meet-semilattice can be made into a complete lattice by defining joins in terms of meets.

Definition 7.5.3. For all families $([\underline{S}_i])_{i\in I}$ of elements of E, where all $\underline{S}_i \in Sub_{cl}\underline{\Omega(L)}$,

$$\bigvee_{i\in I}[\underline{S}_i] := \bigwedge\{[\underline{T}] \mid [\underline{S}_i] \leq \underline{T} \ \forall \ i \in I\}$$

The meet on the right hand side above is well-defined because E is a complete meet-semilattice, and this is clearly a least upper bound for the family $([\underline{S}_i])_{i\in I}$ with regards to the partial order on E defined above. Thus, E is a complete lattice. Note that in general,

$$\bigvee_{i\in I}[\underline{S}_i] \neq \left[\bigvee_{i\in I}\underline{S}_i\right].$$

It is only known that ϵ preserves meets, and in general ϵ does not preserve joins. If ϵ were to preserve both meets and joins, then it would be a surjective complete lattice homomorphism whose domain $Sub_{cl}\underline{\Omega(L)}$ is distributive. This would imply that its codomain L is also distributive, which is not true in general of orthomodular lattices.

The following is our main result, which is strongest result we could reasonably hope for relating the complete lattice structure of E and L.

Theorem 7.5.4. *There is a complete lattice isomorphism between E and L, and (f, g) is a pair of order isomorphisms.*

Proof. From Lemma 7.5.1 and its proof, we already know that there is a set isomorphism g from E to L with inverse f. We proceed to show that both g and f preserve arbitrary meets and joins, meaning that f and g are complete lattice isomorphisms.

First, we show that f and g preserve arbitrary meets. Consider g, and let $([\underline{S}_i])_{i\in I}$ be a family of elements of E.

$$g\left(\bigwedge_{i\in I}[\underline{S}_i]\right) = g\left(\left[\bigwedge_{i\in I}\underline{S}_i\right]\right) = \epsilon\left(\bigwedge_{i\in I}\underline{S}_i\right) = \bigwedge_{i\in I}\epsilon(\underline{S}_i) = \bigwedge_{i\in I} g\left([\underline{S}_i]\right)$$

Thus, g is a complete meet-semilattice homomorphism from E to L. In order to show that $f : L \to E$ is a complete meet-semilattice homomorphism as well, note that if $(a_i)_{i\in I}$ is a collection of elements of L, then

$$g\left(\bigwedge_{i\in I}[\underline{\delta^o}(a_i)]\right) = \bigwedge_{i\in I} g\left([\underline{\delta^o}(a_i)]\right) = \bigwedge_{i\in I}\epsilon\left(\underline{\delta^o}(a_i)\right) = \bigwedge_{i\in I} a_i$$

Then, as we already know that f and g are set isomorphisms and thus $f \circ g = id$, it follows that

$$f\left(\bigwedge_{i\in I} a_i\right) = f\left(g\left(\bigwedge_{i\in I}[\underline{\delta^o}(a_i)]\right)\right) = \bigwedge_{i\in I}[\underline{\delta^o}(a_i)] = \bigwedge_{i\in I} f(a_i).$$

It only remains to show that f and g preserve arbitrary joins. First, consider g, and let $([\underline{S}_i])_{i\in I}$ be a family of elements of E. Then,

$$\begin{aligned}\bigvee_{i\in I} g([\underline{S}_i]) &= \bigvee_{i\in I} \epsilon(\underline{S}_i) \\ &= \bigwedge\nolimits^{L} \{a \in L \mid \epsilon(\underline{S}_i) \leq a \ \ \forall\ i \in I\}\end{aligned}$$

The superscript on the meet above indicates that the meet is taken in L. As ϵ is surjective, then every $a \in L$ can be written as $\epsilon(\underline{A})$ for some $\underline{A}$ in $Sub_{cl}\underline{\Omega(L)}$. Thus,

$$= \bigwedge\nolimits^{L} \{\epsilon(\underline{A}) \in L \mid \epsilon(\underline{S}_i) \leq \epsilon(\underline{A}) \ \ \forall\ i \in I\}$$

