THE CONVEX HULL OF THE INTERPOLATING BLASCHKE PRODUCTS

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1. Introduction and Notations

In the sequel we prove that if a Blaschke product B is continuous in the closed unit disk except on a closed set $E \subset \mathbb{T}$ of measure zero, then B is contained in $9K$, where K denotes the closed convex hull of the interpolating Blaschke products. Moreover, we show that a generic Blaschke product is contained in $27K$. By the well known theorem of Marshall, this implies that the unit ball of H^∞ is contained in $27K$. The proofs employ a technical result, given in the first section, which may be of some independent interest. The results in the paper improve earlier work in [8],[4] and [10]. We refer to these papers and to the book [3] for further background on the questions treated here.

This paper is based on part of the author's UCLA dissertation completed in 1995 under the direction of John Garnett. I thank him for all his help and encouragement. My thanks also go to the referee and the editor for their help in clarifying the presentation.

$\mathbb{D}$: —— Open unit disk, $\quad \mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$

$\mathbb{T}$: —— Unit circle, $\quad \mathbb{T} \equiv \partial\mathbb{D}$

H^∞: —— The space of bounded analytic functions in $\mathbb{D}$.

L^∞: —— The space of essentially bounded functions on $\mathbb{T}$.

$P_z(w)$: —— The Poisson kernel in $\mathbb{D}$, $\quad P_z(w) = \frac{1-|z|^2}{|1-\overline{w}z|^2}$.

$\rho(z,w)$: —— The "pseudo hyperbolic distance" between z and w in $\mathbb{D}$,

$$\rho(z,w) \equiv |\frac{z-w}{1-\overline{w}z}|$$

$d(z,w)$: —— The hyperbolic distance between z and w in $\mathbb{D}$.

$$d(z,w) \equiv \log \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

$d(X,Y)$: —— For $X \subset \mathbb{D}, Y \subset \mathbb{D}$,

$$(X,Y) \equiv \inf\{d(x,y) : x \in X, y \in Y\}$$

A^ϵ_δ: —— An annulus of thickness 2ϵ about the circle $\{|z| = \delta\}$

$$A^\epsilon_\delta \equiv \{z \in \mathbb{D} : \delta - \epsilon \leq |z| \leq \delta + \epsilon\}$$

I: —— A half open subinterval of $\mathbb{T}$

$$I = \{e^{i\theta} \in \mathbb{T} : \theta_1 \leq \theta < \theta_2\} \qquad \text{for some} \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi.$$

$|I|$: —— $|I| = \theta_2 - \theta_1$, the arc length of I.

$Q = Q(I)$: —— A Carleson "square" in $\overline{\mathbb{D}}$.

$$Q \equiv \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r \leq 1\} \qquad \text{for some} \quad I \subset \mathbb{T}$$

$\ell(Q(I))$: —— $\ell(Q) \equiv |I|$

βQ: —— For $\beta \in \mathbb{R}^+$,

$$\beta Q \equiv \{re^{i\theta} : \frac{\theta_1+\theta_2}{2} - \beta(\frac{\theta_2-\theta_1}{2}) \leq \theta < \frac{\theta_1+\theta_2}{2} + \beta(\frac{\theta_2-\theta_1}{2}), \quad 1 - \beta(\theta_2 - \theta_1) \leq r \leq 1\}$$

$T(Q)$: —— $T(Q) \equiv Q \cap \{z \in \mathbb{D} : |z| < 1 - \frac{\ell(Q)}{2}\}$

u_a: —— For an inner function u and $a \in \mathbb{D}$ the inner function u_a is defined by

$$u_a = \frac{u-a}{1-\overline{a}u}.$$

$B|_U$: —— If B is a given Blaschke product with zeros $\{z_\nu\}$ and U a subset of the open unit disk, then

$$B|_U \equiv \prod_{z_\nu \in U} \frac{-\overline{z_\nu}}{|z_\nu|} \cdot \left(\frac{z - z_\nu}{1 - \overline{z_\nu} z} \right)$$

$\mathcal{I}$: —— The set of interpolating Blaschke products.

K: —— The closed convex hull of $\mathcal{I}$.

$\mathcal{F}$: —— The set of finite products of interpolating Blaschke products.

2. Preliminaries

This section includes some background on the results presented in the paper and on two interesting open questions. Several results recorded here will be referred to later.

A holomorphic function $u \in H^\infty(\mathbb{D})$ is called an inner function if its radial limits (which exist a.e. on $\mathbb{T}$ by Fatou's theorem) satisfy $|u(e^{i\theta})| = 1$ a.e on $\mathbb{T}$.

An example of an inner function is

$$u(z) = e^{i\alpha} z^m \prod_{\nu=1}^{\infty} \frac{-\overline{z_\nu}}{|z_\nu|} \cdot \left(\frac{z - z_\nu}{1 - \overline{z_\nu} z} \right)$$

where the convergence of the infinite product is assured by requiring that

$$\sum_\nu (1 - |z_\nu|) < +\infty,$$

where each zero z_ν is counted with its multiplicity. An inner function of this type is called a Blaschke product.

There are non-constant inner functions with no zeros in $\mathbb{D}$ called singular functions. They are

$$u(z) = \exp\left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\lambda(e^{i\theta}) \right)$$

where $d\lambda$ denotes a positive measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure. Now, by the canonical factorization theorem for H^∞, the only other examples of inner functions in $\mathbb{D}$ are constant multiples of products of the above two types. The following theorem shows that the Blaschke products are uniformly dense in the inner functions. See [3] for details.

Theorem 2.1 (Frostman). *Let $v(z)$ be a non-constant inner function on the unit disc. Then for all ζ with $|\zeta| < 1$, except possibly for a set of logarithmic capacity zero, the function*

$$v_\zeta(z) = \frac{v(z) - \zeta}{1 - \bar{\zeta} v(z)}$$

is a Blaschke product.

