VERTICAL VARIATION OF HARMONIC FUNCTIONS IN UPPER HALF SPACES.

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Abstract. Some results of Bourgain on the radial variation of harmonic functions in the disk are extended to the setting of harmonic functions in upper half spaces.

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1. Introduction

Let f be a bounded analytic function in the unit disk $\mathbb{D}$ and let

$$V(f,\theta) = \int_0^1 |f'(\rho e^{i\theta})|\, d\rho.$$

It was shown in [7] that the set

$$FRV(f) \equiv \{\theta \in \mathbb{T} : V(f,\theta) < \infty\}$$

may have Lebesgue measure zero and may be of first category.

Answering a question from [7], Bourgain showed in [1] that $FRV(f)$ is always of Hausdorff dimension 1 when f is bounded and analytic or even when f is a bounded real harmonic function. By similar arguments, it was shown in [2] that the same conclusion holds if f is a real positive harmonic function.

Here we will show that the technique and results in [1] and [2] can be generalized to real valued harmonic functions in the upper half space $\mathbb{R}^{n+1}$. To be specific, we will replace the notion of radial variation with variation on vertical lines and prove the following theorems.

Theorem A. *If u is a bounded real valued harmonic function in the upper half space $\mathbb{R}^{(n+1)}$ then there is a set $E \subset \mathbb{R}^n$ having Hausdorff dimension n*

such that

$$\int_0^1 |\nabla u(x,y)|\, dy < +\infty$$

for each $x \in E$.

Theorem B. *If* u *is a positive harmonic function in the upper half space* $\mathbb{R}^{(n+1)}$ *then there is a set* $E \subset \mathbb{R}^n$ *having Hausdorff dimension* n *such that*

$$\int_0^1 |\nabla u(x,y)|\, dy < +\infty$$

for each $x \in E$.

The line of reasoning for proving the theorems follows [1] but the results appear not to have been known. The generalization to higher dimensions requires some manipulation of convolution kernels and the asymptotic estimation of certain oscillatory integrals by a technique of Wong, [9], [10]. The lemmas containing the required estimates are given in section 3 and their proofs in section 7.

A construction of P.W. Jones, [5], shows that there are bounded analytic functions f_1 and f_2 in $\mathbb{D}$ such that

$$\int_\gamma (|f_1'| + |f_2'|)\, ds = +\infty$$

for any curve γ in $\mathbb{D}$ joining 0 to $\partial\mathbb{D}$. Using Jones' idea, it is easy to generate examples of bounded complex valued or vector valued harmonic functions in the upper half space which have infinite vertical variation at every point. Simple extensions of examples of the type in [7] show that in the real valued case, the set of finite vertical variation can be of measure zero and of first category. In the above two senses then the results are the best possible.

In [6], Bourgain's result in the disk is used in conjunction with a stopping time construction of Lipschitz sub-domains to prove that for any Bloch function b defined in the disk there is a countable set of points E in $\partial\mathbb{D}$ such that

$$\liminf_{r\to 1} \frac{\text{Re } b(r\xi)}{\int_0^r |b'(\rho\xi)|d\rho} > 0$$

for each $\xi \in E$. The proof depends upon the Riemann mapping theorem to pull Bourgain's result back to the Lipschitz sub-domains. It is an open question whether an analog of the Jones-Mueller result holds for harmonic Bloch functions in upper half spaces and in light of Theorems A and B the question can be reduced to finding a substitute for the Riemann mapping part of the Jones-Mueller argument.

2. Poisson kernels

Let

$$p_y(x) = \int_{\mathbb{R}^n} e^{-2\pi i t\cdot x} e^{-2\pi|t|y}\, dt = \frac{c_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}$$

be the usual n-dimensional Poisson kernel and let $P_N = p_{\frac{1}{N}}$.

We will use the following properties of P_N.

1. $P_N \geq 0$
2. $\int P_N = 1$
3. $|\nabla P_N| \leq CNP_N$ pointwise.
4. If $N_1 > N$ and $t \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |P_N(t+\tau) - P_N(t)| P_{N_1}(\tau)\, d\tau \leq \frac{N}{N_1} \log(2 + \frac{N}{N_1}) P_N(t)$$

5. If $N_1 > N$ and $t \in \mathbb{R}^n$

$$|P_N * P_{N_1} - P_N| \leq (\frac{N}{N_1})^{\frac{1}{2}} P_N(t)$$

The proofs, being exactly as in section 1 of [1] are omitted.

