RANDOM WALK AND BOUNDARY BEHAVIOR OF
FUNCTIONS IN THE DISK.

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ABSTRACT. Simple martingale proofs of some results of Rohde [9, 10] on
the boundary behavior of Bloch functions are presented, making clear their
connection with random walk in the plane.

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1. INTRODUCTION

A function $f$, defined and analytic in the unit disk, is called a Bloch function
if
$$
\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.
$$
We write $f \in B$. The function is said to be in the little Bloch space $B_0$ if

$$
(1 - |z|^2)|f'(z)| \to 0 \quad |z| \to 1 \quad z \in \mathbb{D}.
$$

The following proposition, which establishes a close connection between Bloch
functions and conformal mappings, is well known. (See [1, 2])

**Proposition 1.1.** If $g$ is a univalent function in $\mathbb{D}$ and $f = \log g$ then $f \in B$
and $\|f\|_B \leq 6$. Conversely, if $\|f\|_B \leq 1$ then there exists a univalent function $g$
such that $f = \log g'$.

In this note we use the device of Bloch martingales, developed by Makarov in
[6] and briefly outlined below, to give simple proofs of the following two theorems
of S. Rohde.

**Theorem 1.2 (Rohde).** An inner function in the little Bloch space which is not
a finite Blaschke product has, for each $\delta \in \mathbb{D}$, the non-tangential limit $\delta$ on a set
$E_\delta \subset \mathbb{T}$ with Hausdorff dimension 1.

**Theorem 1.3 (Rohde).** Let $\{Q_n\}$ be a sequence of squares in the plane with
pairwise disjoint interiors, edges parallel to the coordinate axes and of length
a > 0, and such that \( Q_n \) is adjacent to \( Q_{n+1} \) for all \( n \). There is a universal constant \( K > 0 \) such that if \( g \in \mathcal{B} \) has nontangential limits almost nowhere then there exists a set \( E \) with

\[
\dim E \geq 1 - a^{-1} K \| g \|_B
\]

such that, for each \( \zeta \in E \) we can find \( r_n \to 1 \) with

\[
g(r\zeta) \subset Q_n \cup Q_{n+1} \quad \text{for} \quad r_n \leq r \leq r_{n+1}.
\]

Given an arc \( I \subset \mathbb{D} \) let \( z_I = \tau(0) \) where \( \tau \) is the conformal self mapping of the disk which maps \( \partial \mathbb{D} \cup \{ \text{Re} \ z > 0 \} \) onto \( I \). The theorems in section 2 will follow from the next lemma of Makarov. (See [6].)

**Lemma 1.4 (Makarov).** If \( b \in \mathcal{B} \) and \( I \) is an arc on \( \partial \mathbb{D} \) then we have

\[
(*) \quad \left| \frac{1}{|I|} \int_I (b(\zeta) - b(z_I))^n \, d\zeta \right| \leq Cn!\|b\|_{L^2}^n \quad n \geq 1.
\]

Here, the integral is defined as the limit of the integrals of

\[
(b(r\zeta) - b(z_I))^n
\]
as \( r \to 1 \).

**Proof.** We have

\[
\int_0^1 \left( \log \frac{1 + t}{1 - t} \right)^n \, dt = 2 \int_0^\infty x^n \frac{e^x}{(1 + e^{2x})} \, dx \leq 2 \int_{-\infty}^0 x^n e^x \, dx \leq 2n!.
\]

If \( b(0) = 0 \) and \( I = [e^{-i\pi}, e^{i\pi}] \subset \mathbb{T} \) then \((*)\) follows by twice integrating the inequality

\[
|b'(z)| \leq \frac{\|b\|_{L^2}}{1 - |z|^2}
\]
on the imaginary axis from \(-i\) to \( i\) and applying the Cauchy integral theorem. In general, let

\[
g(z) = b(\tau(z)) - b(\tau(0))
\]
where \( \tau \) is the self mapping of \( \mathbb{D} \) used to define the point \( z_I \). We then have

\[
\left| \int_I (b(\zeta) - b(z_I))^n \, d\zeta \right| = \left| \int_{-1}^1 (g(iy))^n \tau'(iy) \, dy \right|
\]
and

\[
(1 - |z|^2) |g'(z)| = (1 - |q|^2) |b'(\zeta)| \quad \zeta = \tau(z).
\]
Now since
\[ |f'(z)| = \frac{1 - |z|^2}{1 + |z|^2} \leq C|I|, \]
(\#) follows from the first case considered, and the proof is complete.