This paper is concerned with those Blaschke products which have simple zeros lying in an interpolating sequence, that is, a sequence of points in $\mathbb{D}$ such that each interpolation problem

$$f(z_\nu) = w_\nu \qquad \nu = 1, 2, \ldots \quad \{w_\nu\} \in \ell^\infty$$

has a solution $f \in H^\infty$. The basic facts on interpolating sequences are contained in the following theorem of Carleson.

Theorem 2.2 (Carleson). *If $\{z_j\}$ is a sequence in the unit disc, then the following conditions are equivalent:*

1. *The sequence is an interpolating sequence.*
2. *There is a $\delta > 0$ such that*

$$\prod_{j, j \neq k} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right| \geq \delta, \qquad k = 1, 2, \ldots$$

3. *The points z_j are separated,*

$$\left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| \geq a > 0, \qquad j \neq k,$$

and there is a constant C such that for every "square" Q we have

$$\sum_{z_j \in Q} 1 - |z_j| \leq C\ell(Q).$$

A Blaschke product with simple zeros on an interpolating sequence is called an interpolating Blaschke product. We write $\mathcal{I}$ for the set of interpolationg Blaschke products, and $\mathcal{F}$ for the set of finite products of interpolationg Blaschke products. It is known, for example, that every uniformly closed algebra between H^∞ and L^∞ is generated by H^∞ and the complex conjugates of a subset of $\mathcal{I}$. See [2] and [6]. The constructive proof of the Douglas-Rudin theorem due to Jones,[5], shows that any unimodular function in $L^\infty(\mathbb{T})$ may be uniformly approximated by a ratio of products in $\mathcal{I}$.

The following two problems are posed in ([3], p.430)

Problem 1: Do the interpolating Blaschke products generate H^∞ as a uniform algebra?

Problem 2: Can every Blaschke product be uniformly approximated by interpolating Blaschke products?

It is known that the Blaschke products do generate H^∞ and in fact that the closed convex hull of the Blaschke products is the unit ball of H^∞, see [7].

Problem 1 was recently settled in the affirmative in [4], following the work in [8] in which any Blaschke product with zeros which accumulate only at a set of measure zero on $\mathbb{T}$ was shown to be in the linear hull of $\mathcal{I}$. The idea in [4] is to carefully factor a general Blaschke product into finitely many subproducts, to each of which a version of the argument of [8] may be applied. This factorization technique solves Problem 1 because of the following lemma from [8].

Lemma 2.1 (Marshall, Stray). *Given a Blaschke product B which is the product of finitely many interpolating Blaschke products and given $\epsilon > 0$ there exists an interpolating Blaschke product B^* such that*

$$\sup_{z\in\mathbb{D}} |B(z) - B^*(z)| < \epsilon.$$

The lemma shows that we may as well pose problems 1 and 2 for the class $\mathcal{F}$ instead of $\mathcal{I}$. A characterization of $\mathcal{F}$ in terms of zero sequences is given in [4].

Lemma 2.2 (Garnett, Nicolau). *Let B be a Blaschke product and let $\{z_\nu\}$ be its zeros, counted with their multiplicities. Then the following are equivalent:*

1. $B = B_1 \dots B_N$ *, and each B_j is an interpolating Blaschke product.*
2. *There is a constant C such that for every "square" Q we have*

$$\sum_{z_j\in Q} 1 - |z_j| \leq C\ell(Q).$$

3. *There exist positive constants ρ_0, δ_0 such that for each z_ν there is a w_ν with*

$$\left|\frac{z_\nu - w_\nu}{1 - \overline{w_\nu} z_\nu}\right| \leq \rho_0$$

and

$$(1 - |w_\nu|^2)|B'(w_\nu)| \geq \delta_0.$$

Problem 2 remains open, as does the following, intermediate between Problem 1 and 2.

Problem 1.5: Is the unit ball of H^∞ the closed convex hull of $\mathcal{I}$?

In [10], K.Øyma was able to show that K, the convex hull of $\mathcal{I}$, has non-empty interior in H^∞. Here we improve this result to show that K contains all H^∞ functions with norm not greater than $\frac{1}{27}$, by expanding on some ideas from the paper [1]. See also the paper [9] for some similar arguments.

In section 4 we will need the following lemma and remarks . They are from the exposition in [3] of Hoffman's theory of the maximal ideal space of H^∞.

Lemma 2.3 (Hoffman). *Let $B(z)$ be an interpolating Blaschke product with zeros $S = \{z_n\}$, and suppose*

$$\inf_n (1 - |z_n|^2)|B'(z_n)| \geq \delta > 0.$$

There exist $\lambda = \lambda(\delta), 0 < \lambda < 1$, and $r = r(\delta), 0 < r < 1$, satisfying

$$\lim_{\delta \to 1} \lambda(\delta) = 1$$

$$\lim_{\delta \to 1} r(\delta) = 1$$

and having the following properties: The set $B^{-1}(\Delta(0,r)) = \{z : |B(z)| < r\}$ is the union of pairwise disjoint domains V_n, $z_n \in V_n$, and

$$V_n \subset \{z : \left| \frac{z - z_n}{1 - \overline{z_n} z} \right| < \lambda\}.$$

$B(z)$ maps each domain V_n univalently onto $\Delta(0,r) = \{w : |w| < r\}$. If $|w| < r$, then

$$B_w(z) = \frac{B(z) - w}{1 - \overline{w} B(z)}$$

is an interpolating Blaschke product having one zero in each V_n.