3. Two lemmas on convolution kernels

To prove theorems A and B we will need two lemmas which give pointwise control of certain convolution kernels. The lemmas are stated here for reference and proved in section 7.

Let ω denote a fixed unit vector in $\mathbb{R}^n$.

Lemma 3.1. *For each positive integer j there is a function Y_j such that*

$$\widehat{Y}_j(\xi) = \left(\widehat{P}_{2^{k-2}}(\xi)\widehat{P}_{2^{k-1}}(\xi - 2^{k-1}\omega)\right)^{-1}$$

on

$$\frac{1}{4}2^{j-1} \leq |\xi| \leq \frac{3}{2}2^{j-1}$$

and

$$|Y_j(t)| \leq \frac{c}{2^j\left(|t|^2 + 4^{-j}\right)^{\frac{n+1}{2}}}$$

where c is some numerical constant.

For the second lemma, let σ_j denote radial multipliers such that

1. $0 \leq \sigma_j(r) \leq 1$
2. $\operatorname{supp} \sigma_j(r) \subset [\frac{1}{4}2^{j-1}, \frac{3}{2}2^{j-1}]$
3. $\sum\limits_j \sigma_j \equiv 1$
4. $|\sigma_j^{(s)}| \leq c2^{-js}, \qquad s = 1, 2, 3, \ldots$
5. $\sigma_j(r)$ vanishes to all orders at the endpoints of its support.
6. $\sigma_j(r) = Q(2^{-(j+1)}r) - Q(2^{-j}r)$ for some fixed C^∞ function Q with compact support.

Lemma 3.2. *For any positive integer j and positive real y, the kernels*

$$Z_{0,j,y}(x) = \int\limits_{\mathbb{R}^n} \frac{|s|}{2^j} e^{-2\pi|s|y} e^{2\pi\frac{2^j}{10}y} \sigma_j(s) e^{-2\pi i s\cdot x}\, ds$$

and

$$Z_{k,j,y}(x) = \int\limits_{\mathbb{R}^n} \frac{s_k}{2^j} e^{-2\pi|s|y} e^{2\pi\frac{2^j}{10}y} \sigma_j(s) e^{-2\pi i s\cdot x}\, ds$$

for $k = 1, 2, \ldots, n$, satisfy

$$|Z_{\ell,j,y}(x)| \leq \frac{c}{2^j\left(|x|^2 + 4^{-j}\right)^{\frac{n+1}{2}}}$$

for $\ell = 0, 1, \ldots, n$. The constant is independent of y and j.

4. Littlewood -Paley estimates for vertical variation

Let f denote the boundary values of the bounded harmonic function u in the upper half space $\mathbb{R}^{n+1}$. Let the multipliers σ_j be as in lemma 3.2. Considering the tempered distribution $\widehat{f}$ we estimate

$$\begin{aligned} |\frac{\partial u}{\partial x_k}| &\leq 2\pi \left| \int s_k e^{-2\pi|s|y} e^{-2\pi i s\cdot x} \widehat{f}(s)\,ds \right| \\ &\leq 2\pi \sum_j 2^j e^{-2\pi \frac{2^j}{10}y} \left| f * \int \frac{s_k}{2^j} e^{-2\pi|s|y} e^{2\pi\frac{2^j}{10}y} \sigma_j(s) e^{-2\pi i s\cdot x}\,ds \right| \end{aligned}$$

and similarly

$$|\frac{\partial u}{\partial y}| \leq 2\pi \sum_j 2^j e^{-2\pi\frac{2^j}{10}y} \left| f * \int \frac{|s|}{2^j} e^{-2\pi|s|y} e^{2\pi\frac{2^j}{10}y} \sigma_j(s) e^{-2\pi i s\cdot x}\,ds \right|.$$

Since

$$\int_0^1 2^j e^{-2\pi\frac{2^j}{10}y}\,dy \leq \frac{10}{2\pi},$$

we find that

$$\int_0^1 |\nabla u(x,y)|\,dy \leq C \sum_{\ell=0}^{n} \sum_j |f * Z_{\ell,j,y}(x)|.$$

