ANNALS OF MATHEMATICS
Vol. 74, No. 1, July, 1961
Printed in Japan

ON LIMITS OF SEQUENCES OF OPERATORS*

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(Received August 3, 1960)

I. Introduction

1. The purpose of this paper is to bring to light a general principle governing the limit of sequences of operators of the type arising naturally in Fourier analysis. This principle will help to explain the reason for occurrences which have become commonplace in certain aspects of analysis, relate some hitherto unconnected results, and dispose of some open problems.

We begin by sketching some background.

For Fourier series of functions of one variable, $f(\theta)$,

$$f(\theta) \sim \sum c_n e^{in\theta} ,$$

a very basic operation is the passage to the so-called conjugate function $\tilde{f}(\theta)$,

$$\tilde{f}(\theta) \sim -i \sum \operatorname{sign}(n) c_n e^{in\theta} .$$

The linear operator which maps $f(\theta)$ to $\tilde{f}(\theta)$ has the property

$$\| \tilde{f}(\theta) \|_2 \leqq \| f(\theta) \|_2 ,$$

and more generally

$$\| \tilde{f}(\theta) \|_p \leqq A_p \| f(\theta) \|_p \qquad 1 < p < \infty . \tag{1.1}$$

However in the fundamental case $p = 1$, this inequality fails. The appropriate substitute result for this case is also a comparison of the "size" of $\tilde{f}(\theta)$ with that of $f(\theta)$. It is as follows. For any α, $\alpha > 0$, let

$$E_\alpha = \{\theta \mid |\tilde{f}(\theta)| > \alpha\} .$$

Then according to a theorem of Kolmogoroff

$$m(E_\alpha) \leqq (A/\alpha) \int_0^{2\pi} |f(\theta)| \, d\theta . \tag{1.2}$$

This result is fundamental in the theory. Moreover, as the Marcinkiewicz interpolation theorem shows (see e.g., [11, Ch. 12]), the result (1.2) together with the elementary result for $p = 2$, leads to the proof of the M. Riesz inequality (1.1). Several proofs of (1.2) are known. However, of special interest to us is the original proof given by Kolmogoroff

* Many of the results of this paper were announced in the Notices of the A.M.S., 1960, Nos. 566-33, and 566-34.

Sponsored in part by N.S.F. Grant G-4943.

(see [6]). Let

$$\tilde{f}(r, \theta) = -i \sum \operatorname{sign}(n) r^{|n|} c_n e^{in\theta} .$$

Then, by a known result, for every $f(\theta) \in L$, $\lim_{r \to 1} \tilde{f}(r, \theta)$ exists for almost every θ. Using this fact of the "existence" of the conjugate operator, Kolmogoroff proved, by an indirect method, that the limit operation satisfies the inequality (1.2).

Before we proceed to some other examples we wish to recall some definitions. An operator $f \to Tf$ is said to be of (strong) *type* (p, p) if it satisfies a norm inequality

$$\| Tf \|_p \leqq A \| f \|_p ,$$

where A does not depend on f. Similarly an operator is said to be of *weak type* (p, p) if for every $f \in L^p$ the function Tf satisfies the following restriction. Let $\alpha > 0$ and $E_\alpha = \{x \mid | Tf(x) | > \alpha\}$, then

$$m(E_\alpha) \leqq (A/\alpha^p) \int |f|^p dm .$$

The constant A does not depend on α or f. It should be noticed that if an operator is of type (p, p) it is also of weak type (p, p) but the converse is not necessarily true.

The notion of weak-type is important because, for various operations, the fact that they are of weak type $((p, p)$ (for some p)) is in effect a reformulation of their most fundamental real-variable property. A related reason is the Marcinkiewicz interpolation theorem, already mentioned. An important special case states the following: If a linear operator is simultaneously of weak type (p_1, p_1) and $(p_2\ p_2)$, where $1 \leqq p_1 < p_2 \leqq \infty$, then it is of (strong) type (p, p) for every p, $p_1 < p < p_2$.

We now return to some other examples.

Let us consider the family of operators T_h,

$$T_h(f)(\theta) = \frac{1}{h} \int_0^h f(\theta + t) dt , \qquad h > 0 .$$

Then by the classical theorem of Lebesgue, if f is integrable $T_h(f)(\theta) \to f(\theta)$, as $h \to 0$, for almost every θ. It was not until much later that Hardy and Littlewood were led to consider their "maximal" function $f^*(\theta)$ defined, in effect, by

$$f^*(\theta) = \sup_{h>0} | T_h(f)(\theta) |$$

for which they proved the following theorem

$$\| f^*(\theta) \|_p \leqq A_p \| f(\theta) \|_p \qquad 1 < p .$$

Implicit in their argument (and later brought to light by F. Riesz) is the inequality

$$m(E_\alpha) = m\{\theta \,|\, f^*(\theta) > \alpha\} \leqq (A/\alpha)\int |f(\theta)|\, d\theta \,.^1$$

This last result (which is really the basic result for $f^*(\theta)$) may be stated by saying that the mapping $f(\theta) \to f^*(\theta)$ is of weak type (1.1). (Of course, the result of Kolmogoroff, (1.2) above, has a similar statement.)

We mention one more example. It deals with the yet unsolved problem of the existence of an $f \in L^2(0, 2\pi)$ whose Fourier series diverges almost everywhere. This result, due to A. P. Calderon (see [11, p. 165, II]) is of a conditional nature. Suppose we knew that every $f \in L^2(0, 2\pi)$ has an almost everywhere convergent Fourier series. Let $S^*(\theta) = \sup_{n \geqq 1} |S_n(f)(\theta)|$, where the S_n are the n^{th} partial sums. Then, we would have

$$m(E_\alpha) = m\{\theta \mid S^*(\theta) > \alpha\} \leqq (A/\alpha^2) \cdot \int_0^{2\pi} |f(\theta)|^2 d\theta \,.$$

2. The examples mentioned above, together with others, illustrate the general result obtained in this paper. Fix p, $1 \leqq p \leqq 2$.[2] Let T_m be a sequence of bounded linear operators of $L^p(0, 2\pi)$ to itself. Suppose, in addition, that each T_m commutes with translations. We now make the crucial assumption that for every $f \in L^p$, $\lim_{m \to \infty}(T_m f)(\theta)$ exists almost everywhere. Let

$$(T^*f)(\theta) = \sup_{m \geqq 1} |(T_m f)(\theta)| \,.$$

Then

$$m(E_\alpha) = m\{\theta \mid T^*f(0) > \alpha\} \leqq (A/\alpha^p) \cdot \int_0^{2\pi} |f(\theta)|^p d\theta \,.$$

A does not depend on α or f.

We shall prove this theorem and several variants in §§ 8–11 below. The theorem described above is most properly formulated for any compact group, and more generally for the homogeneous space of a compact group. The extension is important for the applications given below. Among the cases considered will be the k-torus, which is the underlying space of multiple Fourier series; also the so-called "dyadic group" which is the appropriate space for Walsh-Paley expansions. We now sketch some of these applications.

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1 For a discussion of these facts see [11, pp. 29-33, I].

2 The restriction $1 \leqq p \leqq 2$ is essential for the conclusions reached below. There are, however, variants of our result which hold for more general Banach spaces. These are considered in the appendix.

3. Let $f(x, y)$ be a function of two variables integrable over the torus $0 \leqq x < 2\pi$, $0 \leqq y < 2\pi$, with Fourier expansion

$$f(x, y) \sim \sum a_{nm} e^{inx} e^{imy} .$$

We form (in a formal way) the "double conjugate" series

$$-\sum \operatorname{sign}(n) \operatorname{sign}(m) a_{nm} e^{inx} e^{imy} ,$$

and ask whether this double conjugate exists in a suitable sense.

One natural approach is to consider the Abel means of the above series

$$\tilde{f}(r, x; \rho, y) = -\sum \operatorname{sign}(n) \operatorname{sign}(m) a_{nm} r^{|n|} e^{inx} \rho^{|m|} e^{imy} \tag{3.1}$$

and inquire whether $\lim_{\substack{r \to 1 \\ \rho \to 1}} \tilde{f}(r, x; \rho, y)$ exists.

Now if $f \in L^p$, $p > 1$, (even $f \in L \log L$) it is known that this limit exists almost everywhere.[3] This situation has an analogy with differentiation of double integrals. In fact if $f \in L \log L$, it is known that

$$\lim_{\substack{h \to 0 \\ k \to 0}} \frac{1}{hk} \int_0^h \int_0^k f(x + t, y + s) dt ds = f(x, y) \tag{3.2}$$

for almost every (x, y). However if one merely assumes that $f \in L$ then, as is known, the limit (3.2) may fail to exist almost everywhere. But if we assume that, for example, $h = k$, then the limit (3.2) exists almost everywhere.[4] Going back to double conjugates (3.1), it was believed (in analogy with the above) that $\lim_{r \to 1} f(r, x; r, y)$ exists for almost every (x, y).

Contrary to expectations, we shall show that there exists an $f \in L$ so that $\lim_{r \to 1} \tilde{f}(r, x; r, y)$ fails to exist for all (x, y) in a set of full measure. A refinement of the argument will show that even if we assume that $f \in L(\log L)^{1-\varepsilon}$, $\varepsilon > 0$, this limit may fail to exist almost everywhere. These and other results related to multiple conjugates are given in § 13 below.

4. We shall now consider the divergence of Fourier series in one variable, and the divergence of Walsh-Paley expansions. According to a result of Kolmogoroff, there exists an integrable f whose Fourier series diverges almost everywhere.[5] The proof of this theorem is, to say the least, extremely difficult. Our general results allow us to obtain a simplification and refinement of this result. This refinement may be understood in terms of the following fact.

If $f(x)$ is integrable, then for almost every x

[3] See [10], for these results.

[4] See e.g., [7].

[5] See [11, p. 305, I].

$$S_n(x, f) - S_m(x, f) = O(\log (m - n)) , \qquad m, n \to \infty .$$

In fact the above holds at each point x of the Lebesgue set of f. Our refinement of the Kolmogoroff example is the following: let $\lambda(n)$ be any function tending to zero as $n \to \infty$. Then there exists an integrable f so that the restriction

$$S_n(x, f) - S_m(x, f) = O(\lambda(m - n) \log (m - n))$$

is false for almost every x. This is done in § 14 below. There we shall also obtain the existence of an integrable function whose Walsh-Paley expansion diverges almost everywhere. While there are many analogies between Fourier series and Walsh series, this particular problem is more complicated for Walsh series on two counts. First, if $S_n(x, f)$ are the partial sums of the Walsh expansion, then $\lim_{k\to\infty} S_{2^k}(x, f)$ exists almost everywhere. This is not the case for Fourier series. Secondly, we have estimates for the Dirichlet kernel of Fourier series, whose analogues are not available for Walsh series. However, as will be seen, the method we shall use will work equally well for Walsh and Fourier series.