Remark 1. In the proof of lemma 2.3 the numbers λ and r are chosen according to

1. $\lambda < \dfrac{2\lambda}{1 + \lambda^2} < \delta$
2. $\lambda = \lambda(\delta) \to 1$ *and* $\dfrac{\delta - \lambda}{1 - \lambda\delta} \to 1 \qquad$ *as* $\delta \to 1$
3. $r(\delta) = \dfrac{\delta - \lambda}{1 - \lambda\delta} \lambda \to 1.$

Remark 2. Let $u \in \mathcal{F}$ have the zero sequence $\{z_n\}$ where multiple zeros are repeated and write

$$u = \prod_{i=1}^{N} B_i \qquad B_i \in \mathcal{I}, \quad 1 \le i \le N.$$

then if $\lambda > 0$ is given and $\epsilon(\lambda) > 0$ is sufficiently small, we have by lemma 2.3

$$\{z \in \mathbb{D} : |u(z)| \le \epsilon\} \subset \bigcup_{i=1}^{N} \{z \in \mathbb{D} : |B_i(z)| \le \epsilon^{\frac{1}{N}}\} \subset \bigcup_{n} \{z : \rho(z, z_n) < \lambda\}.$$

If $\lambda > 0$ is sufficiently small then by the density condition on the set $\{z_n\}$ there is an $M > 0$ such that any connected component of $\bigcup_{n} \{z : \rho(z, z_n) < \lambda\}$ is the union of at most M of the disks $\{z : \rho(z, z_n) < \lambda\}$. By the argument principle, $u_a \in \mathcal{F}$ whenever $|a| < \epsilon(\lambda)$. This shows that if v is any inner function, then the set $\{a \in \mathbb{D} : v_a \in \mathcal{F}\}$ is open.

The last (well known) lemma of this section gives us information on the modulus of a Blaschke product in terms of its zero sequence. It's proof is an exercise with the identity of Lagrange:

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2} = 1 - \left|\frac{z-w}{1-\overline{w}z}\right|^2 \qquad \text{for all} \quad z, w \in \mathbb{D}$$

Lemma 2.4. *Let $0 < a < 1$ and let B be a Blaschke product in the disk with zero sequence $\{z_\mu\}$ where multiple zeros are repeated in the sequence. Suppose that*

$$\left|\frac{z - z_\mu}{1 - \overline{z_\mu} z}\right| > a > 0 \qquad \textit{for each} \quad \mu.$$

Then

$$\frac{1}{2} \sum_{\mu} \frac{(1-|z|^2)(1-|z_\mu|^2)}{|1-\overline{z_\mu}z|^2} \le \log \frac{1}{|B(z)|} \le \frac{1}{2} C(a) \sum_{\mu} \frac{(1-|z|^2)(1-|z_\mu|^2)}{|1-\overline{z_\mu}z|^2}.$$

The left hand inequality holds without any condition except $0 < a < 1$ *and we have* $C(a) \to 1$ *as* $a \to 1$

Proof. See ([3], p. 288-289).

3. An approximation theorem

In this section we use ideas from Bishop's characterization of the zero sets of Blaschke products in the little Bloch space to prove a technical theorem. See [1]. The theorem will be used in section 4 to factor Blaschke products in a procedure similar to that used in [4]. If u is any inner function with the canonical factorization

$$u(z) = \prod_{z_\nu} \frac{-\overline{z_\nu}}{|z_\nu|} \cdot \left(\frac{z - z_\nu}{1 - \overline{z_\nu} z} \right) \exp\left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\lambda(e^{i\theta}) \right),$$

we associate with u a measure μ_u defined by

$$\mu_u \equiv \sum (1 - |z_\nu|) \delta_{z_\nu} + d\lambda.$$

Here λ is the non-negative singular measure occuring in the singular factor of u and $\{z_\nu\}$ is the zero set of u. Multiple zeros are repeated in the sequence according to their multiplicity.

Theorem 3.1. *Let* u *be a given inner function and let* $1 > \delta > 0$ *be fixed. Let* $E_\delta = \{z \in \mathbb{D} : |u(z)| = \delta\}$. *The following two statements are equivalent.*

1. *There exists an* $\epsilon > 0$ *such that*

$$u_a \in \mathcal{F}, \quad \textit{for each} \quad a \in A_\delta^\epsilon.$$

2. *There are* $M > 0$ *and* $\eta > 0$ *such that for each* $z \in E_\delta$, *there exists a square* Q *with the following two properties:*
   (a) $d(z, T(Q)) < M$,
   (b) $\frac{\mu_u(Q)}{\ell(Q)} \notin [\frac{1}{2\pi}(\log \frac{1}{\delta} - \eta), \frac{1}{2\pi}(\log \frac{1}{\delta} + \eta)]$.

The theorem and lemma 2.1 imply that if we can factor a given finite Blaschke product into $o(\frac{1}{\delta})$ factors, each of which satisfies condition 2 with

M and η independent of δ then we can solve Problem 2. This fact had been noticed independently by K. Øyma and communicated to the author in June 1996.

Before beginning the proof of the theorem we record, in the following lemma, some remarks from [1].

Lemma 3.1. *There exists a universal constant α with $0 < \alpha < 1$ such that for any inner function u, if $\mu_u(16Q) = 0$ for some square $Q \subset \overline{\mathbb{D}}$ with $\ell(Q) < \frac{1}{32}$, then for any $e^{i\theta} \in Q$ and for $w_n = (1 - 2^{-n}\ell(Q))e^{i\theta}$, we have*

$$|u(w_n)| \geq |u(w_{n-1})|^{\alpha} \qquad \textit{for each} \quad n \geq 1.$$

See ([1], p. 215 - 216) for a proof of the lemma.

Proof of theorem.

