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PURE COMPETITION, COALITIONAL POWER, AND FAIR DIVISION*

BY LLOYD S. SHAPLEY AND MARTIN SHUBIK¹

1. INTRODUCTION

THIS PAPER IS CONCERNED with the conceptual foundations of the theory of economic competition, and with the effect thereon of a new solution concept borrowed from the mathematical theory of games.

In Section 2, we consider three basic principles of distribution in an economic society—pure competition, coalitional power, and fair division—and show how they lead to three different ideas of what might constitute the “solution” of a mathematical model of the marketplace. Two of these solutions, the *competitive equilibrium* and the *core*, have been found to be intimately related, despite sharp differences in heuristic interpretation [6, 4, 25, 21]. Our present purpose is to introduce to economic analysis the third of these solutions, the *value of the game*, and to compare and contrast it with the other two.

Like the core, the value solution presupposes that the market is a collusive, multi-person game. The value solution, however, looks for a unique, equitable compromise among all opposing interests, whereas the core merely delimits a “no man’s land” between unyielding coalitions. The competitive equilibrium, for its part, recognizes no collusion at all. But when the number of traders is large, it can nevertheless be shown under rather general conditions that all three varieties of solution come into agreement, predicting the same outcome, but for different reasons.²

In Sections 3, 4, and 5, which may be read independently of one another, we consider some typical applications. In Section 3, numerical solutions are determined for a simple symmetric market model involving complementary goods (see [18]). In Section 4, the mutual convergence of value, core, and competitive equilibrium is demonstrated for a general class of Edgeworth market games (see [24, 20]). In Section 5, a particular Edgeworth model is viewed as a game without transferable utility and is solved explicitly as a function of market size. This exercise represents the maiden voyage of a new definition of the value, and the technical argument is accordingly spelled out in some detail.

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² Agreement in the limit, for a restricted class of oligopolistic models, has been established for yet another game-theoretic solution: the Nash *noncooperative equilibrium point* (see [21]).

The main definitions of the formal theory are summarized in the Appendix.

2. THREE APPROACHES

Underlying traditional writings and utterances on the virtues of competition and the "invisible hand" of the open marketplace are several fundamentally different approaches to the question of distribution and trade in an economic society. The three that will concern us in this essay are, in brief: (1) *pure competition*: individuals interact only through an impersonal market, which provides them with information and receives their separate decisions; (2) *coalitional power*: individuals join forces and coordinate decisions when it is to their advantage, constrained ultimately only by the countervailing power of opposing coalitions; (3) *fair division*: gains in welfare arising out of economic activity are distributed equitably among all participants, in accordance with their contributions to the economy.

These three conceptual points of departure, seemingly at odds, may or may not lead to conflicting results in application. The mathematical theory of games provides a formal frame of reference within which these ideas may be analyzed, compared, and, to a certain extent, integrated.

2.1. *The competitive equilibrium.* The classical theory of pure competition has demonstrated that, under certain general conditions concerning consumer preferences and production possibilities, there will exist a schedule of equilibrium prices for the goods and services in the economy [27, 10, 2]. If each individual maximizes his own welfare, on the assumption that these prices will prevail and without spending more than he receives, then demand and supply will just balance. The resulting "competitive" allocation of goods and services has the property of *Pareto optimality*: no redistribution could benefit anyone without hurting someone else.

The beauty of this kind of price mechanism, which uses money only as a bookkeeping device, is that it replaces a complicated, joint maximization problem by a set of simple, individual maximization problems, the solutions of which all synchronize to give an optimal result. It is a decentralized decision system with a (weak) welfare property.

A drawback, however, is the absence in the model of a method of generating the equilibrium prices that determine the pattern of trade. One might postulate a central authority, which not only calculates and publishes the prices but also selects a particular price schedule if there should happen to be more than one with the equilibrium property. The informational requirements of such an agency, however, would seem to vitiate any subsequent advantages of decentralization. Alternatively, one might hope that competitive prices would arise out of some dynamic adjustment process, steered by tentative bids and offers from the individual traders. Unfortunately, convergence and stability are not ensured without added assumptions about the economy that exclude many cases of interest [1, 14, 12, 13].

These difficulties are closely related to the possibility of nonuniqueness of the competitive solution, a phenomenon that can occur without visible peculiarities in the data of the model. It must be admitted, however, that even

without uniqueness, the set of competitive allocations (arising from a given initial allocation) is generally a very substantial narrowing of the set of all Pareto optima.

2.2. *The core.* In order to discuss the question of coalitional power, we turn to a game-theoretic solution known as the core. In a general n -person cooperative game, the core may be described as the set of final imputations of wealth that are not "coalitionally dominated" by other possible divisions of welfare. In other words, for an imputation to belong to the core, no group of participants should be able, through collusion, to ensure an outcome of the game that every member of the group would prefer to the given imputation.³

The core concept may be regarded as an extension of the notion of Pareto optimality, taking into account the possibility of independent optimization by subsets of the economy, as well as by the economy as a whole. It may also be regarded as an extension, to groups, of the individualistic principle that says that a man will not accept any redistribution of wealth that worsens his initial position, unless compelled to do so. Thus, the core is both Pareto optimal and individually rational.

In many general n -person games the core is void.⁴ In these cases the intermediate coalitions—between the one-person sets and the all-player set—are too strong and cannot all be satisfied at once. On the other hand, it has been shown [6, 23] that in an economy in which competitive prices exist, the core is not void; indeed, it contains all competitive allocations. This means, somewhat remarkably, that the decentralized "pure competition" equilibrium cannot be upset by collusion among any subset of traders, even though they violate all the rules of pricing, communication, and trading on which the competitive equilibrium in principle depends.

Even more remarkably, when the number of participants in the economy is increased in a suitably homogeneous manner, the core can be shown to shrink, until in the limit only the competitive solution is left [24, 7, 6]. Note that prices play no part in the definition of the core. Nevertheless, in an economy with a large number of participants, the device of giving free rein to coalitions has the effect of generating competitive prices.

When there are only a few participants, the core may be large. For example, in trading between just two individuals, the core must consist of the entire Edgeworth contract curve, there being no intermediate coalitions to raise objections. It is also worth noting that the core can and does exist

³ A simple three-person money-sharing game will illustrate this. Suppose the players are told that (1) they may have \$1 if they can agree how to divide it; (2) failing this, any two of them may exclude the third and get 80¢, again provided they agree on the division; (3) failing this, all get nothing. This game has no core, for no matter how the \$1 might be divided, some two players will get less than 80¢ together.