5. We next turn to multiple Fourier series summed by spherical means. More particularly let

$$f(x) \sim \sum a_n e^{in\cdot x} , \qquad x = (x_1, \cdots, x_k)$$

be a Fourier expansion on the k-torus, and

$$S_R(x, f) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{(k-1)/2} a_n e^{in\cdot x}$$

be the spherical means of index $(k - 1)/2$. The importance of the index $(k - 1)/2$ is due to the fact that for many problems summability of index $(k - 1)/2$ is the proper analogue of convergence, when $k = 1$. See [1] and [9].

In § 15 we shall prove that there exists an f, integrable over the k-torus whose Fourier series is almost everywhere not summable by the means of order $(k - 1)/2$. The result for $k \geqq 2$ has special interest because, as is known by [9], if we assume slightly more than integrability (that is $|f| (\log^+ |f|)^2$ is integrable) then the means of order $(k-1)/2$ converge almost everywhere.

6. As a last application, we shall show that our general result, described in § 2 above, can be used to prove results of a positive nature, and thus its use is not exclusively one of supplying counter-examples. We shall concern ourselves with the "multiplier problem" for Fourier series in one variable. The problem is that of characterizing the sequences

λ_n of multipliers which lead to bounded transformation T on L^p into itself given by

$$Tf(x) \sim \sum c_n \lambda_n e^{inx}$$

when

$$f(x) \sim \sum c_n e^{inx} .$$

The solution of this problem is known only if $p = 1$, $p = 2$, and $p = \infty$.

We shall find conditions which are necessary and nearly sufficient, in terms of the function

$$K(x) \sim \sum \frac{\lambda_n e^{inx}}{in} .$$

These results are given in § 16. Whether these conditions turn out to be the most useful possible (and this is a practical problem) remains yet to be seen.

The author wishes to express his indebtedness to A. Zygmund and G. Weiss for many valuable suggestions which have been incorporated into this paper. In fact, at many points, it would be impossible to disengage their contributions from the results obtained here. This is particularly so in § 11, where the general result is extended to Orlicz spaces, and in § 16 where the multiplier problem is discussed.

II. General theorems

7. *Preliminary lemmas.* We shall consider the following general situation. Let G be a compact group, and M a homogeneous space of G. That is, G acts transitively on M, and the structure imposed on M is the one inherited from G. In particular M has a unique normalized measure dm, which is invariant under G. Examples of this situation of particular relevance are as follows:

(1) $G = M = k$-torus, and G acts on M in the natural way.

(2) $G = M =$ dyadic group, G acting on M in the natural way.[6]

(3) $G =$ group of rotations in n variables and M the unit sphere, with G acting on M in the usual way.

We have mentioned that M has a unique invariant measure dm, normalized by $\int_M dm = 1$. G also has a unique (so-called Haar) measure, dg, normalized by $\int_G dg = 1$.

It is worthwhile to recall the connection of dm with dg. Let E be any Borel set in M. Fix a point $p \in M$, and let E be the corresponding set in G, that is

[6] A more detailed discussion of this case will be given in § 14.

$$\hat{E} = \{g \in G \mid g(p) \in E\} .$$

Then

$$(7.1) \qquad m(E) = \text{Haar measure of } \hat{E} \text{ in } G .$$

LEMMA 1.[7] *Let $E_1, E_2, \cdots, E_n, \cdots$ be a collection of sets in M, with the property that $\sum m(E_n) = \infty$. Then there exists a sequence of elements $g_1, g_2, \cdots, g_n, \cdots$ belonging to G, so that the translated sets $F_1, F_2, \cdots, F_n, \cdots, F_n = g_n(E_n)$, have the following property: Let*

$$F_0 = \limsup F_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} F_n .$$

Then $m(F_0) = 1$. That is, almost every point of M belongs to infinitely many sets F_n.

REMARKS. (i) If the sets E_n were mutually independent then the above would reduce to the Borel-Cantelli lemma (see [2, p. 104]; we then could take all g_i = identity).

(ii) The proof will show that the above conclusion may be obtained for "almost all" sequences $g_1, g_2, \cdots, g_n, \cdots$ and not just a particular sequence $g_1, g_2, \cdots, g_n, \cdots$.

PROOF. We consider two infinite product spaces. First, let $\mathcal{M} = \prod_{k=1}^{\infty} M_k$, where $\mathcal{M}$ is the infinite product of M_k's, each M_k being a copy of M. Thus the points of $\mathcal{M}$ are sequences $\{x_n\}$, where $x_n \in M$. We impose on $\mathcal{M}$ the usual product measure of the measures dm on each M_k. We call this product measure dm, also. Next we consider the infinite product group $\Gamma = \prod_{k=1}^{\infty} G_k$, where Γ is the infinite product of G_k, each G_k is a copy of G. The elements of Γ are sequences $\{g_n\}$, where $g_n \in G$. On Γ we also consider the usual product measure. We notice that Γ is a compact group which acts on $\mathcal{M}$ in a natural way; that $\mathcal{M}$ is therefore a homogeneous space under Γ; and that the product measure of $\mathcal{M}$ and Γ are, in fact, their invariant measures.

Consider now the collection of sets $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n, \cdots$ on $\mathcal{M}$ defined as follows. $\mathcal{E}_n$ = set of all points in $\mathcal{M}$ whose n^{th} coordinate is restricted to lie in E_n. Then $m(\mathcal{E}_n) = m(E_n)$. Let $\mathcal{E}_0 = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \mathcal{E}_n$. We claim that $m(\mathcal{E}_0) = 1$.

In fact, ${}^c\mathcal{E}_0 = \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} {}^c\mathcal{E}_n$, and thus $m({}^c\mathcal{E}_0) \leqq \sum_k m(\bigcap_{n=k}^{\infty} {}^c\mathcal{E}_n)$. But $m(\bigcap_{m=k}^{\infty} {}^c\mathcal{E}_n) = \prod_{n=k}^{\infty} m({}^c\mathcal{E}_n) = \prod_{n=k} (1 - m(\mathcal{E}_n)) = \prod_{n=k}^{\infty} (1 - m(E_n)) = 0$, since $\sum m(E_n) = \infty$.

Therefore $m({}^c\mathcal{E}_0) = 0$, and hence $m(\mathcal{E}_0) = 1$. (The argument up to this point is that of the Borel-Cantelli lemma, applied to the sets $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n, \cdots$).

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[7] A basic special case of this lemma is due to A. P. Calderon. His proof is in [11, p. 165, II].

Let now $\psi(x_1, x_2, \cdots, x_n, \cdots)$ be the characteristic function of the set $\mathcal{E}_0$. Consider the function $f(\gamma, x)$ defined on $\Gamma \times M$ as follows

$$f(\gamma, x) = \psi\big(g_1^{-1}(x), g_2^{-1}(x), \cdots, g_n^{-1}(x), \cdots\big)$$

where $\gamma = \{g_n\} \in \Gamma$, and $x \in M$. Let us apply the relation (7.1) to the case where Γ is the group, $\mathcal{M}$ the homogeneous space, $p = (x, x, x, \cdots x, \cdots)$, and $E = \mathcal{E}_0$. We then have that for each x, $f(\gamma, x) = 1$, for almost every γ. Hence, by Fubini's theorem, for almost every γ, $f(\gamma, x) = 1$ for almost every x. Fix such a γ, $\gamma = \{g_n\}$.

Therefore for almost every $x \in M$, $\gamma^{-1}(x) \in \mathcal{E}_n$ for infinitely many n. That is, for almost every $x \in M$, $g_n^{-1}(x) \in E_n$ for infinitely many n. Hence, for almost every $x \in M$, $x \in g_n(E_n)$, for infinitely many n. This proves the lemma.

We next consider the Rademacher functions $r_n(t)$. Their definition is given by

$$r_n(t) = r_0(2^n t) ,$$

$r_0(t) = 1$ if $0 \leqq t \leqq 1/2$, and -1, if $1/2 < t < 1$, and $r_0(t+1) = r_0(t)$.

These functions are orthonormal over $[0, 1]$ (although not complete). Their importance is due to the fact that they are mutually independent. We shall not make explicit use of this fact, but instead we shall use a property that, in effect, follows from this. Let

$$F(t) = \sum a_n r_n(t) , \qquad \sum |a_n|^2 < \infty \tag{7.2}$$

be a Rademacher series.

LEMMA 2. *Let E be any measurable subset of* $[0, 1]$ *whose Lebesgue measure is not zero. Then there exists an integer N, and a constant A (both depending only on E) so that*

$$\Big(\sum_{n \geqq N} |a_n|^2\Big)^{1/2} \leqq A \operatorname{ess\,sup}_{t \in E} |F(t)| . \tag{7.3}$$

This lemma is well-known, see [11, p. 213, I].

8. *General Theorems*. As before, G is a compact group, M a homogeneous space of G, and dm the G-invariant measure on M. If x is any point of M, and g an element of G, then $g(x)$ denotes the action of g on x. We also define the operator τ_g by

$$(\tau_g f)(x) = f\big(g^{-1}(x)\big) , \qquad g \in G .$$

We shall refer to the family of operators τ_g, $g \in G$ of *translations*.

Now suppose we are given a sequence of linear operators T_m satisfying the following two properties. Fix p, $1 \leqq p \leqq 2$:

(a) Each T_m is a bounded linear operator of $L^p(M)$ into itself.

(b) Each T_m commutes with translations, that is $T_m \tau_g = \tau_g T_m$, all

$g \in G$.

The following theorem is the main result of this paper.

THEOREM 1. *Let* T_m *be a sequence of operators satisfying conditions* (a) *and* (b) *above. Suppose that for every* $f \in L^p(M)$,

(8.1) $$\lim_{m\to\infty} T_m(f)(x)$$

exists for almost every $x \in M$.

Let

$$T^*(f)(x) = \sup_{m\geq 1} |(T_m f)(x)| \ .$$

Then there exists a constant A, *so that if* $\alpha > 0$ *and* $E_\alpha = \{x \mid T^*f(x) > \alpha\}$, *then*

(8.2) $$m(E_\alpha) \leqq (A/\alpha^p)\int |f|^p dm \qquad (A \textit{ is independent of } \alpha \textit{ and } f.)$$

Before proving the theorem, we state some corollaries which follow immediately from the proof of Theorem 1 given below.