For any $z \in \mathbb{D}$ let Q_z be a square which has z at the center of its top edge and let

$$S(z,n) \equiv \left\{ z' \in Q_z : 2^{-(2n)}(1-|z|) \leq (1-|z'|) \leq 2^{-n}(1-|z|) \right\}.$$

Suppose that condition 2 of the theorem fails for the given inner function u. This means that if we are given $\eta > 0$ and $M > 0$, we can find some $z = z(\eta, M) \in E_\delta$ such that all squares Q with $d(z, T(Q)) < M$ satisfy

$$\frac{\mu_u(Q)}{\ell(Q)} \in [\frac{1}{2\pi}(\log\frac{1}{\delta} - \eta), \frac{1}{2\pi}(\log\frac{1}{\delta} + \eta)].$$

With η and δ fixed, let N be a positive integer and let $z = z(\eta, M)$. Factor u as $u = u_1 u_2$ so that μ_{u_2} is supported in $\overline{\mathbb{D}}/NQ_z$. By lemma 2.4, if $N = N(\delta) > 0$ is sufficiently large then $|u_2(w)| > \frac{\delta}{2}$ for all w on the top edge of Q_z. We further require that

$$(\frac{\delta}{2})^{\alpha^N} > e^{-\eta}$$

where α is the constant from lemma 3.1 and that

$$\int\limits_{\mathbb{T}\setminus NQ_z} P_{z'}(e^{i\theta})\,d\theta \leq \frac{\eta}{10\log\frac{1}{\delta}} \qquad \text{for each} \quad z' \in S(z,N).$$

Let $\mathcal{C}_N$ be the collection of $N2^{2N}$ disjoint Carleson squares contained in NQ_z, with

$$\ell(Q_j) = 2^{-2N}\ell(Q_z) \qquad \text{for each} \quad Q_j \in \mathcal{C}_N.$$

Note that any square S whose top half contains a zero has $\frac{\mu(S)}{\ell(S)}$ differing by at least $\frac{1}{2}$ from some square of exactly half the size contained inside of it. So if M is large enough and $\eta < \frac{1}{2}$ then μ_{u_1} is supported in $\bigcup\limits_{\mathcal{C}_N} Q_j$.

We have

$$\begin{aligned}
\left| -\log|u(z')| \right. &- \left. 2\pi\frac{\mu_u(Q_z)}{\ell(Q_z)} \right| \leq \left| -\log|u(z')| + \log|u_1(z')| \right| \\
&+ \left| \log\frac{1}{|u_1(z')|} - \int\limits_{NQ_z} \frac{1-|z'|^2}{|1-\overline{w}z'|^2}\,d\mu_u(w) \right| \\
&+ \left| \int\limits_{NQ_z} \frac{1-|z'|^2}{|1-\overline{w}z'|^2}\,d\mu_u(w) - \sum_{Q_j\in\mathcal{C}_N} \frac{\mu_u(Q_z)}{\ell(Q_z)} \int\limits_{Q_j\cap\mathbb{T}} P_{z'}(e^{i\theta})\,d\theta \right| \\
&+ \left| \sum_{Q_j\in\mathcal{C}_N} \frac{\mu_u(Q_z)}{\ell(Q_z)} \int\limits_{Q_j\cap\mathbb{T}} P_{z'}(e^{i\theta})\,d\theta - 2\pi\frac{\mu_u(Q_z)}{\ell(Q_z)} \right|
\end{aligned}$$

for any $z' \in S(z,N)$. The first and fourth terms on the right are each smaller than η by our choice of N. By application of lemma 2.4 and properties of the Poisson kernel respectively, the second and third terms are each smaller than η if M is sufficiently large. This shows that $||u(z')| - \delta| < 4\eta$ for any $z' \in S(z,N)$. As N can be arbitrarily large, Schwarz's lemma and remark 2 after lemma 2.3 then show that condition (1) of the theorem fails.

For the remaining half of the proof, let Q_z be as before, the square with z at the center of its top edge. Let $I = Q_z \cap \mathbb{T}$ denote the base of Q_z and let S_m denote the union of 2^m disjoint Carleson squares of length $r_m \equiv 2^{-m}\ell(Q_z)$ contained in Q_z. Suppose now that for any given $\tau > 0$ and integer $n > 0$, there exists a point $z_0(\tau, n) \in \mathbb{D}$ such that $|u(z_0)| = \delta$, but $|u(z_0) - u(w)| < \tau$ for all $w \in 2^n Q_{z_0} \cap \{\xi \in \mathbb{D} : |\xi| \leq 1 - 2^{-2n}\ell(Q_{z_0})\}$. By lemma 2.2 and Schwarz's lemma, this is true for any inner function u for which condition (1) of the theorem fails. We will show that given $M > 0$ and $\eta > 0$, if n is large enough and τ small enough, then all squares Q with $d(z_0, T(Q)) < M$ have

$$\frac{\mu_u(Q)}{\ell(Q)} \in [\frac{1}{2\pi}(\log\frac{1}{\delta} - \eta), \frac{1}{2\pi}(\log\frac{1}{\delta} + \eta)].$$

That will finish the proof of the theorem. Choose a point z with $d(z, z_0) < M$. With sufficiently large n and small τ, we have

$$\left|\int_{\mathbb{D}} P_\xi(w)\, d\mu_u(w) + \log|u(z_0)|\right| < \frac{\eta}{4} \quad \text{for each} \quad \xi \in Q_z/S_n.$$

In fact,

$$\int_{\mathbb{D}} P_\xi(w)\, d\mu_u(w) = \sum_{z_\nu} \frac{(1-|\xi|^2)(1-|z_\nu|)}{|1-\overline{z_\nu}\xi|^2} + \int_{\mathbb{T}} \frac{(1-|\xi|^2)}{|1-\overline{z_\nu}\xi|^2}\, d\lambda.$$

and all values of $|u(\xi)|$ are very near δ for a very large hyperbolic distance around z_0. So by lemma 3.1, the zeros z_ν which are outside of $2^n Q_{z_0}$ make a small contribution to the sum, and lemma 2.4 implies

$$\int_{\mathbb{D}} P_\xi(w)\, d\mu_u(w) \sim \log\frac{1}{|u(\xi)|}.$$

Let $\hat{I} = (1 + 2^{-\frac{n}{2}})I$ and let $c = \frac{\eta}{8|\log \delta|}$. For sufficiently large n, we have

$$\int_{\hat{I}} P_{(1-r_n)e^{i\theta}}(w)\,\frac{d\theta}{2\pi} = \frac{1-(1-r_n)^2}{1-|(1-r_n)w|^2} \int_{\hat{I}} \frac{1-|(1-r_n)w|^2}{|1-e^{-i\theta}(1-r_n)w|^2}\,\frac{d\theta}{2\pi} \geq 1-c$$

for all $w \in S_{2n}$ because the second integral is the harmonic measure at $(1-r_n)w$ of the arc $\hat{I}$. Therefore,

$$\frac{\mu_u(Q_z)}{\ell(Q_z)} = \frac{\mu_u(S_{2n})}{\ell(Q_z)} \leq \frac{1}{(1-c)\ell(Q_z)} \int_{S_{2n}} \left\{ \int_{\hat{I}} P_{(1-r_n)e^{i\theta}}(w)\,\frac{d\theta}{2\pi} \right\} d\mu_u(w).$$

$$\leq (1+2c)\frac{|\hat{I}|}{2\pi\ell(Q_z)} \int_{S_{2n}} P_{(1-r_n)e^{i\theta^*}}(w)\,d\mu_u(w)$$