Now change the payoff rules to make the coalitions of two worth 50¢, instead of 80¢. Then there is a core, consisting of all allocations of \$1 that give every pair of players at least 50¢; for example (50¢, 0, 50¢), (15¢, 40¢, 45¢), etc.

⁴ For example, all essential zero-sum games in the sense of von Neumann and Morgenstern have empty cores [26, (280)].

in some economic models in which there is no competitive solution because of the disequilibrating effect of such features as joint production costs or nonconvex or interdependent preferences [23, 22].

2.3. *The value.* The idea of "fair division" in a socio-economic situation calls for the application of yet another solution concept from the theory of games—the value. This solution seeks to evaluate each player's position in the game *a priori*, taking into account both his own strategic opportunities and his bargaining position with respect to gains attainable through collaboration.

The value can be defined most easily when a common measure of utility exists, together with a vehicle, such as money or credit, which permits utility to be transferred freely among the players. In the absence of such transferability, a value can still be defined, but intrinsic rates of utility comparison between the players must simultaneously be derived from the strategic and bargaining possibilities of the game itself, and multiple solutions are possible.

Intuitively speaking, the value solution seeks to impute the proceeds of total cooperation among the participants in a way that takes fair account of each person's contribution to each possible cooperative venture. In calculating the value of an economic game, one must determine the *marginal worth* of an individual to every subset of other individuals and form an average.⁵ Thus, an ability to measure the economic "worth" of a set of individuals is presupposed. If a money (in the sense of the preceding paragraph) is available, then utilities can be measured on a common scale, and the worth of a coalition can be defined as the maximum combined wealth that the coalition can achieve for its members by its own efforts. These worths, determined for every possible subset of players, comprise the *characteristic function* [26], from which the value of the game can be computed by a formula (see Appendix A.1).

Historically, the formula for the value was first derived from postulates of symmetry, Pareto optimality, additivity, and—most crucially—the value's sole dependence on the kind of information conveyed in the characteristic function [17]. Harsanyi, working constructively from a model of the bargaining process, and later Selten, working deductively from postulates on the move and payoff structure of the game, arrived at a value definition that applies the same mathematical formula to a modified characteristic function [8, 15, 16].⁶ In the present economic models without externalities, the modifi-

⁵ We illustrate with a variant of the previous example. Avoiding total symmetry, we now assume that pairs $\bar{12}$, $\bar{13}$, and $\bar{23}$ can divide 50¢, 50¢, and 80¢, respectively, failing a general agreement to divide \$1. Player 1 is in the weakest position. His marginal worth to the coalition $\bar{123}$ is just 20¢; to $\bar{12}$ and $\bar{13}$ he is worth 50¢; and to the "coalition" $\bar{1}$, consisting of himself alone, he contributes nothing. Averaging these numbers (giving equal weight to each size of coalition, not to each coalition) we find Player 1's value to be $23\frac{1}{3}$ ¢. A similar calculation gives $38\frac{1}{3}$ ¢ to Player 2 and the same to Player 3. This imputation happens to be outside the core, since the coalition $\bar{23}$ can do better alone.

⁶ Use of the classical characteristic function [26] implies in effect that a coalition
(Continued on next page)

cation proves to be irrelevant; the classical characteristic function yields the Harsanyi-Selten value and is simpler to work with.

In general, the value may lie outside the core. That is, there may be coalitions that could seize by their own efforts more than they are allotted in the value.⁷ Indeed, the value exists even when the core is empty.

When no "money" is available, evaluation of the game must proceed indirectly. We *assume* that utility is transferable, and then try to arrange matters so that the value imputation does not, in the end, require any net transfer of utility. This seems at first too much to arrange, but there is some freedom of maneuver. Since the individual utility scales are not related *a priori* to each other, or to a common monetary unit, we are free to adjust them separately, before making utility transferable. (Equivalently, we could leave the individual units alone but permit transfer only at prescribed rates of exchange, as in an international money market.) This freedom to rescale (or to prescribe rates of exchange) proves to be just enough to ensure the existence of a value that is feasible—i.e., one that can be achieved with no net transfer of utility. Thus the value solution, in the "no-transfer" theory, can be defined as the set of all feasible "transfer" values.⁸

This is not the place for a formal axiomatic justification of this definition. However, we may remark that its plausibility rests on a form of the "independence of irrelevant alternatives" principle, namely, the assertion that if the solution-with-transfers-permitted can be achieved without transfers, then it must remain a solution when transfers are forbidden. Of course, it could be objected that the *possibility* of transfers may well influence the outcome even in those cases where the ultimate net transfer is nil.

We shall sketch another view of the value definition which may be of interest. It depends on the observation that when a particular outcome of an n -person game is adopted, this implies not only that an interpersonal comparison of utilities has taken place, but implies it in two distinct ways. The first way relates the given outcome to other possible outcomes on the Pareto surface, basing the comparison on the existence of tradeoffs that were available but did not occur. The other way relates the given outcome to the initial positions (and strategic potentials) of the contestants, using it to refer something about the relative intensities of their desires. The first method infers the individuals' weights in the measuring of social welfare; the second their weights in the sharing of social profit. Our present value concept embodies a *principle of equivalence* between these two methods of inter-

always expects the worst so far as actions of outside players are concerned. In the present economic context, this "worst" is simply a boycott, involving no special costs to the outside players. In other contexts, the most damaging threats might be so costly to make that they should be discounted in determining what a coalition is worth. It is therefore advisable, in a general value theory based on a characteristic function, to recast the definition in terms of "optimum threats," in the sense of Nash [11].

⁷ See footnote 5.

⁸ Appendix A.3 has the formal definition. The key idea of using implicitly-determined utility exchange ratios was adapted from a proposal of Harsanyi [9].

personal comparison: the values of the game are just those outcomes for which the two sets of derived weights coincide.

Thus, under this definition, a value of the game represents a kind of equilibrium. It is as though the players introduced a money-of-account as an aid to rational bargaining, but with the prices of the different people's "units" chosen so that when the books are closed all accounts are miraculously in balance. The game itself is called upon to provide an intrinsic, "equitable" comparison among the personal utility units. As previously noted, the comparison factors are not always unique.

