SIAM REVIEW
Vol. 11, No. 3, July 1969

DUALITY THEORY OF LINEAR PROGRAMS: A CONSTRUCTIVE APPROACH WITH APPLICATIONS*

M. L. BALINSKI† AND A. W. TUCKER‡

Introduction. A linear program is a problem of maximizing or minimizing a linear function of many variables subject to a system of linear constraints each of which is a linear equation or linear inequality. The aim of this paper is to present a self-contained, elementary but rigorous, completely constructive account of the theory and the basic computational tool of linear programming.

Theory in linear programming concerns the concept of duality. Traditionally, the duality theory has been established through the development of a series of statements concerning linear inequalities leading to the Farkas lemma [10] and thence to existence and duality theorems. This was the course followed in the 1951 paper of Gale, Kuhn and Tucker [12], which established the duality foreseen by von Neumann and Dantzig in 1947, and in the papers of Goldman and Tucker [13], [26] and in the book of Gale [11]. Here we show that this development can profitably be reversed: after constructively proving one "main theorem for linear programming" by an elementary inductive argument which directly establishes the duality results, a host of transposition theorems, including the Farkas lemma, are easily and immediately derived.

The basic computational tool for solving a linear program is the simplex method of Dantzig [8], first devised in 1947, which is an iterative method based on the Gauss–Jordan or complete elimination idea (see [22] for these connections). It was originally conceived as a purely computational device and proofs for termination of the algorithm in a finite number of iterative steps depended upon special devices such as "perturbing" the constraint set slightly [5] or imposing a lexicographic ordering on vectors appearing in subsequent steps [9]. Finally, a very subtle inductive proof of termination was given by Dantzig [6].

The simplex method and proofs for its termination were given in "extended tableau form." The "condensed tableau" or "schema," introduced by Tucker in 1958, appears, however, to yield considerably greater clarity and ease of exposition for understanding and explaining theory and computation in linear programming [1], [2], [24], [27]. Realization of this fact motivated a search for proofs of termination within this schematic framework and led to the development of the "mutual primal-dual simplex method" of Balinski and Gomory [3]. This method has a natural inductive counterpart which is used in this paper to prove the main theorem.

This paper, then, presents linear programs in schematic form, develops the simplex method in this form, and uses the simplex method as a means of proving

* Received by the editors October 31, 1968. This research was initially supported by Mathematica under contract with the Army Research Office, United States Army, Contract DA-31-124-ARO(D)-31. Subsequent work of the first author was supported by the Army Research Office under Contract DA-31-124-ARO(D)-366. Work of the second author has been supported by the Office of Naval Research under Contract NONR 1858-21.

† Graduate Studies Division, City University of New York, New York, New York 10036.

‡ Department of Mathematics, Princeton University, Princeton, New Jersey 08540.

the duality theory. As such it presents a unified exposition of all of the theory of linear programming. Specifically, basic linear programming theory and computation is developed in the first four sections where only linear programs in canonical form are considered. In § 5, several more general forms are considered, including a "most" general form, and it is shown how these more generally stated programs can be reduced to canonical form and how the theory and computation applies to them. Lastly, in § 6 and § 7, applications of the main theorem are made to linear inequalities and to matrix games. (For a history of the growth of mathematical programming see [7].)

1. Dual linear programs in canonical form.

$$
\text{(1)} \qquad
\begin{array}{l}
x\text{'s} \geqq 0 \\
u \max
\end{array}
\qquad
\begin{array}{c|ccc|c|l}
 & y_{m+1} & \cdots & y_{m+n} & 1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_1 \\
\vdots & \vdots & & \vdots & \vdots & \quad \vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_m \\
\hline
1 & c_1 & \cdots & c_n & d & = v \\
\hline
 & = x_{m+1} & \cdots = & x_{m+n} & = u &
\end{array}
\qquad
\begin{array}{l}
y\text{'s} \geqq 0 \\
v \min
\end{array}
$$

This *schema* (or tableau) is a compact and suggestive means of presenting the following pair of *dual linear programs in canonical form*:

Given

$$
\text{(2)} \qquad
\begin{array}{ccccccc}
x_1 a_{11} & + \cdots + & x_m a_{m1} & + & c_1 & = & x_{m+1}, \\
\vdots & & \vdots & & \vdots & & \vdots \\
x_1 a_{1n} & + \cdots + & x_m a_{mn} & + & c_n & = & x_{m+n}, \\
x_1 b_1 & + \cdots + & x_m b_m & + & d & = & u,
\end{array}
$$

maximize u constrained by $x_1 \geqq 0, \cdots, x_{m+n} \geqq 0$.

Given

$$
\text{(3)} \qquad
\begin{array}{ccccccc}
a_{11} y_{m+1} & + \cdots + & a_{1n} y_{m+n} & + & b_1 & = & -y_1, \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} y_{m+1} & + \cdots + & a_{mn} y_{m+n} & + & b_m & = & -y_m, \\
c_1 y_{m+1} & + \cdots + & c_n y_{m+n} & + & d & = & v,
\end{array}
$$

minimize v constrained by $y_1 \geqq 0, \cdots, y_{m+n} \geqq 0$. The $n + 1$ linear equations in (2) are read from (1) by associating with each sign of equality (=) in the bottom margin of (1) an equation obtained by forming the inner (or scalar) product of the corresponding coefficient *column* with the "independent" symbol column at the left margin of (1) and then equating this inner product to the corresponding "dependent" symbol (x_{m+j} or u) in the bottom margin. The $m + 1$ linear equations in (3) are read from (1) by associating with each sign of equality (=) in the right margin of (1) an equation obtained by forming the inner (or scalar) product of the corresponding coefficient *row* with the "independent" symbol row at the top margin of (1) and then equating this inner product to the corresponding "dependent" symbol ($-y_i$ or v) in the right margin. Thus the schema (1) exhibits two systems of linear equations: a *column system* of $n + 1$ linear equations in the x's and u, and a *row system* of $m + 1$ linear equations in the y's and v. The side conditions in (1), i.e., x's $\geqq 0$, u max and y's $\geqq 0$, v min, give the only additional information needed to specify completely the *column program* (2) and the *row program* (3).

A *solution* of the column system of (1) consists of values of $x_1, \cdots, x_{m+n}$ and u that satisfy the $n + 1$ equations of this system. Clearly the set of solutions of the column system is m-dimensional: values of the "independent" (or "nonbasic") variables $x_1, \cdots, x_m$ can be taken at will and corresponding values of the "dependent" (or "basic") variables $x_{m+1}, \cdots, x_{m+n}$ and u thus determined. A solution of the column system is termed *feasible* if all its x-values are nonnegative; a solution is *optimal* if all x-values are nonnegative and u is maximal. Likewise, a *solution* of the row system of (1) consists of values of $y_1, \cdots, y_{m+n}$ and v that satisfy the $m + 1$ equations of this system. Clearly the set of solutions of the row system is n-dimensional: values of the "independent" (or "nonbasic") variables $y_{m+1}, \cdots, y_{m+n}$ can be taken at will and corresponding values of the "dependent" (or "basic") variables $y_1, \cdots, y_m$ and v thus determined. A solution of the row system is termed *feasible* if all its y-values are nonnegative; a solution is *optimal* if all y-values are nonnegative and v is minimal.

Each of the programs (2) and (3) is said to be in *canonical form*: every equation expresses a dependent variable as a linear combination of a set of independent variables and 1, and all variables, save the dependent "objective functions" u and v, are required to be nonnegative. Thus, as was seen above, one implication of having a linear program in canonical form is that its equation system is *consistent*, that is, the system has solutions. There are, as will be seen in later sections, other forms for stating linear programs. However, some important practical problems are directly formulated in canonical form such as the diet problem and activity analysis problems, and, of greater significance, the simplex method for finding optimal solutions essentially depends on having programs in canonical form.

(The theory of linear programming largely concerns the duality relationship between the row and column programs. One way of seeing "how" the row program results from the column is via formulation of a Lagrangian. To simplify notation we drop subscripts to denote matrices and vectors and let $x' = (x_1, \cdots, x_m)$, $y' = (y_1, \cdots, y_m)$, $x'' = (x_{m+1}, \cdots, x_{m+n})$ and $y'' = (y_{m+1}, \cdots, y_{m+n})$. Then the column program is: max $u = x'b + d$ when $x'A + c = x'' \geqq 0$, $x' \geqq 0$. Form the Lagrangian $v(x, y) = u + xy = u + x'y' + x''y''$, where the variables $y' \geqq 0$ and $y'' \geqq 0$ are Lagrange multipliers restricted to be nonnegative since $x', x'' \geqq 0$. Now let $v(x, y)$ be minimized over $y \geqq 0$ and maximized over $x \geqq 0$. Necessary conditions for an optimum are that $dv/dx' = b + y' + Ay'' = 0$. But this means that

$$v(x, y) = x'b + d + xy = x'b + d - x'(b + Ay'') + (x'A + c)y'' = cy'' + d \equiv v,$$

a function in y only. So, we find the problem: minimize $v = cy'' + d$ when $Ay'' + b = -y' \leqq 0$, $y'' \geqq 0$, which is the row program (3).)

The easy part of the duality relationship is an immediate consequence of a "key equation"

$$v - u = \sum_{1}^{m+n} x_k y_k, \tag{4}$$

which holds for any column and row solutions, feasible or not. In terms of the schema (1), this equation merely states that the inner product of the symbol-column at the left with that at the right margin equals the inner product of the symbol-row at the top with that at the bottom margin. Equation (4) is easily verified by substitution:

$$v - u = \sum_j c_j y_{m+j} - \sum_i x_i b_i = \sum_j \left(x_{m+j} - \sum_i x_i a_{ij} \right) y_{m+j} + \sum_i x_i \left(y_i + \sum_j a_{ij} y_{m+j} \right).$$

If both solutions are feasible, $x_k \geqq 0$ and $y_k \geqq 0$, $k = 1, \cdots, m + n$, and so $v - u = \sum x_k y_k \geqq 0$. Thus, $u \leqq v$ and $u \leqq \max u \leqq \min v \leqq v$. This means that values of v resulting from feasible y-solutions provide upper bounds on $\max u$; similarly, values of u resulting from feasible x-solutions provide lower bounds on $\min v$. If, by some means, x- and y-feasible solutions are obtained having $u = v$, then these must constitute optimal solutions to both the column and row programs, for u has attained an upper and v a lower bound. But this can occur only if, by (4), $x_k y_k = 0$ for every k.