COROLLARY 1. *In place of* (8.1) *we may make the weaker assumption: for every* $f \in L^p$, $\limsup_{m\to\infty} |T_m(f)(x)| < \infty$, *for some set of* x *of positive measure*; (*this set may depend on* f). *Then the conclusion* (8.2) *still holds.*

An alternative formulation of Corollary 1 is the following:

COROLLARY 2. *If the inequality* (8.2) *fails* (*that is, if the operation* $f \to T^*f$ *is not of weak type* (p, p)) *then there exists an* $f \in L^p$, *so that* $\limsup_{m\to\infty} |T_m(f)(x)| = \infty$ *for almost every* x.

Another corollary is

COROLLARY 3. *Let* $\mathcal{S}$ *be a closed subspace of* $L^p(M)$, *invariant under translations. Suppose that the operation* $f \to T^*f$, *restricted to* $\mathcal{S}$, *does not satisfy an inequality of type* (8.2). *Then there exists an* $f \in \mathcal{S}$ *so that* $\limsup_{m\to\infty} |T_m(f)(x)| = \infty$, *for almost every* x.

9. We now pass to the proof of Theorem 1 and its corollaries. The argument is by contradiction. Let us assume that there is no A, $A > 0$, for which the inequality (8.2) holds. Thus for each integer n, the inequality (8.2) is false for n in place of A, for some $f \in L^p$, and some $\alpha > 0$. Hence, after appropriate normalization, there exists a sequence of functions f_n, and a collection of sets E_n, so that

(9.1) $$(T^*f_n)(x) > 1 \ , \qquad x \in E_n$$

(9.2) $$m(E_n) > n\,||f_n||_p^p = n\int_M |f_n|^p dm \ .$$

By taking a sub-collection of the original collection f_n, E_n, with possible repetitions, it is possible to obtain another collection (which we again index by f_n, E_n) and which satisfies the following:

$$(T^* f_n)(x) > 1 \,, \qquad x \in E_n \tag{9.1*}$$

$$\sum_{m=1}^{\infty} m(E_n) = \infty \quad \text{while} \quad \sum_{n=1}^{\infty} \| f_n \|_p^p < \infty \,. \tag{9.2*}$$

Finally, in view of the convergence of the positive numerical series $\sum \| f_n \|_p^p$, we can find a sequence R_n of positive number so that

$$R_n \to \infty \quad \text{but} \quad \sum \| R_n f_n \|_p^p < \infty \,. \tag{9.3}$$

Let us remember that because of Lemma 1, and (9.2*), there exists a sequence of $g_n \in G$ so that if $F_n = g_n(E_n)$, then almost every point of M is contained in infinitely many F_n. Fix such a sequence g_n once and for all.

We now consider series given (formally) by

$$F(x, t) = \sum r_n(t) R_n \tau_{g_n}(f_n)(x) \,. \tag{9.4}$$

Here the $r_n(t)$ are the Rademacher functions (see § 7 above), and the operators τ_{g_n} are translations by the elements g_n.

We shall prove that this series converges in an appropriate sense, and has two properties:

(1) For almost every t, $0 \leqq t \leqq 1$, $F(x, t) = F_t(x)$ as a function of x, is in $L^p(M)$.

(2) For almost every t, $(T^* F_t)(x) = \infty$, for almost every $x \in M$. If we prove (1) and (2) we shall, of course, have obtained a contradiction with the assumptions of the Theorem 1, and thus proved Theorem 1.

Let

$$F_N(x, t) = \sum_{n=1}^{N} r_n(t) R_n \tau_{g_n}(f_n)(x) \,. \tag{9.5}$$

Since this is a finite sum, it is a well-defined measurable function (on the product $M \times I$, I denoting the unit interval).

We consider first:

$$\int_I \int_M | F_{N_1}(x, t) - F_{N_2}(x, t) |^p dx dt \,.$$

We need the following observation. Let

$$F(t) = \sum_{N_1}^{N_2} b_n r_n(t) \,,$$

then

$$\int_0^1 | F(t) |^p dt \leqq \sum_{N_1}^{N_2} | b_n |^p \,, \qquad 1 \leqq p \leqq 2 \,. \tag{9.6}$$

In fact,

$$\int_0^1 |F(t)|^p dt \leqq \left(\int_0^1 |F(t)|^2 dt\right)^{p/2} = (\textstyle\sum_{N_1}^{N_2} |b_n|^2)^{p/2} \leqq \sum_{N_1}^{N_2} |b_n|^p$$

if $1 \leqq p \leqq 2$, which proves (9.6). Applying (9.6) and Fubini's theorem we obtain

$$\int_I \int_M |F_{N_1}(x, t) - F_{N_2}(x, t)|^p dx dt \leqq \textstyle\sum_{N_1}^{N_2} \| R_n f_n \|_p^p .$$

(Notice $\| f_n \| = \| \tau_{g_n}(f_n) \|_p$).

In view of the convergence of the series $\sum \| R_n f_n \|_p^p$, we see that the sequence $F_N(x, t)$ is Cauchy in $L^p(M \times I)$. Hence there exists an $F(x, t) \in L^p(M \times I)$ so that $F_N(x, t) \to F(x, t)$ in the norm of $L^p(M \times I)$. Also there exists a subsequence N_k so that $F_{N_k}(x, t) \to F(x, t)$ in the norm of $L^p(M)$, for almost every $t \in I$.

Now consider $F(x, t)$ as a function of x. By Fubini's theorem $F(x, t) \in L^p(M)$, for almost every $t \in I$. Let us apply the operator T_m (for some fixed m). Then, $T_m(F_{N_k}(x, t)) \to T_m(F(x, t))$, in the $L^p(M)$ norm, for almost every $t \in I$, since each T_m is a bounded operator on $L^p(M)$. Hence,

(9.7) $$T_m(F(x, t)) = \sum r_n(t) R_n T_m(\tau_{g_n}(f_n))(x) .$$

The series above converges in the norm of $L^p(M \times I)$. Moreover there is a subsequence which converges in the norm of $L^p(I)$, for almost every $x \in M$. These facts may be proved as in the argument above.

Our next claim is the following. The function on the left side of (9.7) is, for almost every $x \in M$, in $L^2(I)$, and is represented by the Rademacher series which appears on the right side of (9.7). In view of what has been said above, it is sufficient to notice, that for almost every x,

(9.8) $$\textstyle\sum_n R_n^2 | T_m(\tau_{g_n}(f_n))(x) |^2 < \infty .$$

This may be seen as follows:

First $\sum \| R_n f_n \|_p^p < \infty$, by our construction. Next $\sum \| R_n T_m \tau_{g_n}(f_n) \|_p^p < \infty$, since $\| \tau_{g_n}(f_n) \|_p = \| f_n \|_p$ and each T_m is a bounded operator. We can rewrite this as follows:

$$\textstyle\sum_n \int_M | R_n T_m \tau_{g_n} f_n(x) |^p dm < \infty .$$

Thus

$$\textstyle\sum_n | R_n T_m \tau_{g_n} f_n(x) |^p < \infty , \qquad \text{for almost every } x .$$

Since $1 \leqq p \leqq 2$, (9.8) is proved.

There is an exceptional set of measure zero, for each m. But since our family T_m is countable, we may assume once and for all that we disregard the set of measure zero, so that in its complement $T_m F(x, t)$ as a function of t, is given by the Rademacher series (9.7).

Suppose now, contrary to (2), that $T^*F(x, t) < \infty$ for a set of positive (x, t) measure. Then there exists a set $\mathcal{S}$ of positive measure in $M \times I$, where $T^*F(x, t)$ is bounded say by A. That is, $T^*F(x, t) \leqq A$, $(x, t) \in \mathcal{S}$. And therefore, for every m

$$(9.9) \qquad | T_m F(x, t) | \leqq A , \qquad \text{all } m, (x, t) \in \mathcal{S} .$$

Let now $E_x = \mathcal{S} \cap (x, I)$. E_x may be considered for each $x \in M$, a subset of I. Since $\mathcal{S}$ has positive measure in $M \times I$, E_x is Lebesgue measurable for almost every x, and for a set $M_0 \subset M$, $| E_x | > 0$, (where $| \cdot |$ denotes the Lebesgue measure), $x \in M_0$, and M_0 is of *positive* measure in M. Let us now apply Lemma 2, to the case of the Rademacher series (9.7), remembering (9.8) and (9.9); the set E will be E_x, where $x \in M_0$. We then obtain

$$\left(\sum\nolimits_{n \geqq N(x)} | R_n T_m \tau_{g_n}(f_n)(x) |^2\right)^{1/2} \leqq A(x) , \qquad x \in M_0$$

where $A(x)$ and $N(x)$ are independent of m.

Hence

$$(9.10) \qquad | R_n T_m \tau_{g_n} f_n(x) | \leqq A(x) , \qquad x \in M_0$$

if $n \geqq N(x)$.

Now $T_m \tau_{g_n} = \tau_{g_n} T_m$. Taking the sup (over m) of the left side of (9.10), we obtain

$$R_n(T^* f_n)(g_n^{-1}(x)) \leqq A(x) , \qquad x \in M_0$$

whenever $n \geqq N(x)$.

Now if $x \in F_0$ (see Lemma 1), then x is contained in infinitely many F_n, $F_n = g_n(E_n)$. But $T^*f_n(y_n) > 1$, if $y_n \in E_n$. Therefore $x \in F_0$, implies $R_n T^* f_n(g_n^{-1}(x)) > R_n \to \infty$, for infinitely many n's. Hence if $x \in F_0$, these are infinitely many n so that:

$$R_n T^* f_n(g_n^{-1}(x)) > A(x) .$$

Thus (9.10) implies $x \notin F_0$. This shows that $M_0 \subset {}^cF_0$, and therefore $m(M_0) = 0$, contrary to what was found earlier.

We have therefore proved that the function $F(x, t)$ satisfies the two conditions (1) and (2) above: that $F(x, t) \in L^p(M)$, for almost every $t \in I$, and (2), that $T^*F(x, t) = \infty$, for almost every (x, t). This leads to a contradiction with the hypothesis of Theorem 1, unless an inequality of the type (8.2) holds for some A. This concludes the proof of Theorem 1. Corollaries 1 and 2 are proved by noticing that we have actually constructed an $f \in L^p$ (that is, $F(x, t)$, for almost every $t \in I$), such that $T^*f(x) = \infty$ for almost every x.

Finally we come to Corollary 3. Suppose that the inequality (8.2) is false for some $f \in \mathcal{S}$, whatever the choice of A. Arguing as before we can consider a collection f_n, where now $f_n \in \mathcal{S}$. In view of the fact that $\tau_{g_n}(f_n) \in \mathcal{S}$, if $f_n \in \mathcal{S}$, and the convergence of the series (9.4) in the norm, we obtain $F(x, t) \in \mathcal{S}$, for almost every $t \in I$. Since $T^*F(x, t) = \infty$ for almost every (x, t), we have proved Corollary 3.