A close affinity between the value of the game and the competitive equilibrium is suggested by the money-of-account analogy. They are not the same, however, as can be seen from the fact that the former makes essential use of cardinal utility while the latter does not. A nonlinear, order-preserving transformation of the utilities will generally change the value, but not the competitive equilibrium nor, for that matter, the core.

In the remainder of this paper we apply the value theory to three different models, correlating the value solution in each case with the core and competitive solutions. The three applications are independent of one another. In the Appendix we have assembled some of the important formal definitions.

3. A SYMMETRIC MARKET GAME

Our first model was chosen for the contrast it provides between the value and the other solutions under discussion. We shall find that the value gives an intuitively more satisfactory measure of the "equities" of the situation while avoiding a violent discontinuity exhibited by both the competitive equilibrium and the core. Also, exploiting that discontinuity, we shall obtain a simple example of convergence to *different* limits, by the value and the other solutions, when the set of traders is expanded linearly but not homogeneously.

The model can be formulated in terms of gloves. Each player starts with one glove—either right- or left-handed, and the players may trade them, or buy or sell them for money, without restriction. At the end of the game, an assembled pair is worth \$1 to whoever holds it. For example, there might be an outside market that would pay that price.⁹

The characteristic function of the game, which states the dollar potential of each coalition S , is given by the equation

$$(1) \quad v(S) = \min(|S \cap R|, |S \cap L|).$$

Here R and L are the original sets of owners of right- and left-handed gloves, respectively, and the notation " $|X|$ " means the number of elements of the set X .

Equation (1) expresses a rudimentary form of complementarity between economic units of different types. Traders on the same side of the market stand in the position of perfect substitutes; traders on opposite sides, perfect complements. A further discussion of this characteristic function will be found in [18].

⁹ An alternative formulation, without money, is given at the end of the section.

3.1. *The competitive equilibrium and core.* The three kinds of solutions discussed in Section 2 will now be determined. Let $r = |R|$ and $l = |L|$, and suppose first that $r < l$. Then the equilibrium price of left-handed gloves in a competitive market is zero, and the members of R can acquire complete pairs at no cost. The unique competitive imputation is therefore

$$(2) \quad \begin{aligned} \omega_i &= \$1 \quad \text{for } i \in R, \\ \omega_j &= \$0 \quad \text{for } j \in L. \end{aligned}$$

The case $r > l$ is just the opposite. In the transition case, $r = l$, the equilibrium prices are not unique; we know only that the sum of the two prices must be \$1. A continuum of competitive imputations results, as follows:

$$(3) \quad \begin{aligned} \omega_i(p) &= \$p \quad \text{for } i \in R, \\ \omega_j(p) &= \$(1 - p) \quad \text{for } j \in L, \end{aligned} \quad (0 \leq p \leq 1).$$

The core of the game necessarily contains all competitive imputations. In this case, it happens to contain no other imputations. Indeed, in any non-competitive imputation the least-favored member of R and the least-favored member of L get less than \$1 combined, and can therefore form a blocking coalition. Thus the core is also given by (2) (or its opposite) or (3).

3.2. *The value.* To calculate the value, we shall make use of the "random order" version of the definition (see Appendix A.1), in which an imputation is built up one player at a time by awarding each player the increment that he brings to the coalition consisting of his predecessors. The value of the game is equal to the average of these imputations, over all possible orderings of players [17].

Let $\phi_{\text{right}}(r, l)$ denote the sum of the values to the r members of R . If we consider separately those orderings that end with a member of R (probability $r/(r + l)$), and those orderings that end with a member of L (probability $l/(r + l)$), we obtain the following difference equation:

$$(4) \quad \begin{aligned} \phi_{\text{right}}(r, l) &= \frac{r}{r + l} [\phi_{\text{right}}(r - 1, l) + v(R \cup L) - v(R' \cup L)] \\ &+ \frac{l}{r + l} [\phi_{\text{right}}(r, l - 1)], \end{aligned}$$

where R' is any set satisfying $R' \subset R, |R'| = r - 1$. If we assume that $r > l$ (thereby eliminating the "v" terms in (4)), then the relevant boundary conditions are

$$\phi_{\text{right}}(r, 0) = 0 \quad \text{and} \quad \phi_{\text{right}}(r, r) = \frac{r}{2}, \quad \text{all } r,$$

and the solution of the difference equation for $r \geq l$ is

$$\phi_{\text{right}}(r, l) = \frac{r}{2} - \frac{r - l}{2} \sum_{k=0}^l \frac{r! l!}{(r + k)! (l - k)!}.$$

(The reader may verify this by direct substitution in (4).) This amount is divided equally among the members of R . With the aid of Pareto optimality:

$\phi_{\text{right}} + \phi_{\text{left}} = v(R \cup L) = l$, the values to members of L are easily determined. The complete value imputation for $r \geq l$ is

$$(5) \quad \begin{aligned} \phi_i &= \frac{1}{2} - \frac{r-l}{2r} \sum_{k=0}^l \frac{r!l!}{(r+k)!(l-k)!} & \text{for } i \in R, \\ \phi_j &= \frac{1}{2} + \frac{r-l}{2l} \sum_{k=1}^l \frac{r!l!}{(r+k)!(l-k)!} & \text{for } j \in L. \end{aligned}$$

The case $r \leq l$ is symmetrical. Table 1 gives an idea of how these equations behave for small numbers of traders.

TABLE 1
VALUE TO A MEMBER OF R

$r \backslash l$	0	1	2	3	4	5	6	7	8
1	0	0.500	0.667	0.750	0.800	0.833	0.857	0.875	0.889
2	0	0.167	0.500	0.650	0.733	0.786	0.822	0.847	0.867
3	0	0.083	0.233	0.500	0.638	0.720	0.774	0.811	0.838
4	0	0.050	0.133	0.272	0.500	0.629	0.710	0.764	0.802
5	0	0.033	0.086	0.168	0.297	0.500	0.622	0.701	0.755
6	0	0.024	0.060	0.113	0.194	0.315	0.500	0.616	0.693
7	0	0.018	0.044	0.081	0.135	0.214	0.330	0.500	0.610
8	0	0.014	0.033	0.061	0.099	0.153	0.230	0.341	0.500

The value solution definitely favors the “short” side of the market, individually and collectively. For example, if $l < r$, the members of L , with less than half the population, get more than half the total profit, which is $v(R \cup L) = l$. On the other hand, the “long” side of the market is not totally defeated, as in the other solutions discussed. The value of the game is less abruptly sensitive to the balance between supply and demand than the competitive equilibrium and the core, since it gives some credit for the bargaining position of the group in oversupply.¹⁰ It is not strange that the competitive solution, with its decentralized outlook, fails to recognize collusive bargaining power, but it is a little surprising that the core—a cooperative-game concept—misses it as well.