(for the appropriate θ^* with $e^{i\theta^*} \in \hat{I}$)

$$\leq \frac{1+4c}{2\pi} \int_{\overline{\mathbb{D}}} P_{(1-r_n)e^{i\theta^*}}(w)\,d\mu_u(w)$$

(for sufficiently large n)

$$\leq \frac{1+4c}{2\pi}(-\log|u(z_0)| + \frac{\eta}{4})$$

$$\leq -\frac{1}{2\pi}\log|u(z_0)| + \eta.$$

Now let $\tilde{I} = (1 - 2^{-\frac{n}{2}})I$ and let $r_n = 2^{-n}\ell(Q_z)$. Let N be a positive integer to be chosen below and for any $e^{i\theta} \in \tilde{I}$ let $Q_{e^{i\theta}}$ be the square with $\ell(Q_{e^{i\theta}}) = Nr_n$ and base centered at $e^{i\theta}$. By lemma 2.4 and lemma 3.1, if n is large enough and τ small enough we may choose a large $N < 2^{\frac{n}{2}}$ such that

$$\left| \int_{\overline{\mathbb{D}} \setminus Q_{e^{i\theta}}} P_{(1-r_n)e^{i\theta}}(w)\,d\mu_u(w) \right| < \frac{\eta}{8}.$$

We therefore have for all $e^{i\theta} \in \tilde{I}$

$$\left| \int_{S_n} P_{(1-r_n)e^{i\theta}}(w)\, d\mu_u(w) - \int_{\overline{\mathbb{D}}} P_{(1-r_n)e^{i\theta}}(w)\, d\mu_u(w) \right| < \frac{\eta}{4}$$

and therefore

$$\frac{\mu_u(Q_z)}{\ell(Q_z)} = \frac{\mu_u(S_n)}{\ell(Q_z)} \geq \frac{1}{\ell(Q_z)} \int_{S_n} \left\{ \int_{\tilde{I}} P_{(1-r_n)e^{i\theta}}(w)\, \frac{d\theta}{2\pi} \right\} d\mu_u(w)$$

$$= \frac{1}{\ell(Q_z)} \int_{\tilde{I}} \left\{ \int_{S_n} P_{(1-r_n)e^{i\theta}}(w)\, d\mu_u(w) \right\} \frac{d\theta}{2\pi}$$

$$\geq \frac{|\tilde{I}|}{2\pi \ell(Q_z)} \left(\int_{\overline{\mathbb{D}}} P_{(1-r_n)e^{i\theta^*}}(w)\, d\mu_u(w) - \frac{\eta}{4} \right)$$

(for an appropriate θ^* with $e^{i\theta^*} \in \tilde{I}$)

$$\geq \frac{1 - 2^{-\frac{n}{2}}}{2\pi} (-\log|u(z_0)| - \frac{\eta}{4} - \frac{\eta}{4})$$

$$\geq -\frac{1}{2\pi} \log|u(z_0)| - \eta.$$

We have shown that

$$\left| 2\pi \frac{\mu_u(Q_z)}{\ell(Q_z)} + \log|u(z_0)| \right| < \eta.$$

Since z was an arbitrary point with $d(z, z_0) < M$, this concludes the proof of the theorem.

4. On the size of the closed convex hull of the interpolating Blaschke products

In this section we will use theorem 3.1 to give proofs of the results stated in the introduction. The line of argument follows [8] and [4] and the size of coefficients is estimated in a manner suggested by [10]. We make use of a fixed net of dyadic "squares ". By lemma 2.1 we may assume that a square in the first level of the net containing any zero has $\ell(Q) \leq \frac{\pi}{128}$.

Theorem 4.1. *Let B be a Blaschke product, continuous on $\overline{\mathbb{D}} \setminus E$ where $E \subset \mathbb{T}$ has Lebesgue measure zero. Then B is contained in $9K$.*

Proof. Let $\mathcal{M} = \{Q_\nu\}$ be the set of maximal dyadic squares which contain no zero of B. We have

$$\sum_{Q_\nu \in \mathcal{M}} \ell(Q_\nu) = 2\pi.$$

Let N be an integer which will be made as large as required and let z_ν be the point at the center of the top edge of the square $2^{-N} Q_\nu$. Let $B_N \in \mathcal{I}$ denote the Blaschke product with simple zeros at the points $\{z_\nu\}$ and let

$$B_{N,\sigma}(z) = \prod_{\nu \neq \sigma} -\frac{\overline{z_\nu}}{|z_\nu|} \left(\frac{z - z_\nu}{1 - \overline{z_\nu} z} \right).$$