10. *A variant of the general theorem for $p = 1$.* We shall now obtain a variant of Theorem 1, for the case $p = 1$, which is particularly useful in applications. In order not to add irrelevant technical considerations, we shall limit ourselves (in this section) to the case where $G = M =$ commutative (and compact) group. The argument given below can be extended, with some care, to the general case of G and M considered above.

We need to introduce some notation. $x, y, \cdots$ designate points of M, and $x + y$ is the sum (of the group operation) in M. $L^p(M)$ will designate as before the Lebesgue space with respect to the invariant measure dm. $C(M)$ = class of continuous functions on M, with the "sup" topology. Finally $\mathcal{B}(M)$ will designate the (finite) Borel measure on M, having the usual norm.

We shall consider, as before, a sequence of operators T_m. We shall assume

(a′) Each T_m is a bounded operator from $L^1(M)$ to $C(M)$.

(b′) Each T_m commutes with translations.

It may be proved, although we omit the elementary argument,* that the conditions (a′) and (b′) are equivalent with condition (c′)

(c′) $T_m(f)(x) = \int_M k_m(x - y)f(y)dy$

where k_m is some function in $L^\infty(M)$.

Let us note that such an operator T_m has a natural extension to a bounded operator from the Borel measures, $\mathcal{B}(M)$, to $L^\infty(M)$. Notice that this extension also commutes with translations. We maintain the designation T_m for this extension.

Our result is then

THEOREM 2. *Let T_m be a sequence of operators of the type described above. Suppose that for every $f \in L^1(M)$, $\limsup_{m\to\infty} | T_m(f)(x) | < \infty$, for some set of x of positive measure (which may depend on f). Then there exists an absolute constant A with the following property. Let $d\mu$ be any Borel measure,*

$$(T^*d\mu)(x) = \sup_{m\geq 1} | T_m(d\mu)(x) | \quad \textit{and} \quad E_\alpha = \{x \mid (T^*d\mu)(x) > \alpha\} .$$

* The argument is based on the F. Riesz representation of bounded linear functionals on L^1.

Then

$$m(E_\alpha) \leqq (A/\alpha)\int_M |d\mu| . \tag{10.1}$$

We shall deduce Theorem 2 from Theorem 1, by the aid of the following lemma.

LEMMA. *Let $T_1, \cdots, T_N$, be a finite collection of operators, each satisfying the conditions* (a′) *and* (b′) *above. Let $d\mu$ be a Borel measure, so that $\|d\mu\| \leqq M$. Let $h_m(x) = T_m(d\mu)$. Then there exists a sequence of functions $f_1(x), f_2(x), \cdots, f_k(x), \cdots, f_k(x) \in L^1(M)$, so that $\|f_k(x)\|_1 < M$, and if $h_m^k(x) = T_m(f_k)$, then $\lim_{k\to\infty} h_m^k(x) = h_m(x)$ for $m = 1, \cdots, N$, for almost every x.*

PROOF. Let $l_k(x)$ be a sequence of non-negative continuous functions with the property that $\int_M l_k(x)dx = 1$, and so that the sequence l_k forms an approximation to the identity, in the usual sense. Let $f_k(x) = l_k^* d\mu = \int_M l_k(x-y)d\mu(y)$. Then each $f_k(x)$ is integrable and $\|f_k\|_1 \leqq \|d\mu\| \leqq M$. Moreover, owing to the fact that each T_m commutes with translations, we obtain

$$T_m(f_k) = T_m(l_k^* du) = l_k^*(T_m d\mu) = l_k^* h_m(x) .$$

Now by a well-known argument $l_k^* h_m(x) \to h_m(x)$ in the L^1 norm, as $k \to \infty$. Hence, selecting a subsequence of the l_k (if necessary) we obtain the desired pointwise convergence. This proves the lemma.

PROOF OF THEOREM 2. Let $T_N^*(x) = \sup_{1\leqq m\leqq N} |T_m(d\mu)(x)|$, and let $E_\alpha^N = \{x \mid T_N^*(x) > \alpha\}$. Notice that the sets E_α^N are increasing with N. Now by Theorem 1 (Corollary 1), there exists a constant A, so that if $f \in L^1(M)$

$$m\{x \mid \sup_{1\leqq m\leqq N} |T_m(f)(x)| > \alpha\} \leqq (A/\alpha)\int_M |f(x)|\, dm , \tag{10.2}$$

with A independent of N.

Apply this to the functions f_k, given in the above lemma. We then obtain

$$m(E_\alpha^N) = m\{x \mid T_N^* d\mu)(x) > \alpha\} \leqq (A/\alpha)\int_M |d\mu| .$$

Now let $N \to \infty$, and we obtain Theorem 2.

11. *A variant for Orlicz spaces.* Theorem 2 above is useful in showing that certain limits of sequences of operators may fail to exist when applied to $L^1(M)$. However, in some cases these limits exist when applied to functions in $L^p(M)$, $p > 1$. It is natural in those cases to consider other classes of functions, such as $L \log L$, which are intermediate between L^1 and the L^p. We shall now show how the general theorem of § 8 may be

modified to contain these cases.

We consider an Orlicz space L_Φ. It is characterized by a function Φ satisfying

(1) $\Phi(0) = 0$, $\Phi(t)$ is convex and increasing in $0 \leqq t < \infty$.

(2) $\Phi(2t) \leqq M\Phi(t)$.

Under these hypotheses the space of functions, L_Φ,

$$L_\Phi = \left\{ f \text{ measurable} \,\middle|\, \int_M \Phi(|f|)dm < \infty \right\},$$

can be made into a Banach space *via* the norm, $||\cdot||_\Phi$, $||f||_\Phi = \inf \lambda > 0$,

$$\int_M \Phi\left(\frac{|f|}{\lambda}\right)dm \leqq \Phi(1).$$

The case $\Phi(t) = t^p$, $1 \leqq p$, leads to the usual L^p space. Another case of special interest is $\Phi(t) = t(\log(t+c))^q$, $q \geqq 0$, which leads to the space $L(\log L)^q$.

We shall find it convenient to use the following definitions. An operator T is of type (Φ, Φ) if there exists a constant A so that

$$\int_M \Phi(|Tf|)dm \leqq \int_M \Phi(A|f|)dm. \tag{11.1}$$

Similarly we shall define T to be of weak type (Φ, Φ), if $E_\alpha = \{x \mid |Tf(x)| > \alpha\}$, then

$$m(E_\alpha) \leqq \int_M \Phi\left(\frac{A|f|}{\alpha}\right)dm, \tag{11.2}$$

where A is some constant independent of α or f.

Now in Theorem 1, we made the restriction (which is essential), that $1 \leqq p \leqq 2$. This is a restriction on the character of Φ, and in the general case we shall assume:

(3) $\Phi(t^{1/2})$ is a *concave* function of t, $0 \leqq t < \infty$.[8]

Our general theorem then is

THEOREM 3. *Let T_m be a sequence of operators each of which is of type* (Φ, Φ) *(as defined in* (11.1) *above), and which commutes with translations. Let Φ satisfy the conditions* (1), (2), *and* (3) *above. Suppose that for every f, $f \in L_\Phi$,* $\limsup_{m\to\infty} |(T_m f)(x)| < \infty$, *for x in some set of positive measure. Let* $T^*(f)(x) = \sup_{m>1} |T_m f(x)|$. *Then there exists a constant A, so that*

$$m(E_\alpha) = m\{x \mid T^*f(x) > \alpha\} \leqq \int_M \Phi\left(\frac{A}{\alpha}|f|\right)dm. \tag{11.3}$$

[8] It may be verified that condition (3) on Φ implies condition (2).

The proof of this theorem is so similar to that of Theorem 1 that we shall only discuss these details which are different. We begin by some remarks concerning Φ. Since $\Psi(t) = \Phi(t^{1/2})$ is concave, a simple geometric argument shows that if $\alpha > 1$, $\Psi(\alpha t) \leqq \alpha\Psi(t)$. Hence

$$\Phi(\alpha t) \leqq \alpha^2\Phi(t) \,, \qquad \text{if } \alpha \geqq 1 \,. \tag{11.4}$$

If we take $\alpha = 1/t$, $0 \leqq t \leqq 1$, we obtain

$$\Phi(t) \geqq t^2\Phi(1) \,, \qquad \text{if } 0 \leqq t \leqq 1 \,. \tag{11.5}$$

We also need the following lemma.

LEMMA. *Let $F(t) = \sum_{N_1}^{N_2} b_n r_n(t)$. Suppose $\Phi(t^{1/2})$ is concave, then*

$$\int_0^1 \Phi(| F(t) |)dt \leqq \sum_{N_1}^{N_2} \Phi(| b_n |) \,. \tag{11.6}$$

PROOF OF THE LEMMA.
Let $\Psi(t) = \Phi(t^{1/2})$; owing to the fact that Ψ is concave, we have

$$\int_0^1 \Phi(| F(t) |)dt = \int_0^1 \Psi(| F(t) |^2)dt \leqq \Psi\left(\int_0^1 | F(t) |^2 dt\right) .$$

But

$$\int_0^1 | F(t) |^2 dt = \sum_{N_1}^{N_2} | b_n |^2 \,.$$

Now

$$\Psi\left(\sum_{N_1}^{N_2} |b_n |^2\right) \leqq \sum_{N_1}^{N_2} \Psi(| b_n |^2) = \sum_{N_1}^{N_2} \Phi(| b_n |) \,.$$

This proves (11.6).

We return now to the proof of Theorem 3. As in the proof of Theorem 1, we pick a sequence of functions $f_n \in L_\Phi$, so that

$$(T^* f_n)(x) > 1, \qquad \text{if } x \in E_n \,,$$

and

$$m(E_n) > \int_M \Phi(n \,|\, f_n |)dm \,.$$

Due to the convexity of Φ, $\Phi(n \,|\, f_n |) \geqq n\Phi(| f_n |)$.

Hence by choosing an appropriate subsequence, with possible repetitions, we obtain

$$\sum m(E_n) = \infty \,,$$

while

$$\sum_{n=1}^{\infty} \int_M \Phi(| f_n |)dm < \infty \,.$$

We can also choose constants $R_n \to \infty$, but so slowly so that

$$\sum \int_M \Phi(R_n \,|\, f_n |)dm < \infty \,.$$

Note that $\Phi(R_n |f_n|) \leqq R_n^2\Phi(|f_n|)$ if $R_n \geqq 1$, by (11.4) above.

We take $F(x, t)$ to be the same as (9.4) above.

Next we use (11.6) above to duplicate the argument given by (9.6). Let us now consider (9.7). An argument similar to the argument given above, shows that

$$\sum_n \int_M \Phi(R_n \,|\, T_m\tau_{g_n}(f_n)(x)\,|)\, dm < \infty$$

and hence, $\sum_n \Phi(R_n \,|\, T_m\tau_{g_n}(f_n)(x)\,|) < \infty$, for almost every $x \in M$.