3.3. *Asymptotic behavior.* In Figure 1 we show the effect of altering the ratio of trader types, holding the size of the market fixed. In the second

¹⁰ For example, if $r = l + 1$, then the members of R , faced with total defeat under pure competition, might select two of their number to withdraw from the market, thus turning the tables on L . This behavior would not be Pareto optimal, since only $l - 1$ pairs of gloves could be formed, but the threat is credible enough and might well raise the price for right-handed gloves.

The reader will recognize this as a standard price-support tactic in situations where collusion is possible. Of course, the value of the game does not directly consider such details of process, but it recognizes and measures the coalition potentials that make such maneuvers effective. For further discussion, see [19].

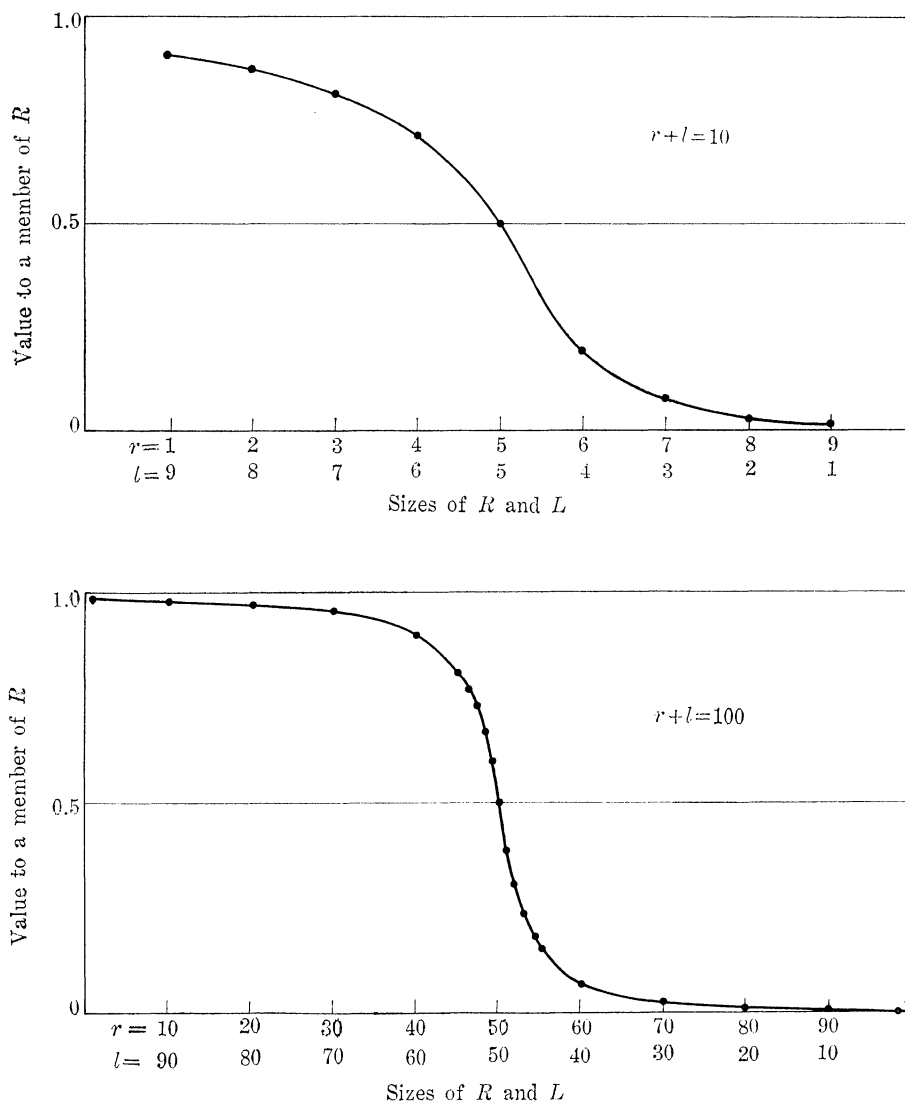


FIGURE 1

VALUE AS A FUNCTION OF COMPOSITION

graph, with ten times as many traders, the slope of the curve in the vicinity of the transition case is noticeably steeper. In the limit, the curve approaches the \square -shape associated with the core and competitive solutions.

The effect of increasing the size of the market while holding the *ratio* of trader types fixed is illustrated in Table 2, for the ratio 2:1. We see that owning a right glove is worth $16\frac{2}{3}\text{¢}$ when there are three traders, but less than 6¢ if there are 30 traders and less than 1¢ if there are 300. The

TABLE 2
VALUE SOLUTIONS FOR TRADER TYPES IN RATIO 2:1

r	l	ϕ_i	ϕ_j	ϕ_{right}	ϕ_{left}
2	1	0.1667	0.6667	0.333	0.667
4	2	0.1333	0.7333	0.533	1.467
6	3	0.1131	0.7738	0.679	2.321
8	4	0.0990	0.8020	0.792	3.208
10	5	0.0884	0.8232	0.884	4.116
20	10	0.0589	0.8822	1.178	8.822
200	100	0.0092	0.9816	1.842	98.158
as	$r = 2l \rightarrow \infty$	$\frac{1}{l} - \frac{9}{l^2}$	$1 - \frac{2}{l} + \frac{18}{l^2}$	$2 - \frac{18}{l}$	$l - 2 + \frac{18}{l}$

general asymptotic formulas for fixed ratio $a : b$, with $a > b$, are

$$\phi_i(ak, bk) = \frac{b}{(a-b)^2k} - \frac{3b(a+b)}{(a-b)^4k^2} + O(k^{-3}) \quad \text{for } i \in R,$$

$$\phi_j(ak, bk) = 1 - \frac{a}{(a-b)^2k} + \frac{3a(a+b)}{(a-b)^4k^2} + O(k^{-3}) \quad \text{for } j \in L.$$

(We omit the derivation.) The rate of convergence of the value to the competitive imputation is of the order of $1/k$. The same rate has been observed in other examples and may be presumed to be typical for homogeneously expanding markets.