Recalling lemmas 3.1 and 3.2, write

$$X_j = P_{2^{j-2}} * (e^{2\pi i 2^{j-1} x\cdot\omega} P_{2^{j-1}})$$

then

$$\begin{aligned} |f * Z_{\ell,j,y}| &= |f * X_j * Y_j * Z_{\ell,j,y}| \\ &\leq |f * X_j| * |Y_j| * |Z_{\ell,j,y}| \\ &\leq C|f * X_j| * P_{2^{j-1}}. \end{aligned}$$

So we have

$$\int_0^1 |\nabla u(x,y)|\,dy \leq C \sum_j |f * X_j| * P_{2^{j-1}}(x).$$

To prove theorems A and B we show in the next two sections that these last expressions on the right are finite on a non- empty closed set. Then we estimate the dimension of these sets in section 7.

5. Proof of Theorem A at a point

As before, let f denote the boundary values of the bounded harmonic function u in the upper half space $\mathbb{R}^{n+1}$. Let ω denote a fixed unit vector in $\mathbb{R}^n$. By the semi-group property of the Poisson kernel we can define

$$f_k = f * P_{2^{k-2}} = f * P_{2^k} * P_{2^k} * P_{2^{k+1}} * P_{2^{k+1}} * \dots$$

For each $x \in \mathbb{R}^n$ and for each angle ψ

$$\int_{\mathbb{R}^n} f_{k+1}(\eta) P_{2^k}(x-\eta)(1+\cos(2\pi 2^k (x-\eta)\cdot\omega+\psi)\,d\eta$$

$$= f_{k+1} * P_{2^k}(x) + \mathrm{Re}\,[e^{i\psi} f_{k+1} * (e^{2\pi i 2^k \omega\cdot x} P_{2^k})(x)]$$

So we may choose $\psi = \psi_x$ such that

$$\int_{\mathbb{R}^n} |f_{k+1}(\eta)| P_{2^k}(x-\eta)(1+\cos(2\pi 2^k (x-\eta)\cdot\omega+\psi_x)\,d\eta$$

$$\geq |f_{k+1} * P_{2^k}(x)| + |f_{k+1} * (e^{2\pi i 2^k \omega\cdot x} P_{2^k})(x)|$$

Now with

$$L(x^k,\eta) = \int P_{2^k}(x^k - x) P_{2^k}(x-\eta)(1+\cos(2\pi 2^k (x-\eta)\cdot\omega+\psi_x)\,dx$$

we have

$$\int |f_{k+1}(\eta)| L(x^k,\eta)\,d\eta \tag{1}$$

$$\geq \quad |f_k(x^k)| + |f_{k+1} * (e^{2\pi i 2^k \omega\cdot x} P_{2^k})| * P_{2^k}(x^k).$$

Suppose we have found points $x^1, \dots, x^k \in \mathbb{R}^n$ and constants $c_j^\ell > c > 0$ for $0 \leq j \leq \ell$ and $0 \leq \ell \leq k$ such that

$$|f_k(x^k)| \geq \sum_{j=0}^{k} c_j^k (|f_j * (e^{2\pi 2^{j-1} x\cdot\omega} P_{2^{j-1}})| * P_{2^{j-1}})(x^k) - C2^{-\frac{j}{2}}. \tag{2}$$

Writing

$$g_j = f_j * (e^{2\pi 2^{j-1} x\cdot\omega} P_{2^{j-1}})$$

and using the properties in section 1 we have

$$\begin{aligned} &\iint |g_j(x)||P_{2^{j-1}}(\eta - x) - P_{2^{j-1}}(x^k - \eta)|L(x^k,\eta)\,dx\,d\eta \\ \leq\ & C2^{-\frac{k-j}{2}}(|g_j| * P_{2^{j-1}})(x^k), \end{aligned}$$

which implies

$$(3)\quad \int (|g_j| * P_{2^{j-1}})(\eta)L(x^k,\eta)\,d\eta \leq (1 + C2^{-\frac{k-j}{2}})(|g_j| * P_{2^{j-1}})(x^k).$$