Now since $\Phi(t) \geqq t^2\Phi(1)$, (see (11.5)) if $0 \leqq t \leqq 1$, we obtain that $\sum_n |\, R_n T_m \tau_{g_n}(f_n)(x)\,|^2 < \infty$, for almost every $x \in M$.

Using this, we see, as before, that $T_m(F(x, t))$ has as a function of t, for almost every x, an expansion as a Rademacher series given by

$$T_m F(x, t) = \sum r_n(t) R_n T_m \tau_{g_n}(t)(x) \,.$$

The rest of the proof is then like that of Theorem 1.

12. *Remarks*. Before we come to our main applications, we wish to clarify some points concerning the above results. First, the theorems above are formulated for a discrete (denumerable) family of operators T_m. However, in some applications one must deal with a family of operators which depend on a parameter with a continuous range. For example, let $(T_h f)(x) = (1/h)\int_0^h f(x+t)dt$, $h > 0$ be the example discussed in the introduction. Let us observe that for every x, $T_h(f)(x)$ is continuous in h, $h > 0$. Hence $\sup_{h>0} |\, T_h f(x)\,| = \sup_m |\, T_{h_m} f(x)\,|$, where h_m is any enumeration of the positive rationals. Also $\lim_{h\to 0}(T_h f)(x)$ exists, implies $\limsup_{m\to\infty} |\, T_{h_m} f(x)\,| < \infty$. This simple device (which is used again in § 15 below) allows us to apply the above results even to certain cases where the auxiliary parameter is continuous.

Another remark, although not so trivial, is the following. The above theorems are formulated for linear operators. We want to show, by an example, how these results may be carried over to a large class of *nonlinear* operations which occur in Analysis. This example will also illustrate Corollary 3 of Theorem 1.

Let $f(\theta)$ be integrable on $(0, 2\pi)$ and let it be of power series type, that is $\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta = 0$ if $n < 0$. Let $\mathcal{S}$ denote the closed subspace of $L^1(0, 2\pi)$ consisting of functions of power series type. Note that $\mathcal{S}$ is invariant under translations ($\mathcal{S}$ is usually referred to as H^1). The function $g^*(\theta)$ is of importance in the theory of H^1 spaces.[9] It is defined as follows. Let $S_n(\theta)$ and $\sigma_n(\theta)$ denote respectively the partial sums and

[9] For a discussion of the function $g^*(\theta)$, see [11, Ch. 15] (where $g^*(\theta)$ is denoted by γ).

Cesaro means of the Fourier series of $f(\theta)$. Then

$$g^*(\theta) = \left(\sum \frac{|S_n(\theta) - \sigma_n(\theta)|^2}{n}\right)^{1/2} .$$

We can consider the non-linear mapping $f(\theta) \to g^*(\theta)$. We shall prove:

THEOREM: *Let* $\alpha > 0$, $E_\alpha = \{\theta \mid g^*(\theta) > \alpha\}$. *Then*

$$m(E_\alpha) \leqq \frac{A}{\alpha}\int_0^{2\pi} |f(\theta)|\, d\theta , \qquad f(\theta) \in H^1 ,$$

where A *is a constant independent of* α *or* f.

PROOF. In fact let α_n^m be an enumeration of the countable collection of all sequences satisfying the following.

(i) Each α_n^m is a complex number whose absolute value is rational, and whose argument is a rational multiple of 2π.

(ii) For each m, $\alpha_n^m = 0$ if n is large enough.

(iii) For each m, $\sum_{n=1}^\infty |\alpha_n^m|^2/n \leqq 1$. Now for each m, define

$$T_m(f)(\theta) = \sum_n \left\{\frac{S_n(\theta) - \sigma_n(\theta)}{n}\right\} \alpha_n^m$$

(this is a finite sum), and

$$(T^*f)(\theta) = \sup_m |T_m f(\theta)| .$$

It is easily verified that $T^*f(\theta) = g^*(\theta)$.

Now it is known that $g^*(\theta) < \infty$, for almost every θ, if $f(\theta) \in H^1$. An application of Corollary 3 of Theorem 1, then, completes the proof of the theorem.

We wish to show now that the theorems above, formulated for $1 \leqq p \leqq 2$, cannot be extended to $p > 2$.[10] We argue as follows.

Let $h(\theta) = |\theta|^{-1/2}(\log(2\pi/|\theta|))^{-1}$, $|\theta| \leqq \pi$. Then $h(\theta) \in L^2(-\pi, \pi)$, but $h(\theta) \notin L^q(-\pi, \pi)$, if $q > 2$. Now $h(\theta) \sim \sum c_n e^{in\theta}$, with $\sum |c_n|^2 < \infty$. It is known that there exists a sequence ε_n, of ± 1, so that

$$f(\theta) \sim \sum \varepsilon_n c_n e^{in\theta} \in L^p , \qquad \text{every } p,\ p < \infty .$$

We let T be the bounded operator on $L^2(-\pi, \pi)$ which is of multiplier type and which consists of multiplication of the Fourier coefficients by ε_n. Let $T_m = \sigma_m \cdot T$, where $\sigma_m =$ Cesaro means of order m. Hence, whenever $f \in L^p$, $p \geqq 2$, $\lim_{m\to\infty} T_m(f)\theta = (Tf)(\theta)$ exists almost everywhere. Suppose now that Theorem 1 had an extension to $p > 2$. Then it would follow that $\|Tf\|_q < \infty$, whenever $2 < q < p$, if $f \in L^p(-\pi, \pi)$. Now take $f \sim \sum \varepsilon_n c_n e^{in\theta}$, then $Tf \sim \sum c_n e^{in\theta} = h(\theta)$, and we obtain a contradiction, since $h(\theta) \notin L^q$, if $q > 2$.

[10] See footnote 2.

III. Applications

13. *Double conjugates of Fourier series.* We begin by considering a special case which already contains the essence of the general result given below in Theorem 5.

Let $f(x, y) \sim \sum a_{nm}e^{inx}e^{imy}$ be integrable over $|x| \leqq \pi$, $|y| \leqq \pi$, and let

(13.1) $$\tilde{f}(r, x; \rho, y) = -\sum a_{nm}\,\mathrm{sign}\,(n)\,\mathrm{sign}\,(m) r^{|n|}e^{inx}\rho^{|m|}e^{imy}\,,$$

be the Abel means of the "double conjugate" series. Then, as is well-known

(13.2) $$\tilde{f}(r, x; \rho, y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q(r, x-t)Q(\rho, y-s)f(t, s)\,dt\,ds$$

where

(13.3) $$Q(r, t) = \frac{r \sin t}{1 - 2r\cos t + r^2}\,.$$

We have:

THEOREM 4. *There exists an $f \in L^1$, so that $\lim_{r\to 1}\tilde{f}(r, x; r, y)$ exists only in a set of measure zero.*

PROOF. We shall show that there exists an $f \in L$ so that $\lim_{m\to\infty}\tilde{f}(r_m, x, r_m, y)$ exists only in a set of measure zero where $r_m = 1 - (1/m)$. Assume the contrary. The operators T_m, $T_m : f(x, y) \to \tilde{f}(r_m, x; r_m, y)$ satisfy the conditions of Theorem 2 in § 10 above. We apply the maximal inequality (10.1) to the case where $d\mu =$ Dirac measure, (that is the measure of mass one, which is entirely concentrated at the origin).

Notice

$$|T^*d\mu(x)| = \sup_{m\geqq 1}\left|\frac{1}{\pi^2}Q\left(1-\frac{1}{m}, x\right)Q\left(1-\frac{1}{m}, y\right)\right|$$
$$\geqq \frac{1}{\pi^2}|Q(1, x)Q(1, y)| = \frac{1}{4\pi^2}\left|\cot\frac{x}{2}\cdot\cot\frac{y}{2}\right| \geqq \frac{A}{|xy|}$$

if $|x| \leqq \pi$, $|y| \leqq \pi$. Now the set (in $|x| \leqq \pi$, $|y| \leqq \pi$) where $(A/|xy)| > \alpha$, has measure of the order $(B/\alpha)\log\alpha$, as $\alpha \to \infty$. This is not $O(1/\alpha)$, as $\alpha \to \infty$. This contradicts (10.1) and proves the theorem.

Without going into detail, we mention that the above argument, with certain changes, proves that the following limits do not exist (except in a set of measure zero) for some $f \in L$.

(a) $$\lim_{\varepsilon\to 0}\frac{1}{\pi^2}\int_{\varepsilon<|t|<\pi}\int_{\varepsilon<|s|<\pi}\cot\left(\frac{t}{2}\right)\cot\left(\frac{s}{2}\right)f(x-t, y-s)\,dt\,ds\,,$$

(b) $\lim_{r\to 1} \frac{1}{\pi^2} \int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} P(r, t)Q(r, s)f(x - s, y - t)\,dt\,ds$.

Here $P(r, t)$ is the Poisson kernel $= 1/2\cdot(1 - r^2)/(1 - 2r\cos t + r^2)$.

(c) $\lim_{m\to\infty} \tilde{\sigma}_m(x, y) - \tilde{f}_m(x, y)$.

Here $\tilde{\sigma}_m(x, y) = (1/\pi^2) \iint \tilde{K}_m(t)\tilde{K}_m(s)f(x - t, y - s)dtds$; $K_m(t)$ are the conjugate Fejer kernels; and $\tilde{f}_m(x, y)$ is the integral in (a) with $\varepsilon = 1/(m + 1)$.

The limit (a) represents another possible approach to the double conjugate. Limit (b) corresponds to the Abel means of a single conjugate. An incorrect statement of the existence of limit (c) is given, without proof, in [8].

Theorem 4 may be refined.

THEOREM 5. *Let $\varepsilon > 0$. There exists an f so that $|f|(\log + |f|)^{1-\varepsilon}$ is integrable, and so that $\lim_{r\to 1} \tilde{f}(r, x; r, y)$ exist only in a set of measure zero.*

PROOF. We argue again by contradiction, this time using Theorem 3 of § 11. Fix $\varepsilon > 0$, and let $\Phi_1(t) = t(\log t)^{1-\varepsilon}$. Notice that if t is large enough, say $t \geqq t_0$, then $\Phi_1(t)$ is convex, increasing, and $\Phi_1(t^{1/2})$ is concave. Now let $\Phi(t) = \Phi_1(t + t_0) - \Phi_1(t_0)$. Then $\Phi(t)$ has these properties for $0 \leqq t < \infty$, and $\Phi(0) = 0$. Also obviously $\Phi(2t) \leqq M\Phi(t)$. We consider the Orlicz space L_Φ of functions so that $\iint \Phi(|f|)dxdy < \infty$. This is easily seen to be identical to the class of functions for which $|f|(\log^+ |f|)^{1-\varepsilon}$ is integrable.