Alternatively, note that

$$\begin{aligned}g\left(\bigvee_{i\in I} \underline{S}_i\right) &= g\left(\bigwedge\nolimits^{E} \{[\underline{T}] \in E \mid [\underline{S}_i] \leq [\underline{T}] \ \ \forall\ i \in I\}\right) \\ &= g\left(\bigwedge\nolimits^{E} \{[\underline{T}] \in E \mid \epsilon(\underline{S}_i) \leq \epsilon(\underline{T}) \ \ \forall\ i \in I\}\right) \\ &= g\left(\left[\bigwedge\nolimits^{Sub_{cl}\underline{\Omega(L)}} \left\{\underline{T} \in Sub_{cl}\underline{\Omega(L)} \mid \epsilon(\underline{S}_i) \leq \epsilon(\underline{T}) \ \ \forall\ i \in I\right\}\right]\right) \\ &= \epsilon\left(\bigwedge\nolimits^{Sub_{cl}\underline{\Omega(L)}} \left\{\underline{T} \in Sub_{cl}\underline{\Omega(L)} \mid \epsilon(\underline{S}_i) \leq \epsilon(\underline{T}) \ \ \forall\ i \in I\right\}\right) \\ &= \bigwedge\nolimits^{L} \{\epsilon(\underline{T}) \in L \mid \epsilon(\underline{S}_i) \leq \epsilon(\underline{T}) \ \ \forall\ i \in I\}\end{aligned}$$

From this, it follows that $\bigvee_{i\in I} g([\underline{S}_i]) = g\left(\bigvee_{i\in I} \underline{S}_i\right)$, meaning g preserves arbitrary joins and thus is a complete lattice homorphism. To see that f also preserves joins, we use the same argument as above. Let $(a_i)_{i\in I}$ be a collection of elements of L, then

$$g\left(\bigvee_{i\in I}[\underline{\delta^o}(a_i)]\right) = \bigvee_{i\in I} g\left([\underline{\delta^o}(a_i)]\right) = \bigvee_{i\in I} \epsilon\left(\underline{\delta^o}(a_i)\right) = \bigvee_{i\in I} a_i$$

Then, as we already know that f and g are set isomorphisms and thus $f \circ g = id$, it follows that

$$f\left(\bigvee_{i\in I} a_i\right) = f\left(g\left(\bigvee_{i\in I}[\underline{\delta^o}(a_i)]\right)\right) = \bigvee_{i\in I}[\underline{\delta^o}(a_i)] = \bigvee_{i\in I} f(a_i).$$

This demonstrates that $g = f^{-1}$ and both are isomorphisms of complete lattices, meaning L and E are isomorphic as complete lattices. □

The above theorem shows that the complete lattice structure of L is represented, in the form of E, in the clopen subobjects of its spectral presheaf. This can be seen as a generalization of Stone's representation theorem for Boolean algebras to the nondistributive case of complete orthomodular lattices. Of course, representing L in $Sub_{cl}\underline{\Omega(L)}$ required the use of the Galois connection $(\underline{\delta^o}, \epsilon)$, which was defined using the lattice structure of L.

A possible next step is to consider whether the orthocomplementation function of L is somehow represented in $Sub_{cl}\underline{\Omega(L)}$, and specifically, in E. One possibility is to consider the Heyting negation or co-Heyting negation of bi-Heyting algebra $Sub_{cl}\underline{\Omega(L)}$ as an orthocomplementation function on E. In either case, one would need to show that the chosen negation is well-behaved with respect to the equivalence classes of E and gives a true complement of each equivalence class, rather than an intuitionistic or paraconsistant complement.

Chapter 8

Conclusion: Generalizing the Spectral Presheaf

The exploration of the spectral presheaf of an orthomodular lattice in the previous sections provides some important clues about what properties of mathematical structures are necessary to define a spectral presheaf and to be able to prove an isomorphism result as in Theorem 5.7.3 and a representation result as in Theorem 7.5.4. Our exploration of the spectral presheaf of an orthomodular lattice has illuminated several key properties which are necessary to obtain such results.

On one hand, to define a spectral presheaf we need nonabelian or nondistributive structures that have some classical duality for their abelian or distributive parts, e.g. Gelfand duality or Stone duality. On the other hand, the poset of abelian/distributive parts, or contexts, must encode sufficient extra information to allow reconstructing algebraic operations also on noncommuting or incompatible elements. In the case of orthomodular lattices, the order structure is encoded by the context category in the sense of Proposition 4.0.5; if $a \leq b$ in an orthomodular lattice L, then there is some Boolean subalgebra $B \in \mathcal{B}(L)$ such that $a, b \in B$. This is not the case in general ortholattices, as we saw in Proposition 4.0.6. This implies that there are non-isomorphic ortholattices that have isomorphic context categories and hence isomorphic spectral presheaves, meaning the isomorphism result of Theorem 5.7.2 doesn't hold for arbitrary ortholattices. This condition also exists for the von Neumann algebra case, but is more subtle and requires the use of additional techniques to explore.