The interesting and difficult part of the duality is that if feasible column and row solutions do exist, then optimal solutions exist with $u = v$ and, thus, with the "complementary orthogonality" property: $x_k y_k = 0$, that is, $x_k = 0$ or/and $y_k = 0$, $k = 1, \cdots, m + n$ (hence $x_k > 0$ implies $y_k = 0$ and vice versa). This is shown below by establishing a rather stronger statement. Namely, given a pair of dual systems (1), each either has or has not a feasible solution. This gives rise to four mutually exclusive possibilities for the dual pair of linear programs (2) and (3):

(i) there exist feasible column and row solutions and hence optimal solutions with $u = v$;

(ii) there exists a feasible column solution, no feasible row solution, and u is unbounded above;

(iii) there exists a feasible row solution, no feasible column solution, and v is unbounded below; and

(iv) there exist no feasible column or row solutions.

For example, consider the simple pair of dual linear programs specified by the schema and marginal information given below:

		y_2	1		
x's $\geqq 0$	x_1	a	b	$= -y_1$	y's $\geqq 0$
u max	1	c	d	$= v$	v min
		$= x_1$	$= u$		

The following numerical assignments lead to the four cases above:

(i) $a = -1, b = 2, c = 3,$

(ii) $a = 1, b = 2, c = 3,$

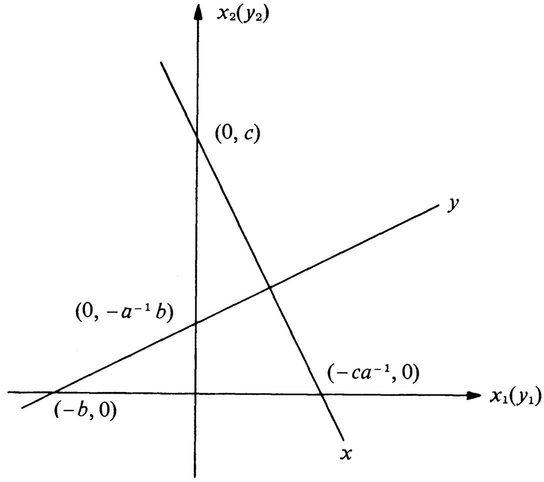


FIG. 1

(iii) $a = -1, b = 2, c = -3$,
(iv) $a = 0, b = 2, c = -3$.

In $m + n = 2$ analytical geometry (see Fig. 1) x's and y's satisfying the column and row equation systems are complementary orthogonal linear manifolds of dimensions $m = 1$ and $n = 1$, respectively. Feasible solutions lie in the nonnegative orthant ($x_k \geqq 0, y_k \geqq 0$). Thus it is easy to see how the four cases arise from the geometry: (i) both manifolds intersect the nonnegative orthant; if not, either (ii) the x-manifold or (iii) the y-manifold intersects the nonnegative orthant or (iv) neither does. These observations, instigated by the key question (4), have led to the aphorism, "duality is complementary orthogonality."

No special significance is attached to an equation system appearing as rows rather than as columns in a schema. In fact, given a schema, if its negative transpose is written without, only, changing the signs of the independent symbols, the same equation systems appear, as is illustrated here:

$$
\begin{array}{c|ccccc|c}
 & \cdots & \eta^* & \cdots & \eta & \cdots & \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
x^* & \cdots & a & \cdots & b & \cdots & = -y^* \\
\vdots & & \vdots & & \vdots & & \vdots \\
x & \cdots & c & \cdots & d & \cdots & = -y \\
\vdots & & \vdots & & \vdots & & \vdots \\
\hline
 & \cdots & = \xi^* & \cdots & = \xi & \cdots &
\end{array}
,\qquad
\begin{array}{c|ccccc|c}
 & \cdots & x^* & \cdots & x & \cdots & \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
\eta^* & \cdots & -a & \cdots & -c & \cdots & = -\xi^* \\
\vdots & & \vdots & & \vdots & & \vdots \\
\eta & \cdots & -b & \cdots & -d & \cdots & = -\xi \\
\vdots & & \vdots & & \vdots & & \vdots \\
\hline
 & \cdots & = y^* & \cdots & = y & \cdots &
\end{array}
.
$$

Thus, taking the negative transpose (with qualification) as defined here of the schema (1), we see that the row becomes the column and the column the row program. If the same row and column marginal information is appended, the identical programs result except that minimize $-u$ replaces maximize u and maximize $-v$ replaces minimize v.

2. Equivalent schemata, pivot transformations and the main theorem. We turn to the problem of determining optimal solutions, or showing none exist (and why), to a pair of dual linear programs as exhibited in the schema (1). Reconsider the complementary orthogonality property: $x_k = 0$ or/and $y_k = 0$ for all k. It suggests a trial and error approach: set some x's equal to 0, the complementary set of y's equal to 0, solve (if possible) for the remaining x's and y's, respectively, and hope that these latter values are nonnegative, i.e., that the solutions are both feasible and hence optimal. But the column (the row) equation system has only m (only n) "degrees of freedom" since it consists of $n + 1$ $(m + 1)$ linearly independent equations in $m + n + 1$ unknowns. Define, then, a *basic* solution of the column program (row program) as a solution to its equations, if existent and unique, having m of the x's set equal to 0 (n of the y's set equal to 0). Call basic (x, u)- and (y, v)-solutions *complementary* if the indices of the m x's and n y's set equal to 0 are complementary subsets of $\{1, 2, \cdots, m + n\}$. Now, search for optimal solutions from among the at most $\binom{m+n}{m}$ complementary basic solutions. This corresponds to trying all partitions of the $m + n$ indices into exclusive subsets of m and n indices. That it is sufficient to limit the search to such trials in analyzing a dual pair of linear programs is implicitly stated in the main theorem and has led to a "combinatorial linear algebra" and the concept of "combinatorial equivalence" [25].

As an example, consider the pair of dual linear programs in canonical form specified by

(5)

		y_3	y_4	y_5	1		
x's $\geqq 0$	x_1	0	1	1	-1	$= -y_1$	y's $\geqq 0$
u max	x_2	-1	8^*	-5	-2	$= -y_2$	v min
	1	1	-3	2	0	$= v$	
		$= x_3$	$= x_4$	$= x_5$	$= u$		

Complementary basic solutions to (5) are immediate: set $x_1 = x_2 = 0 = y_3 = y_4 = y_5$ in the schema to get $u = v = 0$ and $x = (0, 0, 1, -3, 2)$, $y = (1, 2, 0, 0, 0)$. Since the bottom row corresponding to the independent symbol 1 is not nonnegative (excluding the u-value), the (x, u)-solution is not feasible; however, the (y, v)-solution for the right row corresponding to the independent symbol 1 is nonpositive (excluding the v-value). Another trial of the at most $\binom{5}{2} = 10$ trials corresponds to the partition $\{1, 4\}$, $\{2, 3, 5\}$ of the indices $1, \cdots, 5$. The complementary basic solutions may be determined by solving the column system for x_2, x_3, x_5 and u in terms of x_1, x_4 and 1 and solving the row system for y_1, y_4 and v in terms of y_2, y_3, y_5 and 1. The result of this, as is easily verified by some elementary algebra, may be summarized simultaneously in the schema

(6)

	y_3	y_2	y_5	1	
x_1	$1/8$	$-1/8$	$13/8$	$-6/8$	$= -y_1$
x_4	$-1/8$	$1/8$	$-5/8$	$-2/8$	$= -y_4$
1	$5/8$	$3/8$	$1/8$	$-6/8$	$= v$
	$= x_3$	$= x_2$	$= x_5$	$= u$	

in which the column and row equation systems, and hence both programs, of (5) have simply been re-expressed or re-presented in terms of different complementary sets of independent variables. Setting the independent variables in (6) to 0 results in complementary basic feasible solutions since the row and column corresponding to the independent 1's (excluding the southeast corner value) are nonnegative and nonpositive, respectively. Thus, $x = (0, 3/8, 5/8, 0, 1/8)$ and $y = (6/8, 0, 0, 2/8, 0)$, with $u = v = -6/8$, are optimal solutions.

Two schemata, such as (5) and (6), exhibiting column and row equation systems, are said to be *equivalent* if any solution of the column or row equation system of one is a solution of the column or row equation system, respectively, of the other. This equivalent schema together with the information concerning objectives and sign constraints gives a "re-presentation" of the pair of dual linear programs in terms of a different partition into complementary dependent and independent variables obtained by solving the equations of the programs for dependent in terms of independent variables and 1.

More particularly, the transformation from the first schema to the second schema below

$$
\begin{array}{c|ccccc|l}
 & \cdots & \eta^* & \cdots & \eta & \cdots & \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
x^* & \cdots & a & \cdots & b & \cdots & = -y^* \\
\vdots & & \vdots & & \vdots & & \vdots \\
x & \cdots & c & \cdots & d & \cdots & = -y \\
\vdots & & \vdots & & \vdots & & \vdots \\
\hline
 & \cdots = & \xi^* & \cdots = & \xi & \cdots &
\end{array},
$$

(7)

$$
\begin{array}{c|ccccc|l}
 & \cdots & y^* & \cdots & \eta & \cdots & \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
\xi^* & \cdots & a^{-1} & \cdots & a^{-1}b & \cdots & = -\eta^* \\
\vdots & & \vdots & & \vdots & & \vdots \\
x & \cdots & -ca^{-1} & \cdots & d - ca^{-1}b & \cdots & = -y \\
\vdots & & \vdots & & \vdots & & \vdots \\
\hline
 & \cdots & = x^* & \cdots & = \xi & \cdots &
\end{array},
$$

with "pivot" entry $a \neq 0$, is called a *pivot* step. These schemata are equivalent because the second column system is gotten from the first by solving

$$\xi^* = \cdots + x^*a + \cdots + xc + \cdots$$

for $x^* = \cdots + \xi^*a^{-1} + \cdots - xca^{-1} - \cdots$, then using this latter equation to eliminate x^* from all other column equations

$$\xi = \cdots + x^*b + \cdots + xd + \cdots$$
$$= \cdots + (\cdots \xi^*a^{-1} - \cdots - xca^{-1} - \cdots)b + \cdots + xd \cdots .$$