Let $\omega_n \equiv 2^{-(N-n)} Q_\sigma$ and let $a \sim b$ denote the relation $\frac{a}{b} \to 1$ as $N \to \infty$. We have by lemma 2.4

$$\log \frac{1}{|B_{N,\sigma}(z_\sigma)|} \sim \frac{1}{2} \sum_{\nu \neq \sigma} \frac{(1 - |z_\sigma|^2)(1 - |z_\nu|^2)}{|1 - \overline{z_\sigma} z_\nu|^2}$$

$$= \sum_{n \geq N} \sum_{z_\nu \in \omega_n \setminus \omega_{n-1}} \frac{(1 - |z_\sigma|^2)(1 - |z_\nu|^2)}{|1 - \overline{z_\sigma} z_\nu|^2}$$

$$\leq C \sum_{n \geq N} \sum_{z_\nu \in \omega_n \setminus \omega_{n-1}} 2^{-2n} \frac{1 - |z_\nu|^2}{1 - |z_\sigma|^2}$$

$$\leq C' \sum_{n \geq N} 2^{-n}.$$

(Note that the sum over $z_\nu \in \omega_n \setminus \omega_{n-1}$ is vacuous for large n.) Therefore, for a given $\epsilon > 0$, we will have

$$|B_{N,\sigma}(z_\sigma)| = (1 - |z_\sigma|^2)|B_N'(z_\sigma)| > 1 - \epsilon \quad \text{for each} \quad \sigma$$

if $N = N(\epsilon)$ is sufficiently large. Altering the previous notation, we write

$$B_N^w \equiv \frac{B_N - w}{1 - \overline{w}B_N}.$$

By lemma 2.3, there are positive numbers $0 < r_N < 1$ and $0 < \lambda_N < 1$ with $r_N \to 1$ and $\lambda_N \to 1$ as $N \to \infty$ such that $B_N^w \in \mathcal{I}$ for all w with $|w| < r_N$ and such that B_N^w has exactly one zero in each disk $D_\nu = \{z : \rho(z, z_\nu) < \lambda_N\}$. By remark 1 following lemma 2.3 we may assume each D_ν is contained in the square Q_ν.

If $z \in \mathbb{D} \setminus \bigcup_j Q_j$ then the dyadic square Q whose top half contains z has

$$\frac{\mu_{BB_N}(Q)}{\ell(Q)} \geq \frac{\mu_{B_N}(Q)}{\ell(Q)} \geq 2^{-N}.$$

Otherwise, z is hyperbolically near some square with zero mass.

Let $\delta > 0$ satisfy $\frac{1}{2\pi}\log\frac{1}{\delta} = 2^{-(N+1)}$. Then by theorem 3.1 we have

$$(*) \qquad \frac{BB_N + a}{1 + \overline{a}BB_N} \in \mathcal{F} \qquad \text{for each} \quad a \text{ with } |a| = \delta.$$

Now choose $w \in \mathbb{D}$ with $r_N^2 < |w| < r_N$. There is N' depending only on λ_N such that

$$\frac{\mu_{BB_N^w}(Q)}{\ell(Q)} \geq 2^{-N'}$$

for all dyadic squares Q with

$$T(Q) \subset \mathbb{D} \setminus \bigcup_j Q_j.$$

And since all zeros of B_N^w are contained in the disks $\{z : \rho(z, z_\nu) < \lambda_N\}$ it is clear that any point in $\bigcup_j Q_j$ is near some square with zero mass.

So with $\frac{1}{2\pi}\log\frac{1}{\delta'} = 2^{-(N'+1)}$ we have

$$(**) \qquad \frac{BB_N^w + c}{1 + \overline{c}BB_N^w} \in \mathcal{F}, \qquad \text{for each} \quad c \text{ with } |c| = \delta'.$$

Fix a with $|a| = \delta$, and let $\frac{BB_N + a}{1+\overline{a}BB_N} = u \in \mathcal{F}$. Then we have

$$BB_N = -a + (1+|a|)(1-|a|)(u + \overline{a}u^2 + \overline{a}^2u^3 + \dots).$$

By lemma 2.1 we know that $\mathcal{F}$ is contained in the uniform closure of $\mathcal{I}$, so that

$$BB_N \in (1 + 2|a|)K \subset 3K.$$

Similarly, $(**)$ implies

$$BB_N^w \in (1 + 2|c|)K \subset 3K.$$

Now we also have

$$BB_N^w = -wB + (1+|w|)[(1-|w|)(BB_N + \overline{w}B(B_N)^2 + \overline{w}^2B(B_N)^3 + \dots)].$$

By lemma 2.1 again, we know that each term $B(B_N)^n$ is contained in $3K$, so $-wB \in 9K$ and since $|w| \to 1$ as $N \to \infty$, we have $B \in 9K$.

Now we factor a general Blaschke product into two factors in such a way that a version of the above argument may be applied to each.

Theorem 4.2. *If B is a Blaschke product defined in $\mathbb{D}$, then $B \in 27K$.*

Proof. Let B be a Blaschke product with zero set $\{w_n\}$. We may assume that $\{w_n\} \cap \partial T(Q) = \emptyset$ for every Q in a fixed dyadic net. It is clear that any Blaschke product is the uniform limit of products with this property. By lemma 2.1 we may assume that all the zeros satisfy $|w_n| > 1 - \frac{\pi}{128}$ and further, that each dyadic square Q with $\ell(Q) = \frac{\pi}{128}$ satisfies

$$\frac{\mu_B(Q)}{\ell(Q)} < \epsilon_0.$$

Let $\{Q_j^1\} = \mathcal{G}_1$ be the set of maximal dyadic squares with

$$\frac{\mu_B(Q_j^1)}{\ell(Q_j^1)} > 2^M \epsilon_0,$$

where $M > 0$ will later be chosen as large as needed and $\epsilon_0 > 0$ as small as needed. We have

$$\sum_{\mathcal{G}_1} \ell(Q_j^1) \leq 2\pi 2^{-M}.$$

Now let $\{V_k^1\}$ be the set of dyadic squares which are maximal among dyadic squares contained in some $Q_j^1 \in \mathcal{G}_1$ with respect to the property that

$$\frac{\mu(V_k^1)}{\ell(V_k^1)} < \epsilon_0 \, .$$

Because $|B| \to 1$ non-tangentially almost everywhere, we have

$$\sum_{V_k^1 \subset Q_j^1} \ell(V_k^1) = \ell(Q_j^1) \qquad \text{for each} \quad Q_j^1 \in \mathcal{G}_1.$$

To see this, let Q be a square such that $z \in \mathbb{D}$ is contained in $T(Q)$. We then have

$$\log|B(z)| \leq -C \sum_{w_n \in Q} \frac{(1-|z|^2)(1-|w_n|^2)}{|1-\overline{w_n}z|^2}$$

$$\leq -C' \sum_{w_n \in Q} \frac{1-|w_n|}{1-|z|}$$

$$\leq -C' \frac{\mu_B(Q)}{\ell(Q)}$$

and this implies what was claimed.