Another asymptotic result is available. If we let the market expand with the *difference* $d = r - l$ held constant, we obtain the following estimates:¹¹

$$\phi_i(r, r-d) = \frac{1}{2} - \frac{d}{2} \sqrt{\frac{\pi}{r}} + O(r^{-1}) \quad \text{for } i \in R,$$

$$\phi_j(r, r-d) = \frac{1}{2} + \frac{d}{2} \sqrt{\frac{\pi}{r}} + O(r^{-1}) \quad \text{for } j \in L.$$

In other words, if traders are added in equal numbers on both sides of the market, the distinction between "long" and "short" disappears in the limit, and all get equal shares of the profit. This provides an example (if $d \neq 0$) of a linearly, nonhomogeneously expanding market in which the value and the competitive imputation do *not* tend to the same limit.

3.4. *Concluding remarks.* Still another solution concept—the von Neumann-Morgenstern stable sets [26]—has been applied to the symmetric market game (1) by one of the authors [18]. It was found that in the transition case, $r = l$, the unique stable set is the core itself, in other words, the straight line of imputations defined by (3) above. In the other cases, $r > l$ and $r < l$, there

¹¹ The method of derivation, in brief, is to multiply the summands in (5) by $(r+l)!/r!l!$, to obtain truncated sums of binomial coefficients that can be estimated with the aid of Stirling's formula.

are infinitely many stable sets, each one a continuous curve of imputations emanating from the one-point core and possessing certain monotonicity properties. Thus, a “price-like” parameter can always be identified, but most of the stable sets will involve some kind of price discrimination—i.e., unequal treatment of traders of the same type.

A nonsymmetric generalization of the present example, typified by Böhm-Bawerk’s celebrated horse market [5], has also been considered by one of the authors [19]. Again, it was found that the traders that are priced out of the market receive consideration in the value of the game, but not in the core or competitive equilibrium. The stable sets are again continuous, monotonic curves.

A final remark: We could have avoided the use of money in this example if we had made the commodities continuously divisible and given the traders identical (cardinal) utility functions:

$$(6) \quad u^i(x, y) = \min(x, y), \quad \text{all } i \in N.$$

Money would then be superfluous, since the relevant utility transfers could be accomplished (on a constant-sum basis) by transferring bundles containing equal amounts of the two commodities.

4. A CONVERGENCE THEOREM

A proof of the convergence of the value to the competitive solution was given in [20] for a general class of replicated markets with money, as formulated in Appendix A.2. Here we shall focus on a two-sided market with equal tastes, in order to display the main sequence of ideas of that more general proof while avoiding certain secondary complications. Along the way, we shall prove the convergence of the core to the competitive solution.

The two types of traders are distinguished only by their initial commodity bundles, which are, respectively, $(a, 0)$ and $(0, b)$. Let there be km traders of the first type and kn of the second; here k is to be regarded as variable, m and n fixed. The *relative composition* (μ, ν) of the market will thus remain fixed at $\mu = m/(m + n)$, $\nu = 1 - \mu = n/(m + n)$.

All traders are assumed to have the same concave and twice-differentiable utility function of the separable form $u(x, y) + \xi$, where ξ is the net change from the initial money level. The characteristic function of the game will depend only on the numbers s, t of traders of each type in a coalition and will be written $v(s, t)$. Because of equal tastes, total utility for any coalition is maximized by an equal sharing of goods, and we have at once

$$(7) \quad v(s, t) = (s + t)u(\sigma a, \tau b),$$

where (σ, τ) is the relative composition of the coalition, thus: $\sigma = s/(s + t)$, $\tau = 1 - \sigma = t/(s + t)$. Note that v is homogeneous of the first degree: $v(\lambda s, \lambda t) = \lambda v(s, t)$; this holds regardless of the homogeneity of u .

4.1. *The competitive solution.* Pareto optimality can be achieved by allocating $(\mu a, \nu b)$ to everyone, followed by an arbitrary money transfer. In order to support this goods allocation, the competitive prices must be

$$\begin{aligned}\pi_x &= \frac{\partial u(\mu a, \nu b)}{\partial x} && \text{(first good) ,} \\ \pi_y &= \frac{\partial u(\mu a, \nu b)}{\partial y} && \text{(second good) .}\end{aligned}$$

The competitive payoffs are therefore given by

$$(8) \quad \begin{aligned}\omega_1 &= u(\mu a, \nu b) + \nu a \pi_x - \nu b \pi_y && \text{(first type) ,} \\ \omega_2 &= u(\mu a, \nu b) + \mu b \pi_y - \mu a \pi_x && \text{(second type) .}\end{aligned}$$

In these expressions, the first term is the utility of the final holding, the second is the payment received for selling off part of the initial holding, and the third is the money spent on buying the other good.

Now there may be competitive *allocations* other than $(\mu a, \nu b)$, since u need not be strictly concave. But the competitive *imputation* ω is unique, as are the prices.¹² Note also that ω_1 and ω_2 are independent of k . As we change the size of the market, the competitive solution remains fixed.

4.2. *The core.* Next, let us examine the behavior of the core. Expanding u in (7) in a Taylor's series about $u(\mu a, \nu b)$, we have

$$v(s, t) = (s + t)[u(\mu a, \nu b) + (\sigma - \mu)a\pi_x + (\tau - \nu)b\pi_y + O((\sigma - \mu)^2)] .$$

Using (8) and the relation $\sigma - \mu = \nu - \tau = \sigma\nu - \tau\mu$, we have

$$(9) \quad v(s, t) = s\omega_1 + t\omega_2 + (s + t)O((\sigma - \mu)^2) .$$

Moreover, by concavity, the remainder term is either zero or negative. Hence $v(s, t) \leq s\omega_1 + t\omega_2$, and the competitive imputation satisfies every coalition and is an element of the core.