Simultaneously, the second row system is obtained from the first by solving

$$-y^* = \cdots + a\eta^* + \cdots + b\eta + \cdots$$

for $-\eta^* = \cdots + a^{-1}y^* + \cdots + a^{-1}b\eta + \cdots$, and using the latter equation to eliminate η^* from all other row equations

$$-y = \cdots + c\eta^* + \cdots + d\eta + \cdots$$
$$= \cdots + c(\cdots - a^{-1}y^* - \cdots - a^{-1}b\eta - \cdots) + \cdots + d\eta + \cdots .$$

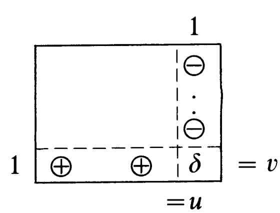
Thus, in going from the first to the second schema, the independent (or dependent) symbols of column and row systems are the same save for one symbol in each. In words, rules for the pivot step are these:

1. Exchange symbols corresponding to the pivot entry a; namely, replace η^* with y^* and $-y^*$ with $-\eta^*$ in the row system, and replace x^* with ξ^* and ξ^* with x^* in the column system.
2. Keep all other symbols (e.g., η, $-y$; x, ξ) unchanged.
3. Replace the pivot a by its reciprocal $1/a$; each entry b in the pivot's row by b/a; each entry c in the pivot's column by $-c/a$; and each entry d in the b's column and the c's row by $(d - bc/a)$.

As is easily verified, the schema (6) is obtained from (5) by a pivot step with pivot entry 8 designated by a star (*).

Beginning with the schema (1) and pivoting, we can generate a sequence of equivalent schemata each of which has "associated" complementary basic solutions obtained by setting the independent variables equal to 0. If a schema having the form

(8)



(where ⊕, ⊖ stand for nonnegative and nonpositive values, respectively) is obtained, its associated complementary solutions are feasible and hence optimal. In a slightly different interpretation, (8), together with information regarding objectives and sign constraints, is simply a representation for the programs of (1) which makes their solution obvious. For, the column basic feasible solution with independent variables set to 0 has $u = \delta$, but the equation for u implies $u \leqq \delta$ since u equals δ plus a nonpositive linear combination of nonnegative constrained (independent) variables. So, $\max u = \delta$. Similar reasoning applies to the row program. Of course, it may not be possible to obtain an equivalent schema (8).

THE MAIN THEOREM FOR LINEAR PROGRAMMING. *Given a schema* (1)

$$\begin{array}{c|ccc|c|l} & y_{m+1} & \cdots & y_{m+n} & 1 & \\ \hline x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_m \\ \hline 1 & c_1 & \cdots & c_n & d & = v \\ \hline & = x_{m+1} & & = x_{m+n} & = n & \end{array}$$

exhibiting a pair of equation systems, there exists an equivalent schema, obtainable in a finite number of pivot steps (*with no pivot in last row or column*), *which has one of the four mutually exclusive forms*:

(9)

(i)
$$\begin{array}{c|ccc|c|l} & & & & 1 & \\ \hline & & & & \ominus & \\ & & & & \vdots & \\ & & & & \ominus & \\ \hline 1 & \oplus & \cdots & \oplus & & = v \\ \hline & & & & = u & \end{array},$$

(ii)
$$\begin{array}{c|ccc|c|l} & & & & 1 & \\ \hline \vdots & & & & & \vdots \\ x_k & \oplus & \cdots & \oplus & + & = -y_k \\ \vdots & & & & & \vdots \\ \hline 1 & \oplus & \cdots & \oplus & & = v \\ \hline & & & & = u & \end{array},$$

(iii)
$$\begin{array}{c|ccc|c|l} & \cdots & y_l & \cdots & & \\ \hline & & \ominus & & \ominus & \\ & & \vdots & & \vdots & \\ & & \ominus & & \ominus & \\ \hline 1 & & - & & & = v \\ \hline & \cdots & = x_l & \cdots & = u & \end{array},$$

(iv)
$$\begin{array}{c|ccccc|c|l} & & \cdots & y_l & \cdots & & 1 & \\ \hline \vdots & & & \ominus & & & & \vdots \\ & & & \cdot & & & & \\ y_k & \oplus & \cdots & \cdot & \cdots & \oplus & + & = -y_k \\ & & & \cdot & & & & \\ \vdots & & & \ominus & & & & \vdots \\ \hline 1 & & & - & & & & = v \\ \hline & & \cdots & = x_l & \cdots & & = u & \end{array}$$

The proof is deferred.

These four forms, when taken to represent the equation systems of a pair of dual linear programs in canonical form (1), correspond to the four possibilities discussed in § 1. Form (i) shows the program to have solutions obtained by setting the complementary independent variables equal to 0 and reading off the nonnegative values for the dependent x's and y's from the row and column, respectively, corresponding to the independent 1's. Form (ii) shows the column program to have feasible solutions obtained by setting x_k equal to any nonnegative value and the remaining independent x's equal to 0; however, this shows u to be unbounded above since an increase in x_k implies an increase in u. At the same time (ii) shows that the row program has no feasible solution since an equation expresses $-y_k$, constrained to be nonpositive, as the sum of a positive constant and a nonnegative linear combination of nonnegative variables. Form (iii) is similar to (ii) except the roles of column and row programs are interchanged. Finally, (iv) shows that column and row programs have no feasible solutions.

Three corollaries, useful in what follows, are immediate consequences of the main theorem.

COROLLARY 1. *Given a schema*

$$\begin{array}{c|ccc|l} & y_{m+1} & \cdots & y_{m+n+1} & \\ \hline x_1 & a_{11} & \cdots & a_{1,n+1} & = -y_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_m & a_{m1} & \cdots & a_{m,n+1} & = -y_m \\ \hline 1 & c_1 & \cdots & c_{n+1} & = v \\ \hline & = x_{m+1} & & = x_{m+n+1} & \end{array} \tag{10}$$

there exists an equivalent schema, obtainable in a finite number of pivot steps (*with no pivots in last row*), *which has one of the two mutually exclusive forms*:

$$\text{(v)} \quad \begin{array}{c|ccc|l} & & & & \\ \hline & & & & \\ & & & & \\ \hline 1 & \oplus & \cdots & \oplus & = v \\ \hline \end{array} \quad , \qquad \text{(vi)} \quad \begin{array}{c|c|l} & y_l & \\ \hline & \ominus & \\ & \vdots & \\ & \ominus & \\ \hline 1 & - & = v \\ \hline & = x_l & \end{array} \quad . \tag{11}$$

Proof. Assume the main theorem is established and identify y_{m+n+1} with 1 and x_{m+n+1} with u in (1). Then, if (i), (iii) or (iv) of (9) results, either (v) or (vi) is obtained directly. If (ii) results and (vi) does not, then we have a schema of form

$$\begin{array}{c|ccc|c|l} & & & & y_{m+n+1} & \\ \hline & \oplus & \cdots & \oplus & +^{*} & \\ \hline 1 & \oplus & \cdots & \oplus & - & = v \\ \hline & & & & = x_{m+n+1} & \end{array} \tag{12}$$

One pivot step with pivot entry designated by the star results in form (v).

The corollary simply states this: either (v) the column equation system of (10) has a solution with x's $\geqq 0$ or (vi) not. Further classical results concerning systems of linear inequalities are immediate consequences of this corollary (see below). Of course, the same corollary holds if the forms in (10) and (11) are altered to their negative transposes. This yields the following corollary.

COROLLARY 2. *Given a schema*

$$\begin{array}{c|ccc|c|l} & y_{m+2} & \cdots & y_{m+n+1} & 1 & \\ \hline x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_{m+1} & a_{m+1,1} & \cdots & a_{m+1,n} & b_{m+1} & = -y_{m+1} \\ \hline & = x_{m+2} & \cdots & = x_{m+n+1} & = u & \end{array} \quad , \tag{13}$$

there exists an equivalent schema, obtainable in a finite number of pivot steps (with no pivots in the last column), which has one of the two mutually exclusive forms

$$
\text{(vii)}\quad
\begin{array}{c|c|}
 & 1 \\
\hline
\quad\quad\quad & \ominus \\
 & \vdots \\
 & \ominus \\
\hline
 & =u
\end{array}\;,
\qquad
\text{(viii)}\quad
\begin{array}{r|ccc|c|l}
 & & & & 1 & \\
\hline
 & & & & & \\
x_k & \oplus & \cdots & \oplus & + & = -y_k. \\
 & & & & & \\
\hline
 & & & & =u &
\end{array}
\tag{14}
$$

A final and immediate conclusion is now given.

COROLLARY 3. *Given a schema*

$$
\begin{array}{r|ccc|l}
 & y_{m+2} & \cdots & y_{m+n+2} & \\
\hline
x_1 & a_{11} & \cdots & a_{1,n+1} & = -y_1 \\
\vdots & \vdots & & \vdots & \quad\vdots \\
x_{m+1} & a_{m+1,1} & \cdots & a_{m+1,n+1} & = -y_{m+1} \\
\hline
 & =x_{m+2} & & =x_{m+n+2} &
\end{array}\;,
\tag{15}
$$

there exists an equivalent schema, obtainable in a finite number of pivot steps, which has one (or both) of the forms

$$
\begin{array}{r|ccc|l}
\hline
\vdots & & & & \vdots \\
x_k & \oplus & \cdots & \oplus & = -y_k, \\
\vdots & & & & \vdots \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|}
\cdots & y_l & \cdots \\
\hline
 & \ominus & \\
 & \vdots & \\
 & \ominus & \\
\hline
\cdots & =x_l & \cdots
\end{array}\;.
\tag{16}
$$

3. Simplex methods and a proof of the main theorem. Consider, now, the problem of taking a schema of form (1) and pivoting on it and each of its successors in such a way as to obtain a finite sequence of equivalent schemata, each containing representations for the pair of dual linear programs (2) and (3) and terminating with a schema having one of the four forms in (9). The highly successful computational, and theoretical, algorithm by which this can be accomplished is the *simplex method* [8] first proposed in 1947 by G. D. Dantzig.