Let

$$R_j^1 = Q_j^1 \setminus \bigcup_{V_k^1 \subset Q_j^1} V_k^1 \qquad \text{for each} \quad Q_j^1 \in \mathcal{G}_1.$$

Define $\mathcal{G}_2 = \{Q_j^2\}$ to be the set of dyadic squares which are maximal among squares contained in V_k^1 for some k with respect to the property that

$$\frac{\mu(Q_j^2)}{\ell(Q_j^2)} \geq 2^M \epsilon_0 \, .$$

We have

$$\sum_{Q_j^2 \subset V_k^1} \ell(Q_j^2) \leq 2^{-M} \ell(V_k^1), \qquad \text{for each} \quad k.$$

We form the set of squares $\{V_k^2\}$ and regions R_j^2 as before and continue to obtain Q_j^n, R_j^n and V_k^n. If $2^M \epsilon_0 < \frac{1}{2}$ our assumptions imply that all zeros of B are contained in the interiors of the regions R_j^n. Let σ and ν index the pairs (n,k). Choose an integer $N > 0$ and let $B_N \in \mathcal{I}$ be the product with one zero $z_\nu = z_{(n,k)}$ at the center of the top edge of each square $2^{-N} V_k^n$ for all n and k. We will also write $V_k^n = V_\nu$. Let

$$B_{N,\sigma} \equiv \prod_{\nu \neq \sigma} -\frac{\overline{z_\nu}}{|z_\nu|} \frac{z - z_\nu}{1 - \overline{z_\nu} z}$$

We claim that given $\epsilon > 0$, if N and M are sufficiently large, then

$$|B_{N,\sigma}(z_\sigma)| \geq 1 - \epsilon.$$

Let S_n denote the square $2^{-(N-n)} V_\sigma$. Then we have

$$\log \frac{1}{|B_{N,\sigma}(z_\sigma)|} \sim \sum_{\substack{z_\nu \in V_\sigma \\ \nu \neq \sigma}} \frac{(1 - |z_\sigma|^2)(1 - |z_\nu|^2)}{|1 - \overline{z_\sigma} z_\nu|^2} + \sum_{n > N} \sum_{z_\nu \in S_n \setminus S_{n-1}} \frac{(1 - |z_\sigma|^2)(1 - |z_\nu|^2)}{|1 - \overline{z_\sigma} z_\nu|^2}.$$

As in the previous proof the sum on the right is as small as we like if N is large enough. The sum on the left is

$$\leq C \sum_{n \leq N} \sum_{z_\nu \in S_n \setminus S_{n-1}} 2^{-2n} \left(\frac{1 - |z_\nu|^2}{1 - |z_\sigma|^2} \right)$$

$$\leq C' \frac{2^N}{\ell(V_\sigma)} \sum_{n \leq N} \sum_{z_\nu \in S_n \setminus S_{n-1}} 2^{-2n} (1 - |z_\nu|^2)$$

$$\leq C'' \frac{2^N}{\ell(V_\sigma)} \sum_{n \leq N} 2^{-2n} \cdot 2^{-(M+N)} \ell(V_\sigma)$$

$$\leq C' \, 2^{-M} \sum_{n \leq N} 2^{-n}$$

and therefore is as small as we like if M is sufficiently large.

As in the proof of the previous theorem there are positive numbers $0 < r_N < 1$ and $0 < \lambda_N < 1$ with $r_N \to 1$ and $\lambda_N \to 1$ as $N \to \infty$ such that

$$B_N^w \equiv \frac{B_N - w}{1 - \overline{w} B_N} \in \mathcal{I} \qquad \text{for each} \quad w \text{ with } |w| < r_N$$

and B_N^w has exactly one zero in each disk $\{z : \rho(z, z_\nu) < \lambda_N\}$. By the remark following lemma 2.3 we may assume that each such disk is contained in the square Q_ν. Taking M to be sufficiently larger than N we may also assume that each D_ν is hyperbolically far from any region R_j^n.

For a fixed Q_j^n let Ω_j^n be the unique dyadic square containing Q_j^n with $\ell(\Omega_j^n) = 2\ell(Q_j^n)$. Notice that by the maximality of the Q_j^n, if there are two adjacent Q_j^n of the same size, say Q_k^n and Q_l^n then $\Omega_k^n \neq \Omega_l^n$.

Let

$$\mathcal{G}_n' = \{\Omega_j^n : Q_j^n \in \mathcal{G}_n\}$$

and write

$$\mathcal{G}_n' = \bigcup_{i=1}^{\infty} \mathcal{H}_i$$

where $\mathcal{H}_1$ is the set of maximal dyadic squares in $\mathcal{G}_n'$ and, for $n > 1$, $\mathcal{H}_n$ is the set of maximal squares in

$$\mathcal{G}_n' \setminus \bigcup_{i=1}^{n-1} \mathcal{H}_i.$$

In the terminology of [3] we have divided each set of squares $\mathcal{G}_n'$ into generations $\mathcal{H}_i$. Note that if $\Omega_j^n \in \mathcal{H}_i$ then

$$\sum_{\{\Omega_k^n \in \mathcal{H}_{i+2} : \Omega_k^n \subset \Omega_j^n\}} \ell(\Omega_k^n) \leq \frac{1}{4} \ell(\Omega_j^n).$$

Define

$$\mathcal{G}_{n,0} = \bigcup_{i=1}^{\infty} \{Q_j^n \in \mathcal{G}_n : \Omega_j^n \in \mathcal{H}_{2i-1}\}$$

and

$$\mathcal{G}_{n,1} = \bigcup_{i=1}^{\infty} \{Q_j^n \in \mathcal{G}_n : \Omega_j^n \in \mathcal{H}_{2i}\},$$

and define

$$W_{n,\alpha} = \bigcup_{Q_j^n \in \mathcal{G}_{n,\alpha}} R_j^n \qquad \alpha = 0, 1$$

and

$$U_\alpha = \bigcup_{n=1}^{\infty} W_{n,\alpha} \qquad \alpha = 0, 1.$$

We will now use these definitions to factor the product B.