Putting together (1), (2), (3) we get

$$\begin{aligned} &\int |f_{k+1}(\eta)|L(x_k,\eta)\,d\eta \\ \geq\ & \int \left(\sum_{j=0}^{k}(1 + C2^{-\frac{k-j}{2}})^{-1}c_j^k(|g_j| * P_{2^{j-1}})(\eta) + \frac{1}{C}(|g_{k+1}| * P_{2^k})(\eta)\right) L(x^k,\eta)\,d\eta. \end{aligned}$$

As L is positive and

$$\int\limits_{|\eta - x^k| > 2^{-\frac{k}{2}}} L(x^k,\eta)\,d\eta \leq C2^{-\frac{k}{2}},$$

we find a point x^{k+1} such that $|x^k - x^{k+1}| < 2^{-\frac{k}{2}}$ and satisfying (2) with k increased to $k+1$. By induction, there is a constant $C > 0$ and a bounded closed interval I centered at x^0 such that the decreasing sequence of closed sets

$$\{x \in \mathbb{R}^n : \sum_{j=0}^{k}(|g_j| * P_{2^{j-1}}(x) \leq C\} \cap I$$

has non-empty intersection. So there is a point $x \in R^n$ such that

$$\sum_{j=0}^{\infty}(|g_j| * P_{2^{j-1}})(x) \leq C$$

and by section 4 such a point satisfies

$$\int_0^1 |\nabla u(x,y)|\,dy \le C' < +\infty.$$

6. Proof of Theorem B at a point

The argument from the previous section only requires a few changes. Recall that any positive harmonic function u in the upper half space $\mathbb{R}^{n+1}$ has the form

$$\int_{\mathbb{R}^n} p_y(x-t)\,d\mu(t) + cy, \quad c \ge 0$$

where μ is a positive Borel measure satisfying

$$\int_{\mathbb{R}^n} \frac{d\mu}{(1+|t|^2)^{\frac{n+1}{2}}}.$$

We may assume that $c = 0$ and now let

$$f_k = \mu * P_{2^k} * P_{2^k} * P_{2^{k+1}} * P_{2^{k+1}} * \dots$$

Choose ψ_x so that

$$\int_{\mathbb{R}^n} f_{k+1}(\eta) P_{2^k}(x-\eta)(1+\cos(2\pi 2^k (x-\eta)\cdot\omega + \psi_x)\,d\eta$$

$$= f_{k+1} * P_{2^k}(x) - |f_{k+1} * (e^{2\pi i 2^k \omega\cdot x} P_{2^k})(x)|.$$

Forming the kernel L as before and assuming inductively that

$$f_k(x^k) \le f_0(x^0) - \sum_{j=0}^{k} c_j^k (|f_j * (e^{2\pi 2^{j-1} x\cdot\omega} P_{2^{j-1}})| * P_{2^{j-1}})(x^k). \tag{4}$$

for points x^k and constants $c_j^k > c > 0$, we find in the same way a point x^{k+1} and constants $c_j^{k+1} > c > 0$ such that (4) holds with k increased to $k+1$. To get a decreasing sequence of compact sets, replace the kernels L

by $L \cdot \chi_{\{|x-\eta|<2^{-\frac{k}{2}}\}}$ and change the induction assumption to

(5)

$$f_k(x^k) \leq \prod_{j\leq k}(1-c2^{-\frac{j}{2}})^{-1} f_0(x^0) - \sum_{j=0}^{k} c_j^k(|f_j * (e^{2\pi 2^{j-1}x\cdot\omega}P_{2^{j-1}})| * P_{2^{j-1}})(x^k).$$

We can do this because all the f_k are positive.

Now we find a point x^{k+1} such that $|x^k - x^{k+1}| < 2^{-\frac{k}{2}}$ and such that (5) holds with k increased to $k+1$.

The rest of the proof is finished as before, replacing f with μ and $\hat{f}$ with $\hat{\mu}$ in section 4.

7. Proofs of the lemmas

7.1. **Proof of lemma 3.1.** It will be enough to consider

$$(\hat{P}_{2^{j-2}}(\xi))^{-1} = e^{2\pi|\xi|2^{-(j-2)}}.$$

We will define a multiplier which is equal to this function on a sufficiently large ball so that the translated multiplier will also have the desired properties. Then the lemma will follow because the convolution of two functions satisfying the required pointwise estimate will be the desired function Y_j.