The idea behind the algorithm is simple. Assume (1) is a schema whose basic row solution is feasible, i.e., the column corresponding to the independent 1 has nonpositive components (excluding the bottom entry) as, for example, in schema (5). Then pivot in such a way as to obtain equivalent schema whose basic row solution is also feasible but has an improved value for v, i.e., a smaller (or at least not greater) southeast corner entry, as, for example, in schema (6). Repeat. In other terms, each pivot is made to find a better trial solution to the row program. A completely analogous procedure can be given for going from one schema with associated column basic feasible solution to another improved one, having the value of u larger (or at least not smaller). This latter approach as a means for solving the row program was first suggested by C. E. Lemke [19]. Although it can be immediately derived from the other (or vice versa) by using the negative transpose observation regarding schemata, we give the explicit rules for both approaches simultaneously below.

A *row* (a *column*) *simplex method* for solving a pair of dual linear programs in canonical form moves by a finite succession of pivot steps from one schema of form (1) with associated row (column) basic feasible solution to another, and so on, until either form (i) or form (iii) (either form (i) or form (ii)) is obtained. The assumption of row (column) feasibility rules out the occurrence of forms (ii) and (iv) (forms (iii) and (iv)). Thus, at any stage, a schema

$$
(17)\qquad
\begin{array}{c|ccc|c|l}
 & y_{\overline{m+1}} & & y_{\overline{m+n}} & 1 & \\
\hline
x_{\bar{1}} & \bar{a}_{11} & \cdots & \bar{a}_{1n} & \bar{b}_1 & = -y_{\bar{1}} \\
\vdots & \vdots & & \vdots & \vdots & \quad\vdots \\
x_{\bar{m}} & \bar{a}_{m1} & \cdots & \bar{a}_{mn} & \bar{b}_m & = -y_{\bar{m}} \\
\hline
1 & \bar{c}_1 & \cdots & \bar{c}_n & \bar{d} & = v \\
\hline
 & =x_{\overline{m+1}} & \cdots = & x_{\overline{m+n}} & =u &
\end{array},
$$

where $\bar{1}, \bar{2}, \cdots, \overline{m+n}$ denotes some permutation of the subscripts $1, 2, \cdots, m+n$, and (17) is equivalent to (1), is at hand with $\bar{b}_i \leqq 0$ for all i (with $\bar{c}_j \leqq 0$ for all j). A *row* (a *column*) *pivot step* is made as follows. If schema (17) is not of form (i), that is, does not exhibit associated basic feasible solutions to both programs, there must exist a $\bar{c}_j < 0$ for some j (a $\bar{b}_i > 0$ for some i). Either (α) every entry in the column of $\bar{c}_j < 0$ is nonpositive (every entry in the column of $\bar{b}_i > 0$ is nonnegative) or (β) not.

(α) The schema is in form (iii) (in form (ii)).

(β) Pivot with pivot entry $\bar{a}_{kj} > 0$ ($\bar{a}_{il} < 0$) satisfying

$$
(18)\qquad \bar{b}_k/\bar{a}_{kj} = \max_s \{\bar{b}_s/\bar{a}_{sj};\, \bar{a}_{sj} > 0\} \qquad (\bar{c}_l/\bar{a}_{il} = \max_s \{\bar{c}_s/\bar{a}_{is};\, \bar{a}_{is} < 0\}).
$$

The choice of pivot entry assures that the new schema again has an associated basic feasible row (column) solution. If $\bar{b}_k < 0$ (if $\bar{c}_l > 0$), the new schema has southeast entry strictly *less* (strictly *greater*) than the previous exhibited value, for $\bar{d}$ in (17) is transformed into $\bar{d} - \bar{b}_k\bar{c}_j/\bar{a}_{kj} < \bar{d}$ (into $\bar{d} - \bar{b}_i\bar{c}_l/\bar{a}_{il} > \bar{d}$). Thus, in this case, the pivot step results in a trial feasible row (column) solution which is strictly better, for its v value is less (u value is greater). An example of a row simplex method is given below.

This last observation provides a proof that a row (column) simplex method must terminate in a finite number of pivot steps *if* it may be assumed that the sequence of equivalent schemata (17) always have all $\bar{b}_i < 0$, i.e., no $\bar{b}_i = 0$ (all $\bar{c}_j > 0$, i.e., no $\bar{c}_j = 0$). For, since the southeast corner entry of every successive schema is strictly smaller (larger), no two of the equivalent schemata in the sequence can have the same set of independent row (column) variables. There being only a finite number of possible schemata, either form (i) or form (iii) (either (i) or (ii)) must be obtained.

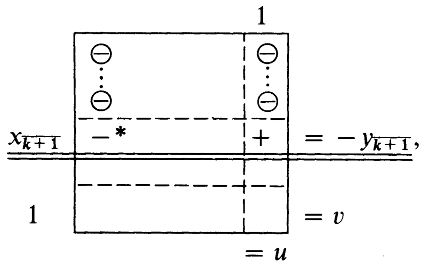
If "degeneracy" occurs, i.e., if schemata (17) in the sequence have some $\bar{b}_i = 0$ (some $\bar{c}_j = 0$), then it is entirely possible for some rules for choice of pivot entry to lead to a subsequence of equivalent schemata which repeats itself, which "cycles." An example exhibiting this phenomenon is shown below. There, the choice of $\bar{c}_j < 0$ is made by the usual computational rule, namely, $\bar{c}_j = \min_k \bar{c}_k$. This example shows that some auxiliary measure of "improvement" needs to be imposed in the event of degeneracy. Many approaches to this problem have been suggested. The proof of the main theorem given below is the inductive counterpart of a constructive procedure which proposes a particularly "natural" measure of improvement [3].

Before proceeding we consider an approach, prerequisite to the method of the proof, for obtaining a schema with associated basic row feasible solution (applying it to the negative transposed schema yields an approach for obtaining a schema with associated basic column feasible solution). Suppose that (17) does not have an associated basic row feasible solution, i.e., $\bar{b}_i > 0$ for some i. Rearrange the rows of the schema so that $\bar{b}_i \leqq 0$ for $i \leqq k$ and $\bar{b}_i > 0$ for $i > k$. Then replacing the $\bar{b}_i$ by their signs, we may exhibit (17) as

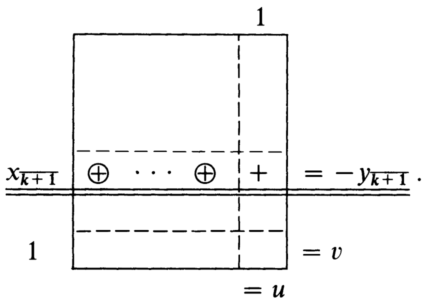
$$
(19)\qquad
\begin{array}{c|ccc|c|l}
 & y_{\overline{m+1}} & \cdots & y_{\overline{m+n}} & 1 & \\
\hline
x_{\bar{1}} & \bar{a}_{11} & & \bar{a}_{1n} & \ominus & = -y_{\bar{1}} \\
\vdots & \vdots & & \vdots & \vdots & \quad\vdots \\
x_{\bar{k}} & & & & \ominus & = -y_{\bar{k}} \\
x_{\overline{k+1}} & & & & + & = -y_{\overline{k+1}} \\
\hline\hline
\vdots & \vdots & & \vdots & \vdots & \quad\vdots \\
x_{\bar{m}} & \bar{a}_{m1} & & \bar{a}_{mn} & + & = -y_{\bar{m}} \\
1 & & & & \bar{d} & = v \\
\hline
 & = x_{\overline{m+1}} & & = x_{\overline{m+n}} & = u &
\end{array}
\; .
$$

In (19) the subschema above the double line may be thought of as specifying a pair of subprograms in canonical form with associated basic feasible row solution with $x_{\overline{k+1}}$ as the temporary independent 1. Therefore, a row simplex method choice of pivot entry can be used directly in (temporarily) minimizing $-y_{\overline{k+1}}$ subject to the constraints as specified in the first k rows. If, after one or several pivot steps on (19), the sign of entry corresponding to $\bar{b}_{k+1}$ becomes nonpositive, then one more row with "correct" sign has been generated and a larger subschema (perhaps the entire schema) can be identified and thought of as specifying a pair of subprograms with associated basic feasible row solution. Otherwise, one of the following forms must be obtained:

(20a)



(20b)

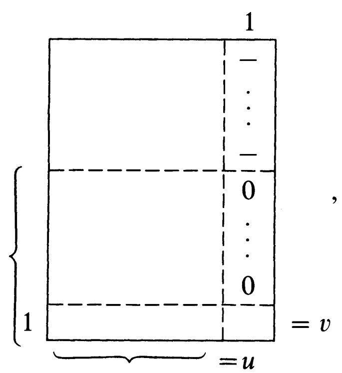


If (20a) occurs, a pivot step with pivot entry starred leads to an equivalent schema whose top $k + 1$ rows (if not others) have $\bar{b}_i \leqq 0$; while if (20b) occurs, no feasible row solution exists. Systematic use of this idea of choosing pivot entries by row (and column) simplex method rules applied to well-defined subschemata is the basis of our method of proof. An example of the above procedure in which a row simplex method is applied follows below.

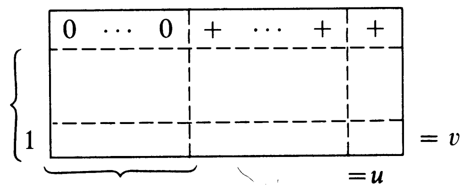
Given schema (1), the proof of the main theorem shows how an equivalent schema having one of the four forms of (9) is obtained in a finite number of pivot steps. The proof is by induction on the number of rows plus columns. If there is either only one row or only one column in (1), one of the forms of the theorem is immediate. Suppose the theorem true, and hence its corollaries true, for the number of rows plus columns less than $m + n + 2$.

Applying the inductive hypothesis to (1) with its last row ignored, we must obtain, by Corollary 2, either of the schemata (after rearrangement of rows and columns according to sign)

(21a)



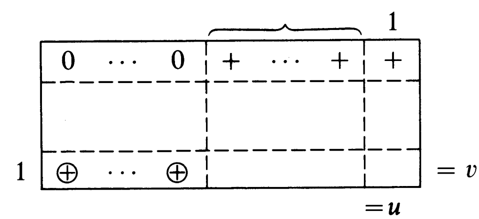
(21b)



equivalent to (1) in a finite number of pivot steps. The induction at this point corresponds to obtaining a schema with associated basic feasible row solution or to showing none exists (compare with the approach for obtaining such schemata given above and (20a), (20b)).