Now let α be any Pareto-optimal imputation, i.e., one with total payoff $v(km, kn)$. If α is not symmetric, i.e., if α gives unequal payoffs to some pair of traders of the same type, and if $k > 1$, then the m worst-treated traders of the first type and the n worst-treated traders of the second type must together get *less* than $v(m, n)$. Hence they can block α . It follows that if $k > 1$, the core is confined to the one dimensional set P of *symmetric Pareto-optimal imputations*. We may parametrize this set by distance from ω , thus: $P = \{\alpha(c) \mid -\infty < c < \infty\}$, where

$$\begin{aligned}\alpha_1(c) &= \omega_1 + c/\mu && \text{(first type) ,} \\ \alpha_2(c) &= \omega_2 - c/\nu && \text{(second type) .}\end{aligned}$$

As we have seen, $\alpha(0) = \omega$ is in the core. We shall not trouble to determine the exact upper and lower bounds for c in the core, which depend somewhat irregularly on m, n , and k . But the convergence of these bounds to zero, and hence the convergence of the core to the competitive solution, can be shown quite easily.

In fact, let Q be a coalition having $km + kn - 1$ members, lacking only

¹² Differentiability is important here. For example, the utility function (6) in Section 3 is not differentiable at $x = y$, and the nonuniqueness of the competitive prices and imputations for the case $r = l$ (see (3)) is the direct result.

one trader of the first type. Then $\alpha(c)$ awards Q the amount

$$v(km, kn) - \omega_1 - c/\mu .$$

If we use (9) to estimate the characteristic function of Q , we obtain

$$v(km - 1, kn) = v(km, kn) - \omega_1 + O(1/k) .$$

Thus, if c is positive, and if k is sufficiently large, then Q can block $\alpha(c)$. Similarly, if c is negative and k large enough, then a coalition lacking just one trader of the second type can block $\alpha(c)$. In the limit, only the competitive imputation $\alpha(0)$ remains unblocked.

4.3. *The value.* Now let us examine the behavior of the value. For any trader, it represents his expected marginal worth to a coalition chosen at random (see Appendix A.1). We may express this (for a trader of the first type) as follows:

$$(10) \quad \phi_1(k) = E\{D_1(s, t)\} ,$$

where E is an averaging operator and D_1 is the finite difference

$$D_1(s, t) = v(s + 1, t) - v(s, t) .$$

The precise form of E could be stated, but it is not relevant here. Indeed, any method of averaging the increments D_1 that will sustain the “almost all” statements in the paragraph following will suffice. The convergence theorem is therefore valid for a whole class of “values” that might be defined.

Now let equation (7) be regarded as defining a function $v(s, t)$ for all positive real numbers s and t ; like u it is twice differentiable. Using homogeneity, a simple Taylor’s expansion, and (7), we have

$$\begin{aligned} D_1(s, t) &= (s + t) \left[v\left(\sigma + \frac{1}{s + t}, \tau\right) - v(\sigma, \tau) \right] \\ &= \frac{\partial v(\sigma, \tau)}{\partial \sigma} + O\left(\frac{1}{s + t}\right) \\ &= \frac{\partial v(s, t)}{\partial s} + O\left(\frac{1}{s + t}\right) \\ &= \frac{\partial}{\partial s} [(s + t)u(\sigma a, \tau b)] + O\left(\frac{1}{s + t}\right) \\ &= u(\sigma a, \tau b) + \tau a \frac{\partial u(\sigma a, \tau b)}{\partial x} - \tau b \frac{\partial u(\sigma a, \tau b)}{\partial y} + O\left(\frac{1}{s + t}\right) . \end{aligned}$$

The last line closely resembles the formula for ω_1 in (8). Indeed, if $s + t$ is large, and if (σ, τ) is close to (μ, ν) , then $D_1(s, t)$ will approach the competitive payoff ω_1 , as required. But if k is large enough, then “almost all” coalitions will be large, and “almost all” coalitions will have compositions approximating (μ, ν) . The latter will be recognized as a form of the law of large numbers. The precise statement is as follows: Given any $\varepsilon > 0$, a number k_ε can be chosen so large that for any $k \geq k_\varepsilon$ a randomly chosen coalition will, with probability at least $1 - \varepsilon$, have size $s + t$ large enough and composition (σ, τ)

near enough to (μ, ν) to ensure that $|D_1(s, t) - \omega_1| \leq \varepsilon$. Hence we may write

$$(11) \quad \phi_1(k) = (1 - \varepsilon)(\omega_1 + \delta(k)) + \varepsilon C_1(k), \quad \text{all } k \geq k_\varepsilon,$$

where $|\delta(k)| \leq \varepsilon$, and $C_1(k)$ is the appropriately weighted average of the $D_1(s, t)$ for the "exceptional" coalitions—i.e., those occurring with probability ε that may be too small or have compositions too far from (μ, ν) .

Let us consider the implications of (11). If $C_1(k)$ were known to be bounded, we could conclude that $\phi_1(k) \rightarrow \omega_1$, and we would be through. A lower bound for $C_1(k)$, namely, $v(1, 0)$, follows at once from the definition of D_1 and the superadditivity of the characteristic function. An upper bound for $C_1(k)$ cannot be deduced so directly, however, unless restrictions are imposed on the behavior of $u(x, y)$ near the boundaries of the positive quadrant. But there is a trick that takes us around this difficulty. We observe that both the value and the competitive imputation are Pareto optimal; hence we have

$$(12) \quad m\phi_1(k) + n\phi_2(k) = m\omega_1 + n\omega_2 \quad \text{for all } k.$$

The lower bound on $C_1(k)$ tells us that $\liminf \phi_1(k) \geq \omega_1$, i.e., that no limit point (finite or infinite) of the sequence $\{\phi_1(k)\}$ is less than ω_1 . Hence, by (12), we have $\limsup \phi_2(k) \leq \omega_2$. To complete the proof, we merely return to (10) and repeat the whole argument with types 1 and 2 interchanged, obtaining $\limsup \phi_1(k) \leq \omega_1$ and $\liminf \phi_2(k) \geq \omega_2$. In this way we finally establish the convergence of the value imputation $\phi(k)$ to the competitive imputation ω .

To sum up the essential idea of the proof: The partial derivatives of the characteristic function $v(s, t)$, show that the marginal value of a player to a *large, nearly balanced* coalition is substantially equal to his competitive payoff. However, if the economy is big enough, almost all coalitions are large and nearly balanced.