Let $\hat{S}_{2^{k-2}}$ be a C^∞ function with derivatives satisfying the pointwise bound

$$|\hat{S}^{(m)}| \leq C2^{-m|k|}$$

and such that

$$\hat{S}_{2^{k-2}}(\xi) = \begin{cases} e^{2\pi|\xi|2^{-(k-2)}} & \text{if } 0 \leq |\xi| \leq \frac{3}{2}2^{(k+1)}, \\ e^{48\pi}e^{-2\pi|\xi|2^{-(k-2)}} & \text{if } |\xi| \geq \frac{7}{4}2^{(k+1)}. \end{cases}$$

Let

$$\hat{T}_{2^{k-2}} = e^{48\pi}e^{-2\pi|\xi|2^{-(k-2)}} - \hat{S}_{2^{k-2}}.$$

Then

$$\hat{T}_{2^{k-2}} = 2e^{24\pi}\sinh(24\pi - 2\pi|\xi|2^{-(k-2)})$$

on the set $[-\frac{3}{2}2^{k+1}, \frac{3}{2}2^{k+1}]$, vanishes to infinite order at $\pm\frac{7}{4}2^{k+1}$ and has

$$|\hat{T}^{(m)}_{2^{k-2}}| \leq C2^{-m|k|}$$

pointwise for each m. Since $\hat{T}$ differs from $\hat{S}$ by a Poisson multiplier, it is enough to estimate $T_{2^{k-2}}$.

Noting that the function $\hat{T}_{2^{k-2}}$ is radial and abusing notation we have

$$\begin{aligned} T_1(x) &= \int\limits_{\mathbb{R}^n} \hat{T}_1(\xi) e^{-2\pi i x\cdot\xi}\, d\xi \\ &= 2\pi |x|^{-(\frac{n-2}{2})} \int\limits_0^\infty \hat{T}_1(r) J_{(\frac{n-2}{2})}(2\pi|x|r)\, r^{\frac{n}{2}}\, dr \end{aligned}$$

where J_ν denotes the Bessel function of order ν, see for example [8] pg. 155.

By the result of Wong in [9], [10], this is

$$= \frac{C}{|x|^{n+1}} + O(\frac{1}{|x|^{n+2}}), \qquad |x| > M_0 \tag{6}$$

for some $M_0 > 0$. Now by the properties of Fourier transforms when composed with dilations

$$T_{2^{k-2}}(x) = \frac{C 2^{-(k-2)}}{|x|^{n+1}} + \ldots \qquad 2^{k-2}|x| > M_0.$$

Since each of the T_{2^j} is bounded by $C2^{nj}$, the required pointwise estimate follows.

For the reader's convenience we briefly explain Wong's proof of (6).

For $x > 0$, $\eta \in \mathbb{R}$ and Re $(\mu + \alpha) > 0$,

$$\int\limits_0^\infty e^{-\eta^2 t^2} J_\alpha(xt) t^{\mu-1}\, dt = \frac{x^\alpha \Gamma(\frac{\mu}{2} + \frac{\alpha}{2}) e^{-\frac{x^2}{4\eta^2}}}{2^{\alpha+1}\eta^{\alpha+\mu}\Gamma(\alpha+1)}\, {}_1F_1(\frac{\alpha}{2} - \frac{\mu}{2} + 1; \alpha + 1; \frac{x^2}{4\eta^2})$$

where ${}_1F_1$ denotes Kummer's hypergeometric function. (See [4] pg. 50 or compute the integral on Mathematica.)