If (21b) holds, applying Corollary 1 to the subschema of (21b) whose rows and columns are bracketed (and are less than $m + n + 2$ in number), we must obtain a schema either of form (ii) or (iv), or of form

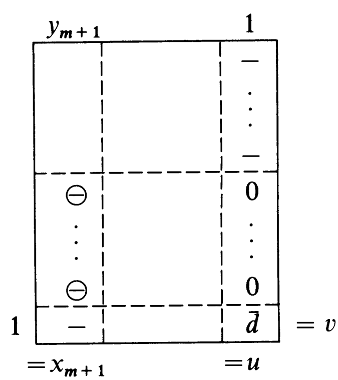
(22)



equivalent to (1) in a finite number of pivot steps. One pivot step on (22) with pivot entry in the top row of the bracketed columns and $\bar{a}_{ij}$ chosen so that $\bar{c}_j/\bar{a}_{ij} = \min_s \{\bar{c}_s/\bar{a}_s; \bar{c}_s < 0\}$ obtains (ii). The induction at this point corresponds to obtaining a schema with associated basic feasible column solution or to showing none exists (compare with the approach for obtaining such schemata above and the negative transpose of (20a), (20b)).

If (21a) holds, applying Corollary 1 to the subschema of (21a) whose rows and columns are bracketed (and are less than $m + n + 2$ in number), we must obtain a schema either of form (i), (iii) or of form (after rearrangement of columns)

(23)



equivalent to (1) in a finite number of steps, where the column corresponding to the independent variable y_{m+1} contains one or more positive entries. But a row simplex method pivot step applied to (23) with pivot in the column corresponding to y_{m+1} results in an equivalent schema with southeast corner entry strictly *less.* Continued application of a row simplex method or of the inductive hypothesis applied to a schema of form (21a) then assures that the southeast corner entry of successive schemata never increases and always strictly decreases within finite numbers of pivot steps. Therefore, as in the "nondegenerate" case, either an equivalent schema of form (i) or of form (iii) must be obtained in a finite number of steps. The induction at this point corresponds to showing no column feasible solution exists or to obtaining a schema which permits strict improvement in the value of the southeast corner entry in one pivot step.

This completes the proof. It should be noted, however, that to every application of the inductive hypothesis there corresponds a constructive set of rules for choice of pivot entry which achieve the same results. These rules derive from row and column simplex method pivot entry choice rules applied to appropriate subschemata, sub-subschemata, etc. [3]. The idea behind this is simple: faced with a schema (21a) in which it is possible to pivot without strictly decreasing the value of the southeast corner entry, switch the objective or measure of improvement and ask of the subschema bracketed in (21a) whether or not it has a feasible column solution. This latter can be done constructively by using a column simplex method (as outlined above in using a row simplex method for finding a schema with associated basic feasible row solution), unless zeros appear in the bottom row. Reapply, then, the same idea by asking of a sub-subschema whether or not it has a feasible row solution by using a row simplex method, and so forth. Of course, if no degeneracies occur at all, then the constructive counterpart of the proof is a straightforward row simplex method.

Example 1. *Solution by the row simplex method.* Consider the pair of dual linear programs in canonical form exhibited in the schema and marginal information:

(a)		y_1	y_2	y_3	1		
	x_4	-3	-1^*	0	1	$= -y_4$	
x's $\geqq 0$	x_5	-4	-3	-1	6	$= -y_5$	y's $\geqq 0$
u max	x_6	-1	-2	1	3	$= -y_6$	v min
	1	-1	1	1	0	$= v$	
		$= x_1$	$= x_2$	$= x_3$	$= u$		

We first obtain a row feasible solution, by the approach outlined above, in schema (d); then by a row simplex method, pivot to termination. Schemata are lettered in sequence.

(b)

	y_1	y_4	y_3	1	
x_2	3	−1	0	−1	$= -y_2$
x_5	5	−3	−1*	3	$= -y_5$
x_6	5	−2	1	1	$= -y_6$
1	−4	1	1	1	$= v$
	$=x_1$	$=x_4$	$=x_3$	$=u$	

(c)

	y_1	y_4	y_5	1	
x_2	3	−1	0	−1	$= -y_2$
x_3	−5	3*	−1	−3	$= -y_3$
x_6	10	−5	1	4	$= -y_6$
1	1	−2	1	4	$= v$
	$=x_1$	$=x_4$	$=x_5$	$=u$	

(d)

	y_1	y_3	y_5	1	
x_2	4/3	1/3	−1/3	−2	$= -y_2$
x_4	−5/3	1/3	−1/3	−1	$= -y_4$
x_6	5/6*	5/3	−2/3	−1	$= -y_6$
1	−7/3	2/3	1/3	2	$= v$
	$=x_1$	$=x_3$	$=x_5$	$=u$	

(e)

	y_6	y_3	y_5	1	
x_2	−4/5	−1	1/5*	−6/5	$= -y_2$
x_4	1	2	−1	−2	$= -y_4$
x_1	3/5	1	−2/5	−3/5	$= -y_1$
1	7/5	3	−3/5	3/5	$= v$
	$=x_6$	$=x_3$	$=x_5$	$=u$	

(f)

	y_6	y_3	y_2	1	
x_5	−4	−5	5	−6	$= -y_5$
x_4	−3	−3	5	−8	$= -y_4$
x_1	−1	−1	2	−3	$= -y_1$
1	−1	0	3	−3	$= v$
	$=x_6$	$=x_3$	$=x_2$	$=u$	

The last schema shows that there are no optimal solutions (case (iii) of the main theorem). [Note that one pivot step could have got from (a) to (f).]

Example 2. *Cycling*. Consider the example, which grew out of a classroom example of H. W. Kuhn's similar to one of E. M. L. Beale, specified by the schema and marginal information:

		y_4	y_5	y_6	y_7	1		
	x_1	-2	-9	1	9	0	$= -y_1$	
x's $\geqq 0$	x_2	$1/3$	1	$-1/3$	-2	0	$= -y_2$	y's $\geqq 0$
u max	x_3	2	3	-1	-12	-2	$= -y_3$	v min
	1	-2	-3	1	12	0	$= v$	
		$= x_4$	$= x_5$	$= x_6$	$= x_7$	$= u$		

The schema is in row feasible form and we apply a row simplex method in which the column of the pivot entry is chosen as that having the "most negative" bottom entry (while this is the common rule used in computation, see [17] for a survey with computational experimentation of alternate choice rules) and the pivot entry as the topmost entry satisfying the "ratio test" (18). It should be noted that if the row corresponding to $-y_3$ is dropped altogether, the resulting example again exhibits cycling, but for a row program which is "homogeneous," that is, in which the constants in the column of 1 are identically zero. Schema (g) is identical to (a), save for arrangement of columns.

(a)

	y_4	y_5	y_6	y_7	1	
x_1	-2	-9	1	9	0	$= -y_1$
x_2	$1/3$	1^*	$-1/3$	-2	0	$= -y_2$
x_3	2	3	-1	-12	-2	$= -y_3$
1	-2	-3	1	12	0	$= v$
	$= x_4$	$= x_5$	$= x_6$	$= x_7$	$= u$	

(b)

	y_4	y_2	y_6	y_7	1	
x_1	1^*	9	-2	-9	0	$= -y_1$
x_5	$1/3$	1	$-1/3$	-2	0	$= -y_5$
x_3	1	-3	0	-6	-2	$= -y_3$
1	-1	3	0	6	0	$= v$
	$= x_4$	$= x_2$	$= x_6$	$= x_7$	$= u$	

(c)

	y_1	y_2	y_6	y_7	1	
x_4	1	9	-2	-9	0	$= -y_4$
x_5	$-1/3$	-2	$1/3$	1*	0	$= -y_5$
x_3	-1	-12	2	3	-2	$= -y_3$
1	1	12	-2	-3	0	$= v$
	$=x_1$	$=x_2$	$=x_6$	$=x_7$	$=u$	

(d)

	y_1	y_2	y_6	y_5	1	
x_4	-2	-9	1*	9	0	$= -y_4$
x_7	$-1/3$	-2	$1/3$	1	0	$= -y_7$
x_3	0	-6	1	-3	-2	$= -y_3$
1	0	6	-1	3	0	$= v$
	$=x_1$	$=x_2$	$=x_6$	$=x_5$	$=u$	

(e)

	y_1	y_2	y_4	y_5	1	
x_6	-2	-9	1	9	0	$= -y_6$
x_7	$1/3$	1*	$-1/3$	-2	0	$= -y_7$
x_3	2	3	-1	-12	-2	$= -y_3$
1	-2	-3	1	12	0	$= v$
	$=x_1$	$=x_2$	$=x_4$	$=x_4$	$=u$	

(f)

	y_1	y_7	y_4	y_5	1	
x_6	1*	9	-2	-9	0	$= -y_6$
x_2	$1/3$	1	$-1/3$	-2	0	$= -y_2$
x_3	1	-3	0	-6	-2	$= -y_3$
1	-1	3	0	6	0	$= v$
	$=x_1$	$=x_7$	$=x_4$	$=x_5$	$=u$	

(g)

	y_6	y_7	y_4	y_5	1	
x_1	1	9	-2	-9	0	$= -y_1$
x_2	$-1/3$	-2	$1/3$	1	0	$= -y_2$
x_3	-1	-12	2	3	-2	$= -y_3$
1	1	12	-2	-3	-2	$= v$
	$=x_6$	$=x_7$	$=x_4$	$=x_5$	$=u$	

4. Existence, duality and complementary slackness. We are now in the position to deduce the central theorems of linear programming.