We consider the sequence of operators $T_m : f(x, y) \to \tilde{f}(r_m, x; r_m, y)$, where $r_m = 1 - 1/m$. Each T_m satisfies an inequality of the form,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi(|T_m f|)dxdy \leqq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi(|A_m f|)dxdy$$

as an elementary application of Jensen's inequality to (13.2) shows. Under our assumptions, $\lim_{m\to\infty} T_m f$ exists almost everywhere for every $f \in L_\Phi$, and therefore the conditions of Theorem 3 are satisfied. Hence if $\hat{f}(x, y) = \sup_{m\geqq 1} |\tilde{f}(r_m, x; r_m, y)|$, then there exists a constant A so that $m(E_\alpha) \leqq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi(A\,|f(x, y)|/\alpha)dxdy$ where, $E_\alpha = \{(x, y)\,|\,\hat{f}(x, y) > \alpha\}$.

We now take

$$f_\delta(x) \sim \sum \frac{\cos nx}{(\log n)^\delta}\,, \qquad \tilde{f}_\delta(x) \sim \sum \frac{\sin nx}{(\log n)^\delta}\,, \qquad \delta > 0\,.$$

Then, as is known, (see [11, p. 189, I]),

$$f_\delta(x) = O\left(|x|^{-1}\left(\log \frac{1}{|x|}\right)^{-1-\delta}\right), \qquad \text{as } |x| \to 0 \tag{13.4}$$
$$= \text{bounded elsewhere in } |x| \leqq \pi\,,$$

while

$$\tilde{f}_\delta(x) \cong A\,|x|^{-1}\Big(\log\frac{1}{x}\Big)^{-\delta}, \qquad \text{as } |x| \to 0 . \tag{13.5}$$

Let $f_\delta(x, y) = f_\delta(x) f_\delta(y)$. Then if $\delta > \varepsilon$, $f_\delta(x, y) \in L_\Phi$, as a simple calculation shows. We further specify δ, by $\delta < 1$.

Now

$$\begin{aligned}\hat{f}(x, y) &= T^*(f_\delta(x, y)) \geqq \lim_{m\to\infty} |\tilde{f}(r_m, x; r_m, y)| \\ &= |\tilde{f}_\delta(x)\tilde{f}_\delta(y)| \geqq A\,|x|^{-1}\Big(\log\frac{1}{|x|}\Big)^{-\delta} |y|^{-1}\Big(\log\frac{1}{|y|}\Big)^{-\delta},\end{aligned}$$

if both x and y are small enough. Hence

$$\{(x, y)\,|\,\hat{f}(x, y) > \alpha\} \subset \Big\{(x, y)\,\Big|\,|x| \leqq x_0, \qquad |y| \leqq y_0,$$

and

$$A\,|x|^{-1}\Big(\log\frac{1}{|x|}\Big)^{-\delta} |y|^{-1}\Big(\log\frac{1}{|y|}\Big)^{-\delta} > \alpha\Big\}.$$

A straight-forward, although cumbersome, calculation yields that the latter set has measure $\geqq (B/\alpha)\cdot(\log\alpha)^{1-\delta}$, as $\alpha \to \infty$. Thus

$$m(E_\alpha) \geqq \frac{B}{\alpha}\cdot(\log\alpha)^{1-\delta}, \qquad \text{as } \alpha \to \infty . \tag{13.6}$$

Now the conclusion of Theorem 3 implies, as we have mentioned,

$$m(E_\alpha) \leqq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi\Big(\frac{A\,|f_\delta(x, y)|}{\alpha}\Big)dx\,dy .$$

However, due to the convexity of Φ,

$$\begin{aligned}\iint \Phi\Big(\frac{A\,|f_\delta(x, y)|}{\alpha}\Big)dx\,dy &\leqq \frac{1}{\alpha}\iint(A\,|f_\delta(x, y)|)\,dx\,dy \\ &= \frac{A}{\alpha}, \qquad \text{if } \alpha \geqq 1 .\end{aligned}$$

This shows that $m(E_\alpha) = O(1/\alpha)$ as $\alpha \to \infty$, which contradicts (13.6), if $\delta < 1$, and concludes the proof of Theorem 5.

Similar results may be obtained for the limits (a), (b), and (c) discussed above, but the argument is somewhat more delicate. We mention also that Theorem 5 has a straight-forward extension to higher dimensions.[11]

14. *Divergence of Fourier and Walsh series.* We consider first Fourier series of one variable. We shall denote by $S_m(x) = S_m(x, f)$ the partial sums of the Fourier expansion of $f(x)$, $0 \leqq x \leqq 2\pi$, and more generally,

[11] The condition then is that $|f|(\log^+|f|)^{k-1-\varepsilon}$ is integrable over the k-torus, where $\varepsilon > 0$.

$S_m(x) = S_m(x, d\mu)$ will denote the partial sums of the Fourier-Stieltjes expansion of a Borel measure $d\mu$. Our result is as follows.

THEOREM 6. *Let $\lambda(n)$, be any sequence, decreasing to zero, as $n \to \infty$. Then there exists an integrable function $f(x)$, so that the restriction*

$$S_n(x, f) - S_m(x, f) = O(\lambda(m - n) \log (m - n))$$

is false for almost every x.

PROOF. Let us consider the countable family of operators Δ_r, defined by

$$\Delta_r(f)(x) = \frac{S_n(x, f) - S_m(x, f)}{\lambda(m - n) \log (m - n)}, \qquad r = (m, n) .$$

The operators Δ_r satisfy condition (a') and (b′) of § 10. Hence in view of Theorem 2, it will be sufficient to prove the following lemma.

LEMMA. *There exists an absolute constant $A > 0$, with the following property. Let k be an integer. Then there exists a measure $d\mu$, $\int_0^{2\pi} |d\mu| = 1$, so that* if $\Delta_k^*(x) = \sup_{\substack{n,m \\ n-m=k}} | S_n(x, dr) - S_m(x, d\mu) |$, *then*

$$\Delta_k^*(x) \geqq A \log k , \qquad \textit{for almost every } x . \tag{14.1}$$

PROOF. We set $d\mu = (1/N) \sum_{j=1}^N d\mu_j$, where $d\mu_j$ is the Dirac measure translated to the point x_j, and where the points $x_1, \cdots, x_N$, are not yet specified. Clearly $\int_0^{2\pi} | d\mu | = 1$. An easy calculation shows

$$S_n(x, d\mu) - S_m(x, d\mu) = \frac{1}{N} \sum_{j=1}^N \frac{\cos \frac{l}{2}(x - x_j) \sin \frac{k}{2}(x - x_j)}{\sin \left(\frac{x - x_j}{2}\right)} \tag{14.2}$$

whose $k = n - m$, and $l = n + m + 1$.

Now k is fixed; but l is at our disposal: it is any integer larger than k, subject only to the condition that if k is odd, l must be an even integer, and if k is even, l must be an odd integer. Let us assume that k is odd. We now choose the x_j so that they are linearly independent over the rationals, and so that the x_j are close (in a manner to be specified) to $j2\pi/N$. As is easily seen, almost every x has the property that $x - x_1$, $x - x_2, \cdots, x - x_N$, are irrational mod 2π, and linearly independent over the rationals.

For such x, we apply Kronecker's theorem[12] to the right side of (14.2)

[12] For Kronecker's theorem see [4, Ch. 23]. We have been informed by A. Zygmund that J. P. Kahane has also independently applied these notions to the problem of divergence of Fourier Series. His work will appear elsewhere.

and obtain,

$$\sup_{\substack{n,m \\ n-m=k}} | S_n(x, d\mu) - S_m(x, d\mu) | = \frac{1}{N} \sum_{j=1}^{N} \frac{| \sin (k/2)(x - x_j) |}{\left| \sin \frac{(x - x_j)}{2} \right|}$$

Now if the x_j were chosen close enough to $j2\pi/N$ and N large enough, then the sum on the right would exceed half of its integral-analogue. Hence

$$\sup_{\substack{n,m \\ n-m=k}} | S_n(x, d\mu) - S_m(x, d\mu) | \geqq \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin (k/2)(x - y)}{\sin \left(\frac{x - y}{2} \right)} \right| dy \geqq A \log k \ .$$

This proves the lemma.

The lemma shows that the operator

$$d\mu \rightarrow \sup_{m,n} \left| \frac{S_k(x, d\mu) - S_m(x, d\mu)}{\lambda(m - n) \log (m - n)} \right| ,$$

cannot satisfy the maximal inequality (10.1). Thus Theorem 6 is proved.

We now consider divergence of Walsh series. As will be seen, the argument that we shall give will, in many ways, be similar to the proof of Theorem 6 above. First we need to consider some preliminary matters.

The Walsh-Paley functions are obtained by completion of the Rademacher functions $r_n(t)$, (which we have considered in § 7 above). In fact we define the Walsh-Paley functions $\Psi_n(t)$ by $\Psi_0(t) = 1$, and $\Psi_n(t) = r_{n_1}(t) r_{n_2}(t) \cdots r_{n_k}(t)$, where $n = 2^{n_1} + 2^{n_2} \cdots + 2^{n_k}$. Since the work of Fine [3], we know that the Walsh-Paley functions are most properly considered as the characters of a compact commutative group G, the so-called dyadic group.

G is the infinite direct product of the group of the elements, 0 and 1, in which the group operation is addition mod 2. Hence G may be considered as the totality of sequences ε_n, $n = 1, 2, \cdots$ where $\varepsilon_n = 0$ or 1. It is possible to set up a measure-preserving map from G to a full subset on the interval [0, 1], with Lebesgue measure. The mapping consists in assigning to each sequence $\{\varepsilon_n\}$ the number $x = \sum_{n=1}^{\infty} \varepsilon_n / 2^n$. If x and y are two numbers of [0, 1], we denote by $x \dotplus y$ the result of mapping x and y to the group G, performing the group addition in G, and then taking the inverse map back to [0, 1]. Hence $x \dotplus y$ is defined for almost every pair (x, y) and is again a number in the interval. If Ψ_n is a Walsh-Paley function, then

$$\Psi_n(x \dotplus y) = \Psi_n(x) \Psi_n(y) \ .$$

Considering the above, we identify Lebesgue measurable functions on [0, 1] with (Haar) measurable functions on G, and thus also $L^1(0, 1)$ with $L^1(G)$.