Kummer's function satisfies

$${}_1F_1(0; c; z) \equiv 1, \qquad z > 0$$

and

$${}_1F_1(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \qquad z \to \infty.$$

So if $\frac{\alpha}{2} - \frac{\mu}{2} + 1 = 0$ then

$$\lim_{\eta \to 0} \int_0^\infty e^{-\eta^2 t^2} J_\alpha(xt) t^{\mu-1}\, dt = 0$$

and otherwise

$$= \frac{\Gamma(\frac{\mu}{2} + \frac{\alpha}{2}) 2^{\mu-1}}{\Gamma(\frac{\alpha}{2} - \frac{\mu}{2} + 1) x^\mu}.$$

Writing

$$\hat{T}_1(r) = c_0 + c_1 r + c_2 r^2 + \epsilon(r)$$

where $\epsilon(r) = O(r^3)$ as $r \to 0^+$, it follows that

$$T_1(x) = \frac{C}{|x|^{n+1}} + \frac{C'}{|x|^{n+2}} + \delta(|x|)$$

where

$$\delta(|x|) = 2\pi |x|^{-(\frac{n-2}{2})} \lim_{\eta \to 0} \int_0^\infty e^{-\eta^2 r^2} J_{(\frac{n-2}{2})}(xr) \epsilon(r) r^{\frac{n}{2}}\, dr. \tag{7}$$

We require the following well known properties of Bessel functions

1. $\frac{d}{dt}[t^{\nu+1} J_{\nu+1}(t)] = t^{\nu+1} J_\nu(t)$
2. $J_\alpha(t) \sim \frac{t^\alpha}{2^\alpha \Gamma(\alpha+1)} \qquad t \to 0$
3. $J_\alpha(t) = \sqrt{\frac{2}{\pi t}} \cos(t - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + O(t^{-\frac{3}{2}}) \qquad t \to \infty$

Define inductively

$$\begin{aligned} \epsilon_0(r) &= \epsilon(r) r^{\frac{n}{2}} \\ \epsilon_{j+1} &= \epsilon_j'(r) - (\frac{n-2}{2} + j + 1)\epsilon_j(r) \frac{1}{r}. \end{aligned}$$

Using the properties $1 - 3$ and integrating the integral in (7) repeatedly by parts gives

$$|x|^{\frac{n-2}{2}} \delta(|x|) = \frac{2\pi}{|x|^m} \lim_{\eta \to 0} \int_0^\infty e^{\eta^2 r^2} \epsilon_m(r) J_{(\frac{n}{2}+m-1)}(2\pi |x| r)\, dr$$

where m is an integer such that $\frac{n+5}{2} < m < \frac{n+8}{2}$. Using $1-3$ again and the definition of T shows that

$$\delta(|x|) = o(\frac{1}{|x|^{n+2}})$$

and completes the proof of (6).

7.2. **Proof of lemma 3.2.** A change of variables shows that

$$Z_{0,j,y}(x) = 2^{nj} Z_{0,1,2^j y}(2^j x).$$

So it is enough to show that

$$|Z_{0,1,y}| \leq \frac{C}{|x|^{n+1}}$$

for a constant C which is independent of y.

But this follows by integrating by parts $m = [\frac{n}{2} + 2]$ times using the above property 1 of Bessel functions, the definition of σ and the fact that $y^m e^{-2\pi r y} e^{2\pi \frac{y}{10}}$ is bounded independently of y when $r \in [\frac{1}{8}, \frac{3}{4}]$. For $k > 0$, notice that

$$2^j Z_{k,j,y}(x) = \frac{\partial}{\partial x_k} \int_{\mathbb{R}^n} e^{-2\pi|s|y} e^{2\pi \frac{2^j}{10} y} \sigma_j(s) e^{-2\pi i s \cdot x}\, ds.$$

The pointwise estimate follows again from repeated integration by parts.

8. Dimension estimates

Recall *Polya's criterion* that a real nonnegative even function $\phi(t)$ on $\mathbb{R}$ with $\phi(0) = 1$ and which is convex on $(0, \infty)$ can be represented as

$$\phi(t) = \int_0^\infty (1 - \frac{|t|}{|s|})^+ \nu(ds)$$

where ν is a probability measure on $(0, \infty]$. Since $(1-|t|)^+$ is the Fourier transform of a positive function, so is any ϕ satisfying Polya's criterion. See for example [3] pg.87.

Let ϕ be a function satisfying Polya's criterion which is C^∞ in $(0,\infty)$, supported in $[-1,1]$, vanishes to all orders at 1 and is greater than $\frac{1}{10}$ on $(\frac{1}{8},\frac{3}{4})$. We may require further that

$$\phi = 1 - ct \qquad t \in (0, \frac{1}{16})$$

for some $c > 1$.