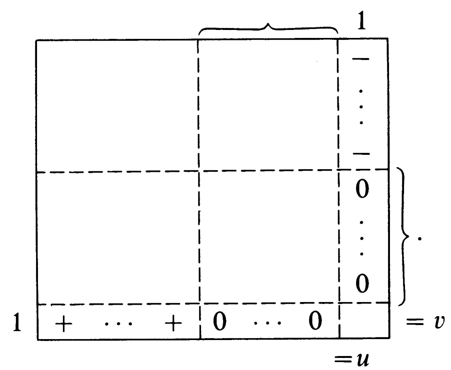
EXISTENCE THEOREM (see [12]). *A necessary and sufficient condition that one of a pair of dual linear programs in canonical forms* (2) *and* (3) *have optimal solutions is that both have feasible solutions.*

DUALITY THEOREM (see [12]). *If feasible solutions exist to both of a pair of dual linear programs in canonical forms* (2) *and* (3), *then optimal solutions exist with* $u = v$.

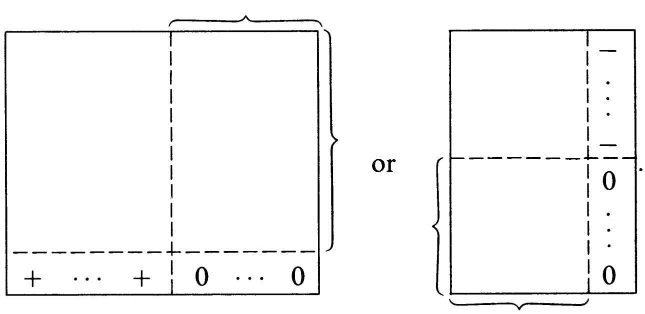
These theorems are immediate consequences of the main theorem since either form (i) is obtained or form (ii), (iii) or (iv) is obtained, whereas the forms are mutually exclusive.

COMPLETE COMPLEMENTARY SLACKNESS THEOREM. *If feasible solutions exist to both of a pair of dual linear programs in canonical forms* (2) *and* (3), *then optimal solutions* $x^* = (x_1^*, \cdots, x_{m+n}^*)$ *and* $y^* = (y_1, \cdots, y_{m+n}^*)$ *exist satisfying* $x_k^* y_k^* = 0$ *and* $x_k^* + y_k^* > 0$ *for each* k; *that is* $x_k^* > 0$ $(y_k^* > 0)$ *if and only if* $y_k^* = 0$ $(x_k^* = 0)$.

This is easily proved. By the main theorem, an equivalent schema of form (i) can be obtained which, after rearrangement of rows and columns, may be displayed as

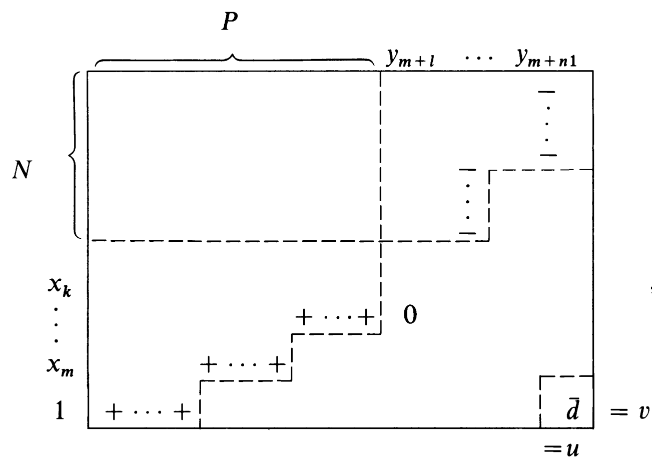


Apply Corollary 3 to the subschema indicated by the brackets and rearrange the rows and columns so that the subschema has one of the two forms



Apply again the same corollary to the second subschema indicated and continue to repeat the same construction until a subschema is obtained which is composed of zeros or is empty. This results in a schema equivalent to (1) having a form

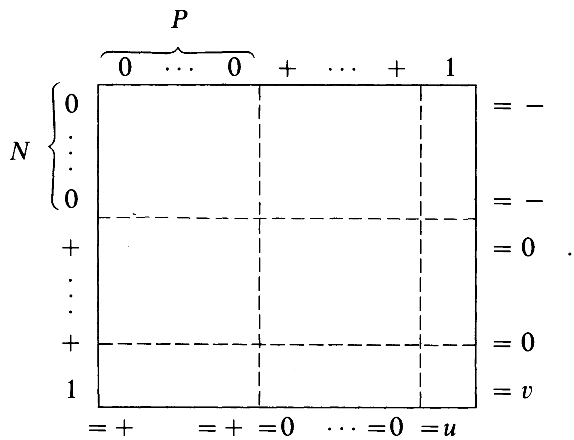
(24)



,

where the columns corresponding to P and not N are "positively tailed," i.e., each column contains a positive entry which is the last nonzero entry of the column; the rows corresponding to N and not P are "negatively tailed," i.e., each row contains a negative entry which is the last nonzero entry in the row; and the entries below the stairlike structure in the region marked 0 are all 0.

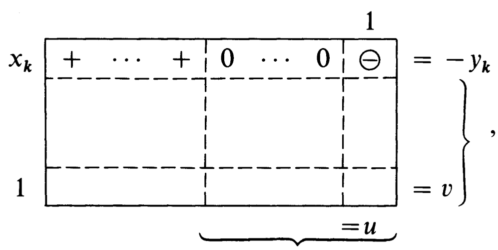
To obtain a solution x^*, y^* as described above, set $x_i^* = 0$ for x_i independent and corresponding to a row of N; and set $y_j^* = 0$ for y_j independent and corresponding to a column of P. Let $x_i = 0$ and $y_j = 0$ for the remaining independent variables temporarily. Then, change x_m by letting $x_m = x_m^* > 0$, with x_m^* small enough so as to leave positive the values of dependent x's corresponding to columns in P whose entries in the row of x_m are not in the 0-region. Repeat for $x_{m-1}, \cdots, x_k$ in that order, that is, change $x_j, j \geqq k$, by letting $x_j = x_j^* > 0$ with x_j^* small enough so as to leave positive the values of dependent x's corresponding to columns in P whose entries in the row of x_j are not in the 0-region. This assignment is clearly possible and maintains the condition that all dependent x's have nonnegative values. Thus we have that $y_j^* = 0$ for independent y_j in P and $x_j^* > 0$ for dependent x_j in P; and $x_k^* > 0$ for independent x_k not in N and $y_k^* = 0$ for dependent y_k not in N (no matter what values are given to independent y's not in P). Repeat a similar construction to obtain $y_{m+l}^* > 0, \cdots, y_{m+n}^* > 0$ for independent y's not in P and corresponding $x_{m+l}^* = 0, \cdots, x_{m+n}^* = 0$ for dependent x's not in P. Schematically we have



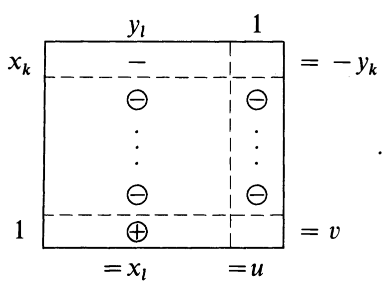
UNBOUNDED CONSTRAINT SET THEOREM. *If at least one program is feasible in a pair of dual linear programs in canonical form, then the constraint set of at least one program of the pair is unbounded.*

The truth of this statement can be construed from the $(m + n)$-space picture of the complementary orthogonal linear manifolds of solutions to row and column systems. More exactly, if (ii) or (iii) of the main theorem occurs, the conclusion is immediate. Otherwise (i) occurs. Apply the main theorem to a subschema excluding the row corresponding to the independent 1 and with the top row considered as the distinguished row. After rearrangement of columns, one of two schemata must result:

(25a)



(25b)



If (25b) is obtained, the row feasible set is unbounded as is seen by letting y_l take arbitrarily large values. If (25a) is obtained, apply the main theorem to the bracketed subschema to get

$$
\begin{array}{c|ccc:ccc:c|l}
 & & & & & & & 1 & \\
\hline
x_k & + & \cdots & + & 0 & \cdots & 0 & \ominus & = -y_k \\
\hdashline
 & & & & & & & \ominus & \\
 & & & & & & & \vdots & \\
 & & & & & & & \ominus & \\
\hdashline
1 & & & & \oplus & \cdots & \oplus & & = v \\
\hline
 & & & & & & & = u &
\end{array}
\quad ,
$$

where x_k can be given arbitrarily large values.

5. General duality and reduction to canonical form. We investigate, now, general formulations of pairs of dual linear programs. We do this in two stages, first considering a "symmetric standard form," then a most "general form," and show how pivoting, together with other simple rules, permits reduction of the programs to a form which can be treated as canonical or which shows "inconsistencies" in the constraints of one or both problems.

Consider the schema

$$
(26) \qquad
\begin{array}{c}
X\text{'s} \geqq 0 \\
u \max
\end{array}
\qquad
\begin{array}{c|ccc|l}
 & -V_2 & Y_2 & 1 & \\
\hline
U_1 & A_{11} & A_{12} & B_1 & = 0 \\
X_1 & A_{21} & A_{22} & B_2 & = -Y_1 \\
1 & C_1 & C_2 & d & = v \\
\hline
 & = 0 & = X_2 & = u &
\end{array}
\qquad
\begin{array}{c}
Y\text{'s} \geqq 0 \\
v \min
\end{array}
\quad ,
$$

where A_{ij} is a matrix of p_i rows and q_j columns and the C's and B's and symbols in the margins are vectors of corresponding dimensions. The schema exhibits two systems of linear equations obtained as before by forming appropriate inner products. Schema (26) together with the additional information specifies a pair of dual linear programs, where the A's, B's, C's and d are given:

Column Program	*Row Program*
Find X's and U's to	Find Y's and V's to
maximize $u = U_1B_1 + X_1B_2 + d$	minimize $v = -C_1V_2 + C_2Y_2 + d$
when	when
$U_1A_{11} + X_1A_{21} + C_1 = 0,$	V_2 unconstrained,
$U_1A_{12} + X_1A_{22} + C_2 = X_2 \geqq 0,$	$Y_2 \geqq 0,$
U_1 unconstrained,	$-A_{11}V_2 + A_{12}Y_2 + B_1 = 0,$
$X_1 \geqq 0.$	$-A_{21}V_2 + A_{22}Y_2 + B_2 = -Y_1 \leqq 0.$

(27)

The programs are said to be in *symmetric standard form*, for each contains equation and inequality constraints in nonnegative constrained and unconstrained, or "free," variables. The terminology is adopted from another form for a pair of dual linear programs which is much considered, the (row) *standard form*: it may be defined in (26) by having $p_2 = 0$ and $q_1 = 0$. The symmetric standard form is of some significance, for the min-cut max-flow theorem of network flow theory may be obtained directly through formulation of the maximum flow problem as a linear program of this form and the minimum cut problem as its dual, and an application of the (generalized) main theorem. In contrast with programs in canonical form there is no guarantee that either program of (27) has a system of linear equations which is consistent (ignoring sign constraints). However, the symmetric standard form contains the canonical form: if $p_1 = q_1 = 0$, then form (1) is obtained. It should be noted that if $p_2 = q_2 = 0$, i.e., no nonnegative constrained variables are present in either program, then no proper optimization problems are obtained: only systems of linear equations remain and these either have no solutions, unique solutions, or linear manifolds of solutions (whence solutions giving either a unique or arbitrary value to an objective).