Now let $f(x) \in L^1$, and let its Walsh-Paley expansion be $f(x) = \sum a_n \Psi_n(x)$, where $a_n = \int_0^1 f(x)\Psi_n(x)dx$. Consider the operation of partial sums

$$S_m(x, f) = \sum_{n=0}^{m} a_n \Psi_n(x) = \int_0^1 D_m(x, y) f(y) dy \ .$$

Here $D_m(x, y) = \sum_{n=0}^{m} \Psi_n(x)\Psi_n(y)$, is the analogue of the Dirichlet kernel. Now if $D_m(x) = \sum_{n=0}^{m} \Psi_n(x)$, then $D_m(x, y) = D_m(x \dotplus y)$. Hence,

$$S_m(x, f) = \int_G D_m(x \dotplus y) f(y) dy \ . \tag{14.3}$$

It is therefore clear that each operator, $f \rightarrow S_m(f)$, commutes with the group operation (of G). Our result is

THEOREM 7. *There exists an $f \in L^1$, so that the sequence $S_m(x, f)$ diverges (as $m \rightarrow \infty$), for almost every x.*

PROOF. We begin by recalling some facts. First, $\lim_{k \rightarrow \infty} S_{2^k}(x, f)$ exists for almost every x. This makes our problem somewhat different from that of Fourier series. Next we quote the following identity for the "Dirichlet kernel."

$$D_m(x) = D_{2^k}(x) + \Psi_1(2^k x) D_n(x) \tag{14.4}$$

where k is the largest integer so that $2^k \leqq m$, and $n = m - 2^k$.

$$\limsup_{n \rightarrow \infty} \int_0^1 | D_n(x) | dx = \infty \ . \tag{14.5}$$ [13]

In view of what has been said about the convergence of the sequence $S_{2^k}(x, f)$, we consider the family of operators

$$\Delta_m(x, f) = S_m(x, f) - S_{2^k}(x, f) \tag{14.6}$$

where k is the largest integer so that $2^k \leqq m$. It is therefore sufficient to show the existence of an $f \in L^1$, so that the sequence $\Delta_m(x, f)$ diverges almost everywhere (as $m \rightarrow \infty$).

Because of Theorem 2, § 10, we consider $\Delta_m(x, d\mu)$ where $d\mu$ is a Borel measure on G. ($d\mu$ may be considered as a Borel measure on [0, 1], whose mass is concentrated at the points of the interval which correspond to points of G.) We shall consider in particular

$$d\mu = \frac{1}{N} \sum_{j=1}^{N} d\mu_j \ ,$$

[13] For the statements contained in (14.4) and (14.5) see [3].

where $d\mu_j$ has its total mass, equal to 1, concentrated at the point h_j, the points h_j to be specified later. On account of (14.5) and (14.6), we obtain

$$\Delta_m(x, d\mu) = \frac{\Psi_1(2^k x)}{N} \sum_{j=1}^N \Psi_1(2^k h_j) D_n(x \dotplus h_j) \ . \tag{14.7}$$

Let us now fix n. Then it is possible to choose N large enough, so that if $| h_j - (j/N) | \leqq 1/2^N$, then

$$\frac{1}{N} \sum_{j=1}^N | D_n(x + h_j) | \geqq \frac{1}{2} \int_0^1 | D_n(x \dotplus t) | \, dt \ . \tag{14.8}$$

For fixed n, (14.8) follows merely by a simple argument of approximating integrals by finite sums, when one observes that D_n is a step function.

Having fixed n, we now fix an N for which (14.8) holds. The requirement on each h_j is that $| h_j - j/N | \leqq 1/2^N$. This requirement can certainly be met by specifying each h_j appropriately in the first N places of its dyadic expansion. We then still have freedom to choose the entries in the dyadic expansion of each h_j, after the N^{th} place, in a perfectly arbitrary manner. We do this as follows.

We enumerate all the N-tuples, whose entries are ± 1. There are 2^N such N-tuples. We choose the $N + 1$ place in the dyadic expansion of h_j, by specifying that

$$\Psi_1(2^{N+1} h_1), \Psi_1(2^{N+1} h_2) \cdots \Psi_1(2^{N+1} h_N)$$

is the first N-tuple of ± 1. That is, we choose the $N + 1$ place in the dyadic expansion of h_j to be 0 or 1, depending whether we wish $\Psi_1(2^{N+1} h_j)$ to be 1 or -1. Similarly we choose the $N + 2$ place, etc. We continue this way until the $N + 2^N$ place. We then complete the dyadic expansion of each h_j (after the $N + 2^N$ place) in an arbitrary way, but so that h_j is not a dyadic rational—that is, so that h_j corresponds to a unique element of G, which is the image of a point in (0, 1).

Let us now go back to (14.7). Then

$$\sup_m | \Delta_m(x, d\mu) | = \frac{1}{N} \sum_{j=1}^N | D_n(x + h_j) | \tag{14.9}$$

for m of the form $m = 2^k + n$, with n fixed. (Recall that for the right choice of k, $\Psi_1(2^k h_j)$ are arbitrarily ± 1.) Combining this with (14.8) we obtain

$$\sup_m | \Delta_m(x, d\mu) | \geqq \frac{1}{2} \int_0^1 | D_n(t) | \, dt \ .$$

If we now use (14.5) above, we see that we have obtained the following conclusion:

Let ρ be any constant, no matter how large; then there exists a Borel measure $d\mu$ on G, so that $\int_G |d\mu| = 1$, and so that

$$\sup_m |\Delta_m(x, d\mu)| \geqq \rho \,, \qquad \text{for all } x \in G \,.$$

This shows that the sequence of operators Δ_m cannot satisfy the conclusion (10.1) of Theorem 2. Hence there exists an $f \in L^1$, so that $\limsup_{m\to\infty} |\Delta_m(x, f)| = \infty$, almost everywhere, that is

$$\limsup_{m\to\infty} | S_m(x, f) - S_{2^k}(x, f) | = \infty \,,$$

almost everywhere. This proves our theorem.

15. *Multiple Fourier series.* Let $f(x) = f(x_1, \cdots, x_k)$ be a function defined on the k-torus, $-\pi < x_i \leqq \pi$, $i = 1, \cdots, k$, and integrable with respect to k-dimensional Lebesgue measure $dx = dx_1 \cdots dx_k$. Consider the Fourier expansion of f

$$f(x) \sim \sum a_n e^{in\cdot x},\ a_n = \frac{1}{(2\pi)^k}\int f(x) e^{-in\cdot x} dx \tag{15.1}$$

where $n = (n_1, \cdots, n_k)$ ranges over the lattice points. We form the Bochner-Riesz means of order $(k-1)/2$, of the expansion,

$$\begin{aligned} S_R(x, f) &= \sum_{|n|<R}\left(1 - \frac{|n|^2}{R^2}\right)^{(k-1)/2} a_n e^{in\cdot x} \\ &= \frac{1}{(2\pi)^k}\int D_R(x-y) f(y) dy \,, \end{aligned} \tag{15.2}$$

where D_R is the kernel corresponding to summability of order $(k-1)/2$.

$$D_R(x) = \sum_{|n|<R}\left(1 - \frac{|n|^2}{R^2}\right)^{(k-1)/2} e^{in\cdot x} \,. \tag{15.3}$$

A similar formalism holds for Borel measures $d\mu(x)$ instead of the case $f(x)dx$, where f is integrable. The importance of the critical index $(k-1)/2$ of summation for multiple Fourier series was first stressed by Bochner in [1]; see also [9].

Our aim is to prove the following:

THEOREM 8. *There exists an $f(x) \in L$, so that* $\limsup_{R\to\infty} |S_R(x, f)| = \infty$, *for almost every x.*

PROOF. Let $S^*(x, f) = \sup_{\infty > R \geqq 0} |S_R(x, f)|$, for $f \in L$. Similarly, let $S^*(x, d\mu) = \sup_{0 > R \geqq 0} |S_R(x, d\mu)|$, for a Borel measure $d\mu$. Let R_m be any enumeration of the rational in $[0, \infty]$. Then $S^*(x, d\mu) = \sup_m |S_{R_m}(x, d\mu)|$.

This follows by observing that for every x, $S_R(x, d\mu)$ is continuous in R, $0 \leqq R < \infty$. We therefore consider the family of operators T_m, defined by $T_m(f)(x) = S_{R_m}(x, f)$, and apply Theorem 2 of § 10.

Let us assume, contrary to the conclusion of the present theorem, that for every $f \in L$, $\limsup_{R\to\infty} | S_R(x, f) | < \infty$ for some set of x of positive measure. A simple argument then shows that $\limsup_{m\to\infty} | S_{R_m}(x, f) | < \infty$, for the same set of x's. The conclusion (10.1) of Theorem 2 would then imply that if $d\mu$ is any Borel measure

$$\sup_{0\leqq R<\infty} | S_R(x, d\mu) | < \infty , \qquad \text{for almost every } x . \tag{15.4}$$

Now let us choose in particular the Dirac measure $d\mu$. Then $S_R(x, d\mu) = D_R(x)$ (see (15.3), and (15.2).) If we know that

$$\limsup_{R\to\infty} | D_R(x) | = \infty , \qquad \text{for almost every } x , \tag{15.5}$$

then we would have a contradiction with (15.4), thus proving our theorem.

However, Bochner has shown (see [1, p. 192]) that $\limsup_{R\to\infty} | D_R(x) | = \infty$, whenever x satisfies the following condition: the countable collection $\{| x - 2\pi n |\}$, (n ranges over the lattice points) is linearly independent over the rationals. ($| x - y |^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2$.)

We need, therefore, only observe that the set of x's for which the collection $\{| x - 2\pi n |\}$ is linearly dependent over the rationals is of measure zero. Bochner shows that this set is nowhere dense, and the same argument also proves the fact that the set is of zero measure. In fact, consider a typical relation of linear dependence

$$\sum_{j=1}^{N} a_j | x - 2\pi n_j | = 0 . \tag{15.6}$$

Here, a_j are integers, and we may assume that $a_1 \neq 0$. Since $| x - 2\pi n_j | = \left(\sum_k (x_k - 2\pi n_{j\cdot k})^2\right)^{1/2}$, the sum on the left side of (15.6) is real-analytic in $x = (x_1, x_2, \cdots, x_k)$, for $x \neq 2\pi n_j$. Since $a_1 \neq 0$, it has a singularity at $2\pi n_1$, and hence is not identically zero. Hence the zeros of such a function must be of measure zero. Finally, the set of all x's is a countable union of such sets, and therefore must be of measure zero. This proves (15.5), and hence our theorem.