We remark that if \( b \in B_0 \) then we may replace the inequality
\[ |b'(z)| \leq \frac{||b||_B}{1 - |z|^2} \]
in the above proof by
\[ |b'(z)| \leq \frac{\beta(1 - |z|)}{1 - |z|} \]
for some function \( \beta = \beta(\delta) \), determined by \( b \), for which \( \beta \to 0 \) as \( \delta \to 0 \) and obtain the inequalities
\[ \left| \frac{1}{|I|} \int_I (b(\zeta) - b(z_I))^n d\zeta \right| \leq Cn!\beta^n(\|b\|_B^n) \quad n \geq 1. \]

For any \( b \in B \) let \( b_r(z) = b(rz) \) for all \( 0 < r < 1 \) and define
\[ b_I = \lim_{r \to 1} \frac{1}{|I|} \int_I b_r(\zeta) \, d\zeta. \]
this limit exists by Cauchy's theorem because, as shown in the proof above, \( b \) is integrable on radii. Let \( \mathcal{F}_n \) denote the increasing sequence of sigma algebras generated by the partition points \( e^{2\pi i 2^{-k}}, 1 \leq k \leq 2^n \) on the probability space \((T, \mathcal{F}, \mu)\). Defining the \( \mathcal{F}_n \) measurable function \( M_n \) by \( M_n |I| = b_I \) it is easily checked that \( M = (M_n, \mathcal{F}_n) \) is a martingale. By lemma 1.4, and the definition of the Bloch space, we have for all \( n \)
1. \( |M_n(\zeta) - b((1 - 2^{-n})\zeta)| \leq C\|b\|_B \quad \zeta \in \partial \mathbb{D} \)
2. \( |\Delta M_n(\zeta)| \leq C\|b\|_B \quad \zeta \in \partial \mathbb{D} \)

We define a Bloch martingale to be a real dyadic martingale such that there exists \( b \in B \) with
\[ \text{Re } b_I = S_n |I| \quad |I| = 2\pi 2^{-n} \]
for each dyadic interval \( I \). The following lemma from [6], which we record without proof, characterizes Bloch martingales.

**Lemma 1.5** (Makarov). *A dyadic martingale \( S \) is a Bloch martingale if and only if the differences \((S|I - S|J)\) are uniformly bounded for all adjacent pairs of dyadic intervals \((I, J)\) with \(|I| = |J|\).*
To make dimension estimates in the next section we will use

**Lemma 1.6 (Hungerford).** Fix $0 < \epsilon < c < 1$. Let $E_0 = T = I_{0,0}$ and for $n > 1$, $E_n = \bigcup I_{n,k}$ where $I_{n,k}$ are disjoint closed arcs such that for each $I_{n,k}$ there is a unique $I_{n-1,j}$ with

1. $I_{n,k} \subset I_{n-1,j}$
2. $|I_{n,k}| \leq \epsilon |I_{n-1,j}|$
3. $\sum_{i(j)} |I_{n,i}| \geq c |I_{n-1,j}|$, where $i(j)$ runs through all indices such that $I_{n,i} \subset I_{n-1,j}$.

Let $E = \bigcap_n E_n$. Then with $\dim E$ denoting the Hausdorff dimension of $E$, we have

$$\dim E \geq 1 - \frac{\log c}{\log \epsilon}.$$ 

Proofs appear in [4] and [8].

2. **Martingale proofs of some results of Rohde**

The behavior of Bloch functions at the boundary of the disk is explained by the following lemma, which is a slight refinement of lemma [5.6] in [6].

**Lemma 2.1.** Let $M = \{M_n\}$ be a complex dyadic martingale determined by a Bloch function $b$ as explained in section 1. Assume that $M_0 = 0$ and that $|\Delta M_n| \leq 1$ for all $n$. Given $0 < \alpha < \frac{\pi}{2}$, there exist $0 < C_\alpha < 1$ and $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$ we have

$$m\left(\left\{|\arg M_t| < \frac{\pi}{2} - \alpha\right\}\right) > C_\alpha$$

where $\tau = \tau_a = \inf\{n : |M_n| \geq a\}$.