Again, the easy part of the duality statement for the programs (27) is immediate. Equating the inner product of the symbol-columns on the left and right of (26) with that of the symbol-rows on the top and bottom, one obtains the "key-equation" $v - u = X_1 Y_1 + X_2 Y_2$ which holds for any values of the variables satisfying the column and the row equations, i.e., for any solutions. Thus, for feasible solutions, $\max u \leqq \min v$ as before.

To obtain the difficult part of the duality statement for (27) the following process is used to reduce this case to the case of programs in canonical form, if possible. The underlying idea is to pivot in order to make all zero marginal symbols independent or, what is the same thing, to make all "free" (unconstrained) variables dependent. Consider, then, the pair of dual linear programs (27) and apply the following rules to (26) and subsequent schemata.

Reduction rules. (i) Pivot on an entry which is in a row (a column) whose equation expresses a dependent marginal zero but is not in the column (the row) whose equation expresses u (v), if possible.

Each application of Rule (i) results in at least one marginal zero becoming independent and its paired free variable u_i of U_1 or/and v_j of V_2 becoming dependent.

(ii) Eliminate the columns or/and row whose equation expresses the dependent free variable u_i or/and v_j from the resulting schema and "file" the equation(s) for u_i or/and v_j. Return to Rule (i).

Each application of Rule (ii) results in a schema which contains one less column or/and row and a file which contains at least one more equation. It is permissible to drop these for the following reason. If a column (a row) corresponding to a dependent free u_i (v_j) is dropped, the linear program of the column (the row) system is not changed since u_i (v_j) is unconstrained; in other terms, the equation serves only to give the value of u_i (of v_j) in terms of other variables. The linear program of the row (column) systems is unchanged as well since the column (the row) which is dropped is a column (a row) of coefficients attached to an independent marginal zero. The file which is constructed by application of the reduction rules simply enables computation of the values of free variables eliminated from the schema by Rule (ii), once the remaining variables have been assigned values.

If no pivoting as described in Rule (i) is possible, a schema of form

(28)
$$\begin{array}{c|ccc|l} & -V_{\bar 2} & Y_{\bar 2} & 1 & \\ \hline U_{\bar 1} & 0 & 0 & \bar{B}_1 & = 0 \\ X_{\bar 1} & 0 & \bar{A}_{22} & \bar{B}_2 & = -Y_{\bar 1} \\ 1 & \bar{C}_1 & \bar{C}_2 & \bar{d} & = v \\ \hline & =0 & =X_{\bar 2} & =u & \end{array}$$

holds, where $(X_{\bar 1}, X_{\bar 2})$ and $(Y_{\bar 1}, Y_{\bar 2})$ are some permutation of (X_1, X_2) and (Y_1, Y_2), and $U_{\bar 1}$ and $V_{\bar 2}$ are subsets of variables of U_1 and V_2, possibly void.

Either $\bar{C}_1 = 0$ and $\bar{B}_1 = 0$, or not. In the former case both equation systems are consistent; the row equations corresponding to $U_{\bar 1}$ and the column equations corresponding to $-V_{\bar 2}$ simply say $0 = 0$, and variables of $U_{\bar 1}$ and $V_{\bar 2}$ only enter with coefficients 0. Thus, in this case, (28) is equivalent to the schema in canonical form

(29)
$$\begin{array}{c|cc|l} & Y_{\bar 2} & 1 & \\ \hline X_{\bar 1} & \bar{A}_{22} & \bar{B}_2 & = -Y_{\bar 1} \\ 1 & \bar{C}_2 & \bar{d} & = v \\ \hline & =X_{\bar 2} & =u & \end{array},$$

and the reduction is complete.

If, on the other hand, $\bar{C}_1 \neq 0$ or/and $\bar{B}_1 \neq 0$ in (24), then the column or/and row system of equations is inconsistent. If one system is consistent and has feasible solutions while the other is inconsistent, then, since the values of $U_{\bar 1}$ or $V_{\bar 2}$ can be arbitrarily set, the value to be optimized is unbounded. Thus the "effect" of inconsistency merges with that of infeasibility. Note that the possibility that one or/and the other program has no nonnegative constrained variables ($p_2 = 0$ or/and $q_2 = 0$) is included.

Consider, now, a most general form as given in the schema

(30)
$$\begin{array}{c} X\text{'s} \geqq 0 \\ \max u = U_1 B_1 + U_2 B_2 \end{array} \quad \begin{array}{c|ccc|l} & -V_2 & Y_2 & B_2 & \\ \hline U_1 & A_{11} & A_{12} & A_{13} & = -B_1 \\ X_1 & A_{21} & A_{22} & A_{23} & = -Y_1 \\ C_1 & A_{31} & A_{32} & A_{33} & = V_1 \\ \hline & =C_2 & =X_2 & =U_2 & \end{array} \quad \begin{array}{c} Y\text{'s} \geqq 0 \\ \min v = C_1 V_1 + C_2 V_2 \end{array},$$

where A_{ij} is a matrix of p_i rows and q_j columns and the symbols in the margins are vectors of corresponding dimension. The schema exhibits two systems of linear equations obtained as before and, together with the additional information, defines a pair of dual linear programs. Each program is said to be in *general form*: nonnegative constrained (X's and Y's) and free or unconstrained in sign (U's and V's) variables are both present; and equations expressing "dependent" constrained and free variables and constants in terms of "independent" constrained and free variables and constants are all present as well. Clearly, all previously discussed forms are contained in the general form. In fact, significantly more is contained since (see § 7) the matrix game problem can be written directly in the form (30), and the minimax theorem obtained through application of the (generalized) main theorem.

It is a simple manipulative matter to see that the general form (30) may be reduced to the symmetric standard form (26). The following schema together with the file for V_1 and U_2 is equivalent to (30), since the same equation systems are represented, save for the appearance of added redundant equations:

$$\begin{array}{c|ccc|l} & -V_2 & Y_2 & 1 & \\ \hline U_1 & A_{11} & A_{12} & A_{13}B_2 + B_1 & = 0 \\ X_1 & A_{21} & A_{22} & A_{23}B_2 & = -Y_1, \\ 1 & C_1A_{31} - C_2 & C_1A_{32} & C_1A_{33}B_2 & = v \\ \hline & =0 & =X_1 & =u & \end{array}$$

where $V_1 = -A_{31}V_2 + A_{32}Y_2 + A_{33}B_2$ and $U_2 = U_1A_{13} + X_1A_{23} + C_1A_{33}$. The equations for V_1 and U_2 can be filed since they express only unconstrained variables.

Note that we can make (30) *include cases in which the U's or/and V's are vacuous merely by setting* $u = 0$ *or/and* $v = 0$ *in the side information and then having the u-column or/and v-row in the reduced schema above consist entirely of zero entries. The insertion or deletion of such a row or column can be made at any time since it is invariant under pivoting.*

To summarize, the foregoing permits statement of the following two theorems pertaining to programs in general form and corresponding to the conclusions for programs in canonical form.

THEOREM. *Given a pair of dual linear programs in general form* (30), *either* (i) *both have feasible solutions and hence optimal solutions with* $\max u = \min v$, *or* ((ii) *and* (iii)) *one system has feasible solutions giving the objective unbounded values while the other system has no feasible solution (due to inconsistency or sign constraints) or* (iv) *neither system has feasible solutions.*

COMPLETE COMPLEMENTARY SLACKNESS THEOREM. *Given a pair of dual linear programs in general form* (30), *each of which has feasible solutions, there exist optimal solutions* (X^*, U^*) *and* (Y^*, V^*) *with* $X^* + Y^* > 0$, $X^* \cdot Y^* = 0$.

6. Transposition theorems. It is possible to regard the various classical transposition theorems concerning solvability of linear inequality systems as corollaries of the generalized main theorem or complete complementary slackness theorem of § 5. This reverses the roles of these theorems as they have previously appeared in the literature. For example, the original paper [12], the basic series of papers [13], [26] on the theory of linear programming, or the book [11] follow the route of first establishing a series of transposition theorems and then deriving from these the general duality theorems. We show below how some of the classical statements are simple consequences of the theorems given above.

GORDAN'S THEOREM (1873) (see [14]). $AY = 0$ *holds for some nontrivial* $Y \geqq 0$ *if and only if* $UA > 0$ *for no* U.

Consider the pair of programs defined in the following schema with marginal information, where $C < 0$:

		Y		
$X \geqq 0$	U	A	$= 0$	$Y \geqq 0$ $\min v$ $(C < 0)$.
	1	C	$= v$	
		$= X$		

The row system has a feasible solution ($Y = 0$). Thus, *either* the column system has a feasible solution, meaning $UA + C = X \geqq 0$ or $UA \geqq -C > 0$ and $\min v = 0$, whence $Y = 0$; *or* the column system has no feasible solution, meaning $\min v$ is unbounded below for feasible Y. Stiemke's theorem (1915) (see [23]) is the same as Gordan's but for asking $Y > 0$ instead of $Y \geqq 0$, and $UA \geqq 0$, $UA \neq 0$ instead of $UA > 0$.