Let $\hat{\Phi}$ denote the radial Fourier multiplier in $\mathbb{R}^n$ such that $\hat{\Phi}(\xi) = \hat{\Phi}(|\xi|) = \phi(|\xi|)$ and let $\Phi_N(t)$ denote the function whose Fourier transform is $\hat{\Phi}(\frac{\xi}{N})$.

By Polya's criterion and Wong's technique from the proof of lemma 3.1 Φ_N is a positive function satisfying the pointwise bound

$$\Phi_N(t) \leq \frac{C}{N(N^{-2} + |t|^2)^{\frac{n+1}{2}}}.$$

Now define $K_N = P_N * \Phi_N$. Then K_N satisfies the properties 1-5 in section 2 and has its Fourier transform supported in the ball of radius N. Since ϕ is linear in $(0,\frac{1}{16})$, there is a constant $c' > 0$ such that the Fourier transform of

$$K_{2^k} * K_{2^k} * K_{2^{k+1}} * K_{2^{k+1}} \dots$$

is bounded below by c' on $[\frac{1}{4}2^{(k-1)}, \frac{3}{2}2^{(k-1)}]$.

All the previous arguments can now be made with K replacing P. In particular, lemma 3.1 is proved in a similar way.

Following Bourgain, we note that in each of the inductive arguments for theorems A and B, the average of

$$\sum_{j \leq k} |f_j * (e^{2\pi i 2^{j-1} x\cdot\omega} K_{2^{j-1}})| * K_{2^{j-1}}$$

with respect to the probability density Ω_k defined by

$$\Omega_k(\eta) = \int L_1(x_1,x_2)L_2(x_2,x_3)\dots L_{k-1}(x_{k-1},\eta)\, dx_1 dx_2 \dots dx_{k-1}$$

$$L_k(x,y) = \int K_{2^k}(x-\eta)K_{2^k}(\eta-y)(1+\cos(2\pi 2^k(\eta-y)+\psi_\eta\, d\eta$$

is uniformly bounded in k.

Given $\epsilon > 0$ we alter the construction as follows: Replace L_k with

$$L_k(x,y) = \int K_{2^k}(x-\eta)K_{2^k}(\eta-y)(1+\epsilon\cos(2\pi(2^k+[\frac{2^k}{10}])(\eta-y)+\psi_\eta)\,d\eta$$

and replace Ω_k with the densities obtained from the new L.

Replace the Littlewood-Paley expressions which control the vertical variation with

$$\sum_{j\le k} |f_j * (e^{2\pi i(2^{j-1}+[\frac{2^{j-1}}{10}])x\cdot\omega} K_{2^{j-1}})| * K_{2^{j-1}}$$

Neither of these changes effects the argument but now because of the shift by $[\frac{2^{j-1}}{10}]$ the densities Ω_k have a weak-$*$ limit which is a probability measure μ on $\mathbb{R}^n$. This can be seen by integration of the Ω_k against a fixed function with compactly supported Fourier transform.

The integral of

$$\sum_0^\infty |f_j * (e^{2\pi i(2^{j-1}+[\frac{2^{j-1}}{10}])x\cdot\omega} K_{2^{j-1}})| * K_{2^{j-1}}$$

with respect to μ is finite, and if γ is a function with $\hat{\gamma}$ supported in $[\frac{-2^k}{20}, \frac{2^k}{20}]^n$ then

$$\begin{aligned} |\int \gamma\, d\mu| &\le C\int \Omega_k(\eta)[|\gamma| * (K_{2^k} * K_{2^k} * K_{2^{k+1}} * K_{2^{k+1}} \dots](\eta)\,d\eta \\ &\le C(1+\epsilon)^k \|\gamma\|_{L^1} \end{aligned}$$

Approximating the characteristic function of a cube Q with sidelength $|Q|$ in L^1 norm by functions with compactly supported Fourier transform now shows that

$$\mu(Q) \le |Q|^{n-\frac{\log(1+\epsilon)}{\log 2}}$$

which easily implies that the Hausdorff dimension of the support of μ is greater than $n - \frac{\log(1+\epsilon)}{\log 2}$. By section 5, this shows that theorems A and B hold on a set with Hausdorff dimension n.

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