**Proof.** By familiar properties of the Fejer kernel, if we are given $\eta > 0$ we may choose $m(\eta)$ so that if $0 \leq \rho(t) \leq 1$ is nondecreasing and left continuous on $[-\pi, \pi]$ then

$$\sum_{\nu = -m}^{m} \frac{m + 1 - |\nu|}{m + 1} \int_{-\pi}^{\pi} \rho(t)e^{-i\nu t} \, dt \geq 1 - \eta$$

$$\Rightarrow \int_{-\frac{\pi}{2} - \alpha}^{\frac{\pi}{2} - \alpha} \rho(t) \, dt > C_\alpha > 0.$$
For each dyadic interval $I$ we have by lemma 1.4,
\[
[(b - b_I)^n]_I \leq C(n) n \|b\|^n_{25}
\]
and
\[
[(b - b_I)^n]_I = (b^n)_I + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (b^{n-k})_I b^k_I
\]
\[
= (b^n)_I - b^n_I + \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} [(b^{n-k})_I - b^{n-k}_I] b^k_I
\]
where the last equality follows from the identity
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
\]
Letting $M^n$ denote the martingale determined by $b^n$ and with $0 \leq n \leq m(\alpha)$, we have
\[
\int M_r \ dm = 0
\]
and
\[
\int M^n_r \ dm = - \int [(M - M_r)^n]_r \ dm
\]
\[
+ \int \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} [(M^{n-k})_r - M^{n-k}_r] M^k_r \ dm.
\]
Applying (***) and (*) to the terms in square brackets, we have by induction that
\[
|\int M^n_r \ dm| \leq C(m) + C'(m) a^{n-2} \quad 0 \leq n \leq m(\alpha).
\]
Let $\rho(t), \ -\pi \leq t \leq \pi$ be the distribution density of $\arg M_r$. From the above argument we have
\[
\left| \int_{-\pi}^{\pi} e^{i n t} \rho(t) \ dt \right| \leq C(m) a^{-2} \quad 0 \leq n \leq m(\alpha)
\]
so that
\[
\sum_{m = -m}^{m} \frac{m + 1 - |\nu|}{m + 1} \int_{-\pi}^{\pi} \rho(t) e^{-i \nu t} \ dt \geq 1 - C(m) a^{-2} \geq 1 - \eta
\]
if $a$ is large enough. By the first paragraph of the proof this implies that
\[
m \left\{ \left| \arg M_r \right| < \frac{\pi}{2} - \alpha \right\} \geq C_\alpha
\]
and completes the proof of the lemma.

We now use Lemma 2.1 to prove the theorems mentioned in the introduction.

Recall that there are singular inner functions $S$ in the little Bloch space since there are singular measures with integrals in the little Zygmund class, ([7],[5],[11]). By a theorem of Frostman (see [3]), the functions

$$\frac{S - \lambda}{1 - \lambda S} \quad \lambda \in \mathbb{D}$$

are non-finite Blaschke products except for a set of $\lambda \in \mathbb{D}$ with logarithmic capacity zero. Hungerford showed in [4] that the zeros of such products must always accumulate on a set with Hausdorff dimension 1. This result was strengthened by Rohde who proved the following theorem [10].

**Theorem 2.2** (Rohde). An inner function in the little Bloch space which is not a finite Blaschke product has, for each $\delta \in \mathbb{D}$, the non-tangential limit $\delta$ on a set $E_0 \subset \mathbb{T}$ with Hausdorff dimension 1.

**Proof.** Fix $\delta \in \mathbb{D}$ and choose $N$ such that $2^{-N} < \frac{1}{4}(1 - |\delta|)$. Let $D_n$ denote the disk of radius $2^{-(N+n)}$ centered at $\delta$. Since $b$ is not a finite Blaschke product, given $\eta > 0$ we can find a dyadic arc $I$ with $|I| < \eta$, $|b_I - \delta| < \eta$ and $(1 - |z|^2)|b'(z)| < \eta$ for all $z$ with $|z| > 1 - \frac{|I|}{2\pi}$. Taking $\eta > 0$ sufficiently small and applying Lemma 2.1 to the function $\frac{b_I}{\eta}$, we can find a dyadic interval $E_0 = I^0$ with $b_{I^0} \in D_2$ and $|I^0| < \eta$. Because $|b| \to 1$ nontangentially almost everywhere, the set of maximal dyadic subintervals $\{J_j\}$ with $b_{J_j} \notin D_0$ has