5. A MARKET WITHOUT TRANSFERABLE UTILITY

In order to illustrate the application of the theory of games without transferable utility, we shall present a simple two-sided Edgeworth market in which both the core and the value can be determined explicitly, but in which there is not so much symmetry that the solutions are uninteresting. In addition to providing working experience with the new value concept, the example will illustrate several theoretical points.

Let there be n traders on each side of the market. Let there be two goods in trade, but no money or credit. Let the initial holdings be

$$\begin{aligned} (1, 0) & \quad (\text{first type}), \\ (0, 1) & \quad (\text{second type}), \end{aligned}$$

and let the (cardinal) utility functions be

$$(13) \quad \begin{aligned} u_1(x, y) &= \sqrt{xy} && (\text{first type}), \\ u_2(x, y) &= \sqrt{x^2 + y^2 + 23xy} && (\text{second type}). \end{aligned}$$

The number 23 is only a convenience; all that really matters is that these func-

tions are concave, homogeneous of degree 1, and symmetric in the two goods.

Figure 2 is the "Edgeworth box" for this market [7, 24]. The origin O represents the allocation that gives $(0, 0)$ to type 2, and hence $(1, 1)$ to type 1. The opposite corner C' is the "origin" for type 1, and R is the initial or no-trade point. The segment CC' is Edgeworth's *contract curve* for the case of two traders (i.e., $n = 1$). The unique competitive allocation ω , which gives all traders $(1/2, 1/2)$, is represented by the point W ; this is independent of n . (The point V represents the value allocation for $n = 1$, which will be determined later.)

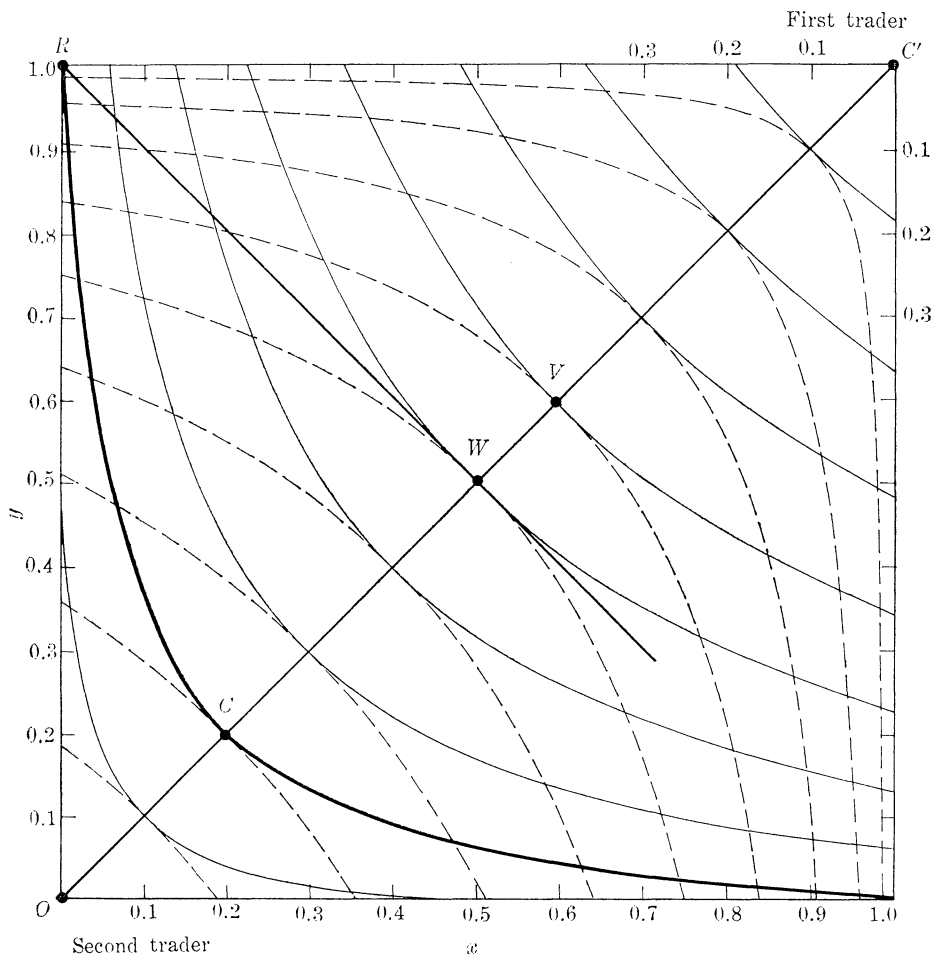


FIGURE 2
THE EDGEWORTH BOX

We emphasize that for $n > 1$, the Edgeworth box serves to represent only the symmetric allocations—those in which traders of the same type are treated alike.

5.1. *The core.* We first discuss the core. For $n = 1$ the core comprises the whole contract curve CC' . More precisely, the core is the image of CC' in the utility space under the mapping (13), as shown in Figure 3.

For $n > 1$, we first observe that all imputations in the core must be symmetric, since any nonsymmetric imputation can be profitably blocked by a two-man coalition consisting of one least-favored trader of each type. We may therefore transfer our attention from the $2n$ -dimensional space of all imputations to the two-dimensional subspace of symmetric imputations.

Because of the homogeneity of (13), the Pareto-optimal symmetric imputations lie along a straight line (OC' in Figure 3); its equation is

$$(14) \quad 5u_1 + u_2 = 5.$$

The core, for each n , is a subset of this line. It remains to discover which points on the line can be blocked. As in the model of Section 4, the most efficient blockers are coalitions that have almost, but not quite, the same relative composition as the market as a whole.

To verify this, let $r(S)$ denote the ratio of first types to second types in an arbitrary coalition S , and let $(1 - t, 5t)$, $0 \leq t \leq 1$, be an arbitrary point on OC' , representing the symmetric imputation α_t . Then a routine calculation reveals that in order for S to block α_t it is necessary and sufficient that $r(S)$ lie strictly between 1 and a certain critical ratio r_t , given by

$$r_t = \frac{t(t^2 - 0.04)}{(1 - t)(t - t^2 - 0.04)}.$$

Note that $r_{1/2} = 1$, showing that the competitive imputation $\alpha_{1/2}$ cannot be blocked, since there is no number strictly between 1 and 1. However, any other α_t will be blocked by some coalition if n is sufficiently large.