Let

$$B_\alpha = B|_{U_\alpha}, \quad \alpha = 0, 1$$

and let

$$B_{N,n,\alpha} = \prod_{\{\nu=(n,k), j : V_k^n \subset G_j^n \in \mathcal{G}_{n,\alpha}\}} -\frac{\overline{z_\nu}}{|z_\nu|} \left(\frac{z - z_\nu}{1 - \overline{z_\nu} z} \right), \qquad \alpha = 0, 1$$

and

$$B_{N,\alpha} = \prod_n B_{N,n,\alpha} \ , \qquad \alpha = 0, 1.$$

Then $B = B_0 B_1$ and $B_N = B_{N,0} B_{N,1}$.

Let

$$B_{n,\alpha}^* = B_\alpha|_{W_{n,\alpha}} \cdot B_{N,n,\alpha}$$

and let

$$B_\alpha^* = \prod_n B_{n,\alpha}^* = B_\alpha B_{N,\alpha}.$$

We claim that

$$\frac{B_\alpha^* - \delta}{1 - \overline{\delta} B_\alpha^*} \in \mathcal{F}, \qquad \alpha = 0, 1$$

for all δ with $\frac{1}{2\pi}\log\frac{1}{|\delta|} = \frac{7}{8}2^{-N}$. This will follow from theorem 3.1 once we show that every point in $\mathbb{D}$ is within bounded hyperbolic distance of some dyadic square Q with

$$\frac{\mu_{B^*_\alpha}(Q)}{\ell(Q)} \notin [\frac{3}{4}2^{-N}, 2^{-N}).$$

Let α be fixed. If $Q^n_j \in \mathcal{G}_{n,\alpha}$ then

$$\frac{\mu_{B^*_{n,\alpha}}(\Omega^n_j)}{\ell(\Omega^n_j)} \leq \left(\frac{2^{-N}+2^M\epsilon_0}{2}\right)+\left(\frac{1}{4}\right)\left(\frac{2^{-N}+2^M\epsilon_0}{2}\right)+\left(\frac{1}{4}\right)^2\left(\frac{2^{-N}+2^M\epsilon_0}{2}\right)+\cdots$$

$$\leq \frac{4}{6}(2^{-N}+2^M\epsilon_0).$$

Here the factor $\frac{1}{4}$ is provided by the "odd-even" splitting within each set $\mathcal{G}_n$.

Now summing over all the sets $\mathcal{G}_n$ we have

$$\frac{\mu_{B^*_\alpha}(\Omega^n_j)}{\ell(\Omega^n_j)} \leq (1+2^{-M}+2^{-2M}+\cdots)\cdot\frac{4}{6}(2^{-N}+2^M\epsilon_0) < \frac{3}{4}2^{-N}$$

if ϵ_0 is small.

Therefore if z is not in any R^n_j with $Q^N_j \in \mathcal{G}_{n,\alpha}$ then it is hyperbolically near the top half of a dyadic square Q with $\frac{\mu_{B^*_\alpha}(Q)}{\ell(Q)} < \frac{3}{4}2^{-N}$. This is because the point z is inside some square V_ν but outside any R^n_j. Moving a fixed hyperbolic distance away from the zero z_ν if necessary, there are only squares with $\frac{\mu}{\ell} < \frac{3}{4}2^{-N}$ directly below.

On the other hand, if $z \in R^n_j$ with $Q^n_j \in \mathcal{G}_{n,\alpha}$ then z is contained in the top half of some dyadic square Q with $\frac{\mu_{B^*_\alpha}(Q)}{\ell(Q)} \geq 2^{-N}$. The claim is proved.

Following the proof of the previous theorem, we may replace B_N by $B^w_N = \frac{B_N - w}{1-\overline{w}B_N}$ where $r_N^2 < |w| < r_N$. The zeros of B_N and B^w_N are in a natural one to one correspondence since each product has exactly one zero in the disk

$$\{z : \rho(z, z_\nu) < \lambda_N\}.$$

We let $B^w_{N,\alpha}$ be the factors of B^w_N corresponding to the factors $B_{N,\alpha}$ of B_N. Then there exists an N' depending only on λ_N such that

$$\frac{B_\alpha B^w_{N,\alpha} - \eta}{1 - \overline{\eta} B_\alpha B^w_{N,\alpha}} \in \mathcal{F}, \qquad \alpha = 0, 1$$

for all η with $\frac{1}{2\pi} \log \frac{1}{|\eta|} = \frac{7}{8} 2^{-N'}$. This follows by a repetition of the previous calculation.

We obtain as before

$$B_\alpha B_{N,\alpha} \in (1 + 2|\delta|)K \quad \text{and} \quad B_\alpha B^w_{N,\alpha} \in (1 + 2|\eta|)K$$

therefore

$$BB_N \in (1 + 2|\delta|)^2 K \quad \text{and} \quad BB^w_N \in (1 + 2|\eta|)^2 K.$$

It follows that

$$-wB = BB^w_N - (1 - |w|^2)(BB_N + \overline{w} B(B_N)^2 + \overline{w}^2 B(B_N)^3 + \dots)$$

and therefore, by lemma 2.1, that $-wB \in 27K$. Since $|w| \to 1$ as $N \to \infty$ we have $B \in 27K$.

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