$$\sum_j |J_j| = |I^0|$$

and given $\epsilon > 0$, if $\eta > 0$ is small enough then since $\eta$ controls $(1 - |z|^2)|b'(z)|$ which in turn controls $\Delta M_n$, we have $|J_j| < \epsilon|I^0|$ for all $j$. We apply Lemma 2.1 in each $J_j$ to obtain $0 < C < 1$ and a set of dyadic intervals $E_1 = \{I_k^1\}$ with $b_{I_k^1} \in D_2$ for all $k$ and

$$\sum_k |I_k^1| \geq C|I^0|.$$  

We continue the construction in the obvious way, obtaining $E_0, E_1, \ldots, E_N$. Because $(1 - |z|^2)|b'(z)| \to 0$ as $|z| \to 1$, with $N_1$ sufficiently large we may alter the construction by switching to the disks $(D_1, D_3)$ in place of $(D_0, D_2)$ and keep the same numbers $\epsilon > 0$ and $C > 0$. In general, for $n > 1$, when $N_n$ is sufficiently large
we may change the construction by switching to the disks \((D_n, D_{n+2})\) in place of 
\((D_{n-1}, D_{n+1})\), keeping the same \(\epsilon, C > 0\). By lemma 1.6 the set \(E = \bigcap_{k} E_{N_k}\) has

\[
\dim E \geq 1 - \frac{\log C}{\log \epsilon}
\]

and by lemma 1.4 and the remark following it, \(b \to \delta\) non-tangentially at each point of \(E\). Since \(\epsilon > 0\) may be chosen arbitrarily small in the above argument, the theorem is proved. \(\square\)

**Theorem 2.3 (Rohde).** Let \(\{Q_n\}\) be a sequence of squares in the plane with pairwise disjoint interiors, edges parallel to the coordinate axes and of length \(a > 0\), and such that \(Q_n\) is adjacent to \(Q_{n+1}\) for all \(n\). There is a universal constant \(K > 0\) such that if \(b \in B\) has nontangential limits almost nowhere then there exists a set \(E\) with

\[
\dim E \geq 1 - a^{-1}K\|g\|_{\mathcal{B}}
\]

such that, for each \(\zeta \in E\) we can find \(r_n \to 1\) with

\[
b(r_\zeta) \subset Q_n \cup Q_{n+1} \quad \text{for } r_n \leq r \leq r_{n+1}.
\]

**Proof.** We may assume that \(\|b\|_{\mathcal{B}} \leq 1\). The theorem is then interesting for large values of \(a\). Let \(Q'_n\) denote the square of edge length \(\frac{a}{2}\) concentric with \(Q_n\) and with parallel edges. Let \(D_n\) denote the disk of radius \(\frac{a}{2}\) concentric with the square \(Q_n\). Consider any two adjacent squares \(Q_n\) and \(Q_{n+1}\). Let \(R_n\) denote the smallest rectangle containing \(Q'_n \cup Q'_{n+1}\). For sufficiently large \(a > 0\) independent of \(b \in B\), if \(b_I \in D_n\) then by the assumption on the function \(b\), finitely many applications of lemma 2.1 show that there exists a universal constant \(0 < C < 1\) and a collection \(\{I_j\}\) of dyadic subintervals of \(I\) with \(\sum |I_j| \geq C|I|\), such that for each \(j\) we have \(B_{I_j} \in D_{n+1}\) and \(b_I \in R_n\) for all dyadic \(J\) with \(I_j \subset J \subset I\). Because \(\|b\|_{\mathcal{B}} \leq 1\) we have \(|I_j| \leq e^{2^{-|I|}}|I|\) for all \(j\). Again, by the assumption on \(b\), we find after finitely many applications of lemma 2.1, a dyadic interval with \(b_{I_0} \in D_0\). Constructing the appropriate nested sequence of intervals contained in \(I_0\), the proof is completed by applying lemma 1.6 and lemma 1.4 with \(n = 1\). \(\square\)

**References**


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