Additionally, any mathematical structure we choose must be covered by its contexts; otherwise, there can be no hope of reconstructing it up to isomorphism from its spectral presheaf. In the lattice case, this is ensured by considering only orthocomplemented lattices L; every element $a \in L$ has a complement a', and thus $\{0, a, a', 1\}$ is a Boolean subalgebra of L containing a. In the algebra case, any von Neumann algebra is covered by its abelian subalgebras; for any element a, the subalebra generated by a is abelian and contains a. Such a result must also hold for any structures on which we wish to define a spectral presheaf.

In Chapter 7, we defined a daseinisation map for an orthomodular lattice L that ultimately allowed us to construct a representation of orthomodular lattice L in the clopen subobjects of L's spectral presheaf modulo an equivalence relation. In order to define this daseinisation map and its adjoint, we had to consider complete orthomodular lattices and complete Boolean subalgebras because the definition of the daseinisation map contains infinite meets. Rui Soares Barbosa, in recent unpublished work, has shown how to define a daseinisation map for non-complete lattices. In future work, we are planning to consider this generalization, which will allow to us define daseinisation and hence an analog of Theorem 7.5.4 and Stone's representation theorem for arbitrary orthomodular lattices.

8.1 Compact Lie groups

One possible next step is to consider the spectral presheaf of a compact Lie group. Lie groups and their associated Lie algebras are closely related to many concepts in quantum mechanics. For example, they can be used to model the symmetry schemes of quantum particles or used to simplify quantum mechanical calculations such as solutions to the Schrödinger equation [7].

Beyond connections to quantum mechanics, compact Lie groups have mathematical structure that suggests the definition of a spectral presheaf would be possible. A compact Lie group has a context category consisting of its compact abelian Lie subgroups, ordered by inclusion. There is then a duality, known as Pontryagin (or Pontrjagin) duality, between a locally compact abelian Lie group and its dual group. The dual group has as elements the characters of G, which are group homomorphisms from G to the circle group T, as well as some additional topological structure. More information about dual groups and Pontryagin duality can be found in Chapter 4 of [28]. There is a clear analogy between Pontryagin duality, Stone duality, and Gelfand duality; all three involve considering topological spaces whose elements correspond to maps from the algebra or group to some fixed simple object.

Another important aspect of the spectral presheaf of a von Neumann algebra that one can define 'flows' on the spectral presheaf and interpret these flows as time-evolution of a quantum system [12]. There is a possible connection between the maximal compact connected abelian Lie subgroups of a compact Lie group, called *maximal tori*, and flows on the spectral presheaf of a Lie group. Any two maximal tori are conjugate to each other by some element of the Lie group, and hence maximal tori can be 'connected' by a flow. More information on maximal tori can be found in [3]. The possible existence of a structure with a nice relationship with flows is another motivation for considering the spectral presheaf of a Lie group as a next step.

8.2 Sober spaces and spatial frames

The duality between Boolean algebras and Stone spaces is a specific instance of the more general duality between spatial frames and sober spaces. A frame is a complete Heyting algebra, and a spatial frame is one that *has enough points* in the sense that the map sending the frame to the frame of open sets of its points is an isomorphism. A sober space is a topological space in which every irreducible closed subset is the closure of a unique point. There is a dual equivalence between the category of spatial frames and the category of sober spaces; more information on these structures and the dual equivalence between them can be found in [20].

It may be possible to exploit this dual equivalence to define a notion of spectral presheaf that is more general than the spectral presheaf of an orthomodular lattice. However, it is currently unclear which non-distributive 'quantum lattices' can be assembled from spatial frames such that the context category of a 'quantum lattice' has spatial frames as its objects while at the same time the order relations in the context category are sufficiently rich to reconstruct the 'quantum lattice.' Certain quantales may be candidates for such 'quantum lattices.'

A future goal is to develop some coherent framework that characterizes exactly what properties a mathematical structure must have in order to have a well-defined spectral presheaf for which duality and representation results can be derived. We hope to be able to provide a uniform way to prove isomorphism results such as Theorem 5.7.3 and construct daseinisation maps as in Section 7.3. In this dissertation, we have begun the exploration into generalizing the spectral presheaf, and we hope and expect that future research will lead to a more comprehensive picture.

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