16. *Multiplier transformations.* We consider Fourier expansions in one variable

$$f(x) \sim \sum c_n e^{inx} . \tag{16.1}$$

The class of multiplier transformations is defined as follows. Suppose λ_n is a given two-way infinite sequence. We say that λ_n is of type (L^p, L^p), if whenever (16.1) is the Fourier series of an $f \in L^p$, then

(16.2) $$Tf \sim \sum c_n \lambda_n e^{inx}$$

is the Fourier series of a function in L^p, and the mapping $f \to Tf$ is a bounded transformation of L^p to L^p. An important problem is one of characterizing the sequences λ_n of type (L^p, L^p). Without loss of generality, we shall assume $\lambda_0 = 0$, and that the λ_n are bounded. We then form the generating function $K(x)$ given by,

(16.3) $$K(x) \sim \sum_{n \neq 0} \frac{\lambda_n e^{inx}}{in} .$$

We shall seek to characterize the sequences λ_n in terms of the function $K(x)$. Let us first recall a basic fact about multiplier transformation. It is this: If a sequence λ_n is of type (L^q, L^q), for some q, $2 \leqq q \leqq \infty$, then it is also of type (L^p, L^p) for each p, $q' \leqq r \leqq q$; here q' is the index conjugate to q, $1/q' + 1/q = 1$. There is a similar statement if $1 \leqq q \leqq 2$. See [11, p. 177, I].

Let us now consider a class of measurable functions, which we shall denote by V_q, $2 \leqq q < \infty$. A function $K(x)$ belongs to V_q, if and only if $K \in L^q$, and

$$\sup \left\| \sum K(b_k - x) - K(a_k - x) \right\|_q < \infty .$$

Here, the summation is taken over any finite collection of non-overlapping intervals, and the "sup" is taken over all such collections of intervals. V_∞ may be defined to be the class of function of bounded variation. Obviously $V_2 \supset V_q \supset V_\infty$, if $2 < q < \infty$.[14] We then have

LEMMA 1. *A necessary and sufficient condition that the multiplier sequence λ_n be of type (L^∞, L^∞), (and hence of type (L^r, L^r), for all r $1 \leqq r \leqq \infty$) is that the function $K(x)$ defined by* (16.3) *belongs to V_∞.*
This is merely a re-phrasing of a well-known fact.

The following lemma is completely elementary, but seems to have escaped attention.

LEMMA 2. *A necessary and sufficient condition that the sequence λ_n be of type (L^2, L^2) (that is, λ_n are uniformly bounded) is that $K(x) \in V_2$.*

PROOF. Suppose first that the sequence λ_n is uniformly bounded $|\lambda_n| \leqq M$. Since $K(x) \sim \sum' (\lambda_n e^{inx})/(in)$, then

$$K(b_k - x) - K(a_k - x) \sim \sum' \frac{\lambda_n}{in} \left(\sum e^{inb_k} - e^{ina_k} \right) e^{inx} .$$

Thus by Parseval's relation

[14] The classes V_q were first considered by Kaczmarz in [5], also in connection with multiplier transformations. Lemma 2 below can also be immediately deduced from his results.

$$\begin{aligned}&\|\sum K(b_k-\lambda)-K(a_k-x)\|_2^2\\&=2\pi\sum\frac{|\lambda_n|^2}{|n|^2}|\sum e^{inb_k}-e^{ina_k}|^2\leqq 2\pi M^2\sum\frac{1}{|n|^2}|\sum_k e^{inb_k}-e^{ina_k}|^2\\&=M^2\|\sum K_0(b_k-x)-K_0(a_k-x)\|_2^2\leqq M^2 2\pi\|\sum K_0(b_k-x)-K_0(a_k-x)\|_\infty^2\\&\leqq M'<\infty\,.\end{aligned}$$

Here $K_0(x)\sim\sum' e^{inx}/(in)$, and we have used the fact that $K_0(x)$ is of bounded variation. Thus $K(x)\in V_2$.

Conversely, suppose $K(x)\in V_2$. Let $f(x)$ be any trigonometric polynomial and let $F(x)=K^*f=(1/2\pi)\int_0^{2\pi}K(x-y)f(y)dy=\sum\lambda^n/(in)a_ne^{inx}$, if $f(x)=\sum a_ne^{inx}$. Note that if $\|f\|_2\leqq 1$, then

$$|\sum F(b_k)-F(a_k)|\leqq\frac{1}{2\pi}\sup\|\sum K(b_k-x)-K(a_k-x)\|_2\leqq A<\infty\,.$$

Therefore F is of bounded variation, and its total variation does not exceed $2A$. Hence $\int_0^{2\pi}|F'(x)|dx\leqq 2A$. Let now $f(x)=(1/\sqrt{2\pi})e^{inx}$. Then $F(x)=(1\sqrt{2\pi})\cdot\lambda_n/(in)\cdot e^{inx}$, and hence $|\lambda_n|(|n|/|n|)\sqrt{2\pi}\leqq 2A$. This proves the lemma.

Our result is the following:

THEOREM 9. *Let* $2<q<\infty$.

(a) *Let the multiplier transformation* (16.2) *be of type* (L^r, L^r) *for all* r, $q'\leqq r\leqq q$. *Then the function* $K(x)$ *given by* (16.3) *belongs to* V_q.

(b) *Conversely, suppose that* $K(x)\in V_q$, *then the multiplier transformation is of type* (L^r, L^r) *for all* r, $q'<r<q$.

PROOF. We consider first part (a), which is relatively trivial. We let $p=q'$, that is $1/p+1/q=1$. Define $F(x)=K(x)*f(x)$, whenever $f(x)\sim\sum a_ne^{inx}\in L^p$. Then $F(x)=\sum\lambda_n/(in)\cdot a_ne^{inx}$. If we designate by Tf, $Tf\sim\sum\lambda_na_ne^{inx}$, then $Tf\in L^p$, by assumption, and $F(x)=\int_0^x(Tf)(t)dt$. Suppose now $\|f\|_p\leqq 1$. Then $|\sum F(b_k)-F(a_k)|\leqq\|Tf\|_1\leqq(2\pi)^{1/q}\|Tf\|_p\leqq(2\pi)^{1/q}M$, where M is the bound of T acting on L^p. Therefore $\left|(1/2\pi)\int_0^{2\pi}\sum\big(K(b_k-x)-K(a_k-x)\big)f(x)dx\right|\leqq M$, whenever $\|f\|_p\leqq 1$. Hence $\|\sum K(b_k-x)-K(a_k-x)\|_q\leqq 2\pi M$. This shows that $K(x)\in V_q$.

We now pass to part (b). As before, we let $F(x)=K*f$, whenever $f\in L^p$. Let $F_m(x)=m\{F(x+(1/m))-F(x)\}=m\{K(x+(1/m))-K(x)\}*f(x)$. Since $K(x)\in L^q$ by assumption, the mapping T_m, given by $f\to F_m$, is for each m a bounded transformation from L^p to itself, which commutes with translations. Notice also that if $f\in L^p$, the function $F(x)$ is of bounded variation. In fact,

$$|\sum F(b_k) - F(a_k)| = \frac{1}{2\pi}\left|\int_0^{2\pi}\left(\sum K(b_k - x) - K(a_k - x)\right)f(x)dx\right|$$
$$\leqq \frac{1}{2}\|\sum K(b_k - x) - K(a_k - x)\|_q \|f\|_p \leqq M < \infty\ .$$

Thus $\lim_{m\to\infty} T_m(f)(x) = \lim_{m\to\infty} m\{F(x + (1/m)) - F(x)\} = F'(x)$ exists for almost every x, whenever $f \in L^p$.

An application of Theorem 1, § 8, shows the mapping $T: f \to F'(x)$ is of weak type (p, p). Now if $K(x) \in V_q$, then $K(x) \in V_2$. Hence the mapping $T: f \to F(x)$ is of type (L^2, L^2). Thus an application of the Marcinkiewicz interpolation theorem shows that the mapping T is of type (L^r, L^r) if $p < r \leqq 2$. Then mapping T coincides with the multiplier transformation (16.2) on all trigonometric polynomials, and hence by continuity, on L^r. A well-known duality argument now proves T is of type (L^r, L^r) if $2 \leqq r < q$. This concludes the proof of Theorem 9.[15]

Appendix: Further generalizations

We have shown in § 12 that the general theorems obtained could not be extended as they stood, to L^p spaces where $2 < p$. It is possible, however, to prove a partial extension of our results to a wide variety of Banach spaces. This, we shall now describe.

Our compact homogeneous space M is as before, and we now consider a Banach space B, with norm $\|\cdot\|$, satisfying the following conditions:

(a) every element $f \in B$ is a function in $L^1(M)$.

(b) There exists constant C, so that $\|f\|_1 \leqq C\|f\|$ for $f \in B$, where $\|\cdot\|_1$ denotes the L^1 norm.

(c) The space B is translation invariant, i.e.: if $f(x) \in B$, then $f_g = f(\tau_g(x)) \in B$, $g \in G$, and $\|f\| = \|f_g\|$.

The class of such spaces includes the L^p spaces, the C^k spaces (when these can be formulated), etc., and any of their closed translation-invariant subspaces.

We now suppose we are given a sequence of operators T_m, which for each m are bounded operators of B into itself, and which commute with translations. Moreover we define, for every $f \in B$, $T^*(f)(x) = \sup_{m\geqq 0}(T_m(f)(x))$. The result then reads:

THEOREM 10. *If for every $f \in B$, $T^*(f)(x) < \infty$ for x in a set of positive measure, then*

$$m\{x \mid T^*(f)(x) > \alpha\} \leqq \frac{A}{\alpha}\|f\|\ .$$

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[15] For the statement of Marcienkiewicz's interpolation theorem see [11, Ch. XII]. For the passage from $p < r \leqq 2$ to $2 \leqq r < q$ see [11, p. 177, I].

When $B = L^p(M)$, $2 \leqq p < \infty$, there is a refinement of this result which may be stated as follows:

THEOREM 11. *If for every* $f \in L^p(M)$, $2 \leqq p < \infty$, $T^*(f)(x) < \infty$, *for* x *in a set of positive measure, then*

$$m\{x \mid T^*(f)(x) > \alpha\} \leqq \frac{A}{\alpha^2} \|f\|_p^2 .$$

The proof of Theorem 10 is essentially the same as that of Theorem 1.

The proof of Theorem 11 is again similar to that of Theorem 1, but the following extra argument is needed.

Let $F(t) = \sum b_n r_n(t)$, then $\int_0^1 |F(t)|^p dt \leqq A_p(\sum |b_n|^2)^{p/2}$, if $p < \infty$. (See e.g., [11, p. 213, I]). From this, it follows that if

$$F_{N_1}(x, t) = \textstyle\sum_1^{N_1} R_n f_n(\tau_{g_n}(x))\tau_n(t) ,$$

then

$$\int_0^1 \int_M | F_{N_1}(x, t) - F_{N_2}(x, t) |^p dx\, dt \leqq A_p(\textstyle\sum_{N_1}^{N_2} \| R_n f_n(x) \|_p^2)^{p/2}$$

by Minkowski's inequality, if $2 \leqq p < \infty$. The rest of the proof is then as before.

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