FARKAS'S THEOREM (1902) (see [10]). *$AY = B$ for some $Y \geqq 0$ if and only if $UB \geqq 0$ for all U satisfying $UA \geqq 0$.*

Consider the pair of dual programs specified in the following schema and marginal information:

		Y	1		
$X \geqq 0$ $\max u$	U	A	$-B$	$= 0$	$Y \geqq 0$.
		$= X$	$= u$		

Either the row system has a feasible solution, meaning $AY = B$, $Y \geqq 0$ and $-UB = u \leqq v \equiv 0$ for any feasible U; *or* the row system has no feasible solution, meaning $\max u$ is unbounded above for feasible U, i.e., there exists a feasible $\bar{U}$ so that $-\bar{U}B$ is large or $\bar{U}B < 0$.

A rewording of Farkas's theorem constitutes what is widely referred to as the "theorem of the separating hyperplane:" either $AY = B$ has a nonnegative solution $Y \geqq 0$ or there exists a solution U to the inequalities $UA \geqq 0$, $UB < 0$. This title stems from the following geometric interpretation. Take the columns of A and B to be the coordinates of points in Euclidean space. Then either the point B belongs to the convex polyhedral cone $C = \{AY | Y \geqq 0\}$ spanned by the points corresponding to the columns of A or there exists a hyperplane $H = \{X | UX = 0\}$ with normal vector U which separates B from C since $UA \geqq 0$ whence $UAY \geqq 0$, all $Y \geqq 0$, while $UB < 0$. Notice that given coordinates of a set of points S and of a point P, the simplex method constitutes a constructive procedure by which either to express P has a nonnegative combination of points of S or to construct a separating hyperplane H.

MOTZKIN'S THEOREM (1936) (see [20]). *Either $UA_1 > 0$, $UA_2 \geqq 0$, $UA_3 = 0$ has a solution U or $A_1Y_1 + A_2Y_2 + A_3Y_3 = 0$ for nontrivial $Y_1 \geqq 0$, $Y_2 \geqq 0$ and Y_3 unrestricted.*

Again, the proof follows directly from consideration of the dual programs specified by the following schema and marginal information, where $C_1 < 0$:

		Y_1	Y_2	Y_3		
$X_1, X_2 \geqq 0$	U	A_1	A_2	A_3	$= 0$	$Y_1, Y_2 \geqq 0$
	1	C_1	0	0	$= v$	$\min v$.
		$= X_1$	$= X_2$	$= 0$		

The above have all resulted from the generalization of the main theorem. The following two theorems use the complete complementary slackness property of certain pairs of solutions. Thus, these theorems say something about the character of solutions, rather than about the existence of solutions.

VON NEUMANN'S THEOREM OF THE ALTERNATIVES (1944) (see [30]). *Either* $AY \leqq 0$ *has a nontrivial solution* $Y \geqq 0$ *or there exists a solution* X *to* $XA > 0$, $X \geqq 0$.

Consider the pair of systems described by the following schema and marginal information:

$$\begin{array}{ccccc} & & Y & & \qquad Y, \hat{Y} \geqq 0. \\ & X & \boxed{\quad A \quad} & = -\hat{Y} & \\ X, \hat{X} \geqq 0 & & = \hat{X} & & \end{array}$$

Clearly, both systems have feasible solutions. Thus, there exist solutions X^*, Y^* so that $X^*A + Y^* > 0$. *Either* $X^*A > 0$, implying $Y^* = 0$, *or* some component of X^*A is 0, implying $Y^* \geqq 0$ and nontrivial.

TUCKER'S THEOREM (1956) (see [26]). *If* $A^T = -A$, *i.e., if* A *is skew symmetric, then* $XA \geqq 0$, $X \geqq 0$ *has a solution* X^* *with* $X^* + X^*A > 0$.

Consider the systems specified by the following schema and marginal information, where I represents the identity matrix:

$$\begin{array}{ccccc} & & Y & & \qquad Y, \hat{Y} \geqq 0. \\ & X & \boxed{\quad A + I \quad} & = -\hat{Y} & \\ X, \hat{X} \geqq 0 & & = \hat{X} & & \end{array}$$

Again, both systems have feasible solutions. Thus, there exist solutions X^* and Y^* with $X^*A + X^* + Y^* > 0$. But $Y^* = 0$ since $AY^* + Y^* \leqq 0$ and $Y^* \geqq 0$ implies $0 \geqq Y^*(AY^* + Y^*) = Y^*AY^* + (Y^*)^2 = (Y^*)^2$. The last equality follows because $Y^*AY^* = 0$, which is due to the skew symmetry of A.

7. Matrix games. A matrix game (or two-person zero-sum game in normalized form) is specified by an $m \times n$ matrix $A = (a_{ij})$ of real numbers (or elements in any ordered field) with the rule that in a play of the game players I and II simultaneously choose a row i, $i = 1, \cdots, m$, and a column j, $j = 1, \cdots n$, respectively, with the result that II "pays" I an amount a_{ij}. A *pure strategy* for player I is the choice of some one row of A; for II, the choice of some one column. A *mixed strategy* for I is the choice of a probability vector $(x_1, \cdots, x_m)$, $\sum_i x_i = 1$, $x_i \geqq 0$ (in a great many plays of the game I chooses row i with probability x_i); for II a probability vector $(y_{m+1}, \cdots, y_{m+n})$, $\sum_j y_{m+j} = 1$, $y_{m+j} \geqq 0$ (in a great many plays of the game II chooses column j with probability y_{m+j}). Given mixed strategies, I's "expected gain" against the pure strategy choice of column j by II is $\sum_i x_i a_{ij}$, and II's "expected loss" against the pure strategy choice of row i by I is $\sum_j a_{ij} y_{m+j}$.

The rationale of the theory of games requires player I to choose his mixed strategy to maximize expected gain against any choice of column by II, and player II to choose his mixed strategy to minimize his maximum expected loss against any choice of row by I. Letting

$$u = \min_j \sum_i x_i a_{ij} \quad \text{and} \quad v = \max_i \sum_j a_{ij} y_{m+j},$$

I's objective is to choose x's to maximize u, and II's objective is to choose y's to minimize v. Clearly, $u \leqq v$. Thus, if x's and y's can be found such that $u = v$, they must constitute *optimal strategies.* In that case, the common value $u = v = \sum_{i,j} x_i a_{ij} y_{m+j}$ is called the *value* of the game A.

MINIMAX THEOREM (1928) (see [29]). *There exists a pair of optimal mixed strategies, and value, for any matrix game.*

The players' respective problems constitute a pair of dual linear programs of form (26) or (27), with Player I's problem the column program and II's, the row program:

$$\begin{array}{cc|c:ccc|cc}
 & & -v & y_{m+1} & \cdots & y_{m+n} & & \\
\hline
 & u & 0 & -1 & \cdots & -1 & = -1 & \\
\hdashline
 & x_1 & 1 & a_{11} & \cdots & a_{1n} & = -y_1 & \\
x\text{'s} \geqq 0 & \vdots & \vdots & \vdots & & \vdots & \vdots & y\text{'s} \geqq 0 \\
u \max & & & & & & & v \min \\
 & x_m & 1 & a_{m1} & \cdots & a_{mn} & = -y_m & \\
\hline
 & & =1 & =x_{m+1} & \cdots & =x_{m+n} & &
\end{array}$$

Since both programs clearly have feasible solutions, the main theorem establishes the truth of the minimax theorem. The reduction process of the last section together with a simplex method then yield a method for obtaining optimal strategies and the value of the game.

The matrix game problem has been generalized directly to a "constrained game" problem [4] in which each player in a matrix game has linear inequality (or equation) constraints imposed on his choice of mixed strategies. Suppose player I faces constraints $\sum_i x_i b_{ij} \geqq b_j, j = 1, \cdots, q$, and player II, $\sum_j c_{ij} y_{m+j} \leqq c_i$, $i = 1, \cdots, p$. The requisite schema for this situation is

$$\text{(31)} \qquad \begin{array}{cc|c:ccc:ccc:c|cc}
 & & -v & y_{m+1} & \cdots & y_{m+n} & z_{p+1} & \cdots & z_{p+q} & 1 & & \\
\hline
 & u & 0 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & = -1 & \\
\hdashline
 & x_1 & 1 & a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1q} & 0 & = -y_1 & \\
x\text{'s} \geqq 0, & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & y\text{'s} \geqq 0, \\
 & x_m & 1 & a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mq} & 0 & = -y_m & \\
w\text{'s} \geqq 0 & & & & & & & & & & & z\text{'s} \geqq 0 \\
\hdashline
 & w_1 & 0 & c_{11} & \cdots & c_{1n} & 0 & \cdots & 0 & -c_1 & = -z_1 & \\
u + \bar{u} \max & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & v + \bar{v} \min \\
 & w_p & 0 & c_{p1} & \cdots & c_{pn} & 0 & \cdots & 0 & -c_p & = -z_p & \\
\hdashline
 & 1 & 0 & 0 & \cdots & 0 & -b_1 & \cdots & -b_q & 0 & = \bar{v} & \\
\hline
 & & =1 & =x_{m+1} & \cdots & =x_{m+n} & =w_{p+1} & \cdots & =w_{p+q} & =\bar{u} & &
\end{array}$$

If the inequality constraints imposed on mixed strategies have feasible solutions, then the pair of dual linear programs displayed in (31) also have feasible solutions. For example, if $(x_1, \cdots, x_m)$ satisfies $\sum x_i b_{ij} \geqq b_j$, taking u small enough insures satisfaction of the remaining constraints for any choice of nonnegative $w_1, \cdots, w_p$. Therefore, if the extra constraints have feasible solutions, there must exist optimal solutions. In that case, $(x_1, \cdots, x_m)$ and $(y_{m+1}, \cdots, y_{m+n})$ constitute optimal mixed strategies and the value of the constrained game is defined to be $u + \bar{u} = v + \bar{v}$.

It seems appropriate to make these definitions, for the "expected payoff" of II to I is

$$\sum_{i,j} x_i a_{ij} y_{m+j} = u + \bar{u} = v + \bar{v}.$$

This is immediately verified by using the complementary orthogonality property of optimal solutions:

$$\begin{aligned} 0 &= \sum_{i,j} x_i a_{ij} y_{m+j} + \sum_{i,j} w_i c_{ij} y_{m+j} - \sum_j u y_{m+j} \\ &= \sum_{i,j} x_i a_{ij} y_{m+j} + \sum_i w_i(-c_i) - u \\ &= \sum x_i a_{ij} y_{m+j} - \bar{u} - u. \end{aligned}$$

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