For each n , the coalitions of size $2n - 1$ provide the best type-composition ratios available, namely $(n - 1)/n$ and $n/(n - 1)$. Setting r_t equal to these numbers in turn, and solving for t , gives us the endpoints of the core. We have done this numerically for several values of n , as shown in Table 3 and Figure 3. Asymptotically (last line of Table 3), the length of the core varies inversely with the size of the market.

TABLE 3
ENDPOINTS OF THE CORE

n	C		C'	
1	0.800	1.000	0.000	5.000
2	0.577	2.116	0.421	2.897
3	0.546	2.272	0.453	2.733
4	0.533	2.337	0.467	2.665
5	0.525	2.374	0.474	2.628
10	0.512	2.440	0.488	2.560
as $n \rightarrow \infty$	$\frac{1}{2} + \frac{21}{184n}$	$\frac{5}{2} - \frac{105}{184n}$	$\frac{1}{2} - \frac{21}{184n}$	$\frac{5}{2} + \frac{105}{184n}$

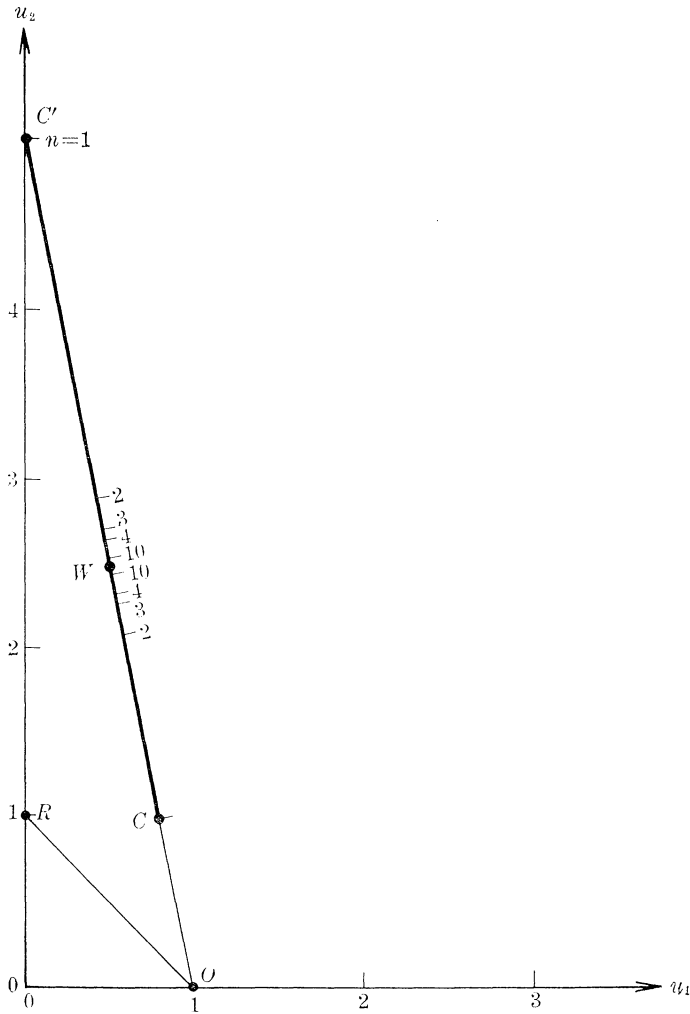


FIGURE 3
 THE UTILITY SPACE (SYMMETRIC PAYOFFS ONLY), SHOWING THE SHRINKING
 OF THE CORE TO THE COMPETITIVE PAYOFF¹³

5.2. *The value.* We now turn to the value. The technique, as explained in Section 2.3, is to introduce a set of “weights” $\{\lambda_i\}$, nonnegative and not all zero, on which to base hypothetical exchanges of utility among the traders. Thus, if α is a payoff vector attainable in the market, we assume that $\alpha + \beta$ is also attainable for any β such that $\sum \lambda_i \beta_i = 0$. Given the weights, we are

¹³ We have used the same letters to denote *allocation* points in the (x, y) -space and the corresponding *payoff* points (imputations) in the (u_1, u_2) -space.

in a position to calculate the “ λ -transfer value” of the game (Appendix A.3). This value is in general infeasible, i.e., not attainable by commodity transfers alone. We must try to find weights that yield a feasible λ -transfer value; this will be, by definition, a *value* of the original market without transferable utility.

We shall first dispose of the possibility of a nonsymmetric solution—one that gives different payoffs to traders of the same type. This would require unequal weights, since the value formula itself is symmetric. Suppose therefore that $\lambda_j < \lambda_k$ where j and k are traders of the same type, and consider how the λ -transfer value might be attained. Regardless of utility transfers, the allocation of goods must maximize the weighted sum of utilities. Any goods of positive utility that j could transfer to k would increase this sum because of the following comparison:

$$(15) \quad \begin{aligned} \lambda_k u(x^j + x^k, y^j + y^k) &\geq \lambda_k [u(x^j, y^j) + u(x^k, y^k)] \\ &\geq \lambda_j u(x^j, y^j) + \lambda_k u(x^k, y^k), \end{aligned}$$

with strict inequality on the second line if $u(x^j, y^j) > 0$. Hence j 's share of the *goods* allocation must be worthless. But his *payoff* in the λ -transfer value is easily seen to be positive, even if $\lambda_j = 0$.¹⁴ It follows that the λ -transfer value can be attained only with the aid of a utility transfer, making it infeasible for the original game. We conclude that only equal weights need be considered for traders of the same type.

With symmetry established, the λ -transfer characteristic function (see Appendix A.3) can be written in the form $v^\lambda(s, t)$, where the pair of integers (s, t) is the type-composition of the coalition in question. A simple fact about this function will be useful.

LEMMA. *If $st \neq 0$, then $v^\lambda(s, t) = v^\lambda(t, s)$.*

PROOF. The inequality (15), which depends on the concavity and homogeneity of (13), shows that any coalition can attain its maximum λ -weighted total utility while concentrating all its goods in the hands of at most two traders—one of each type. For a coalition of composition (s, t) , with neither s nor t equal to 0, there will be an optimal allocation that gives, say, (x, y) to one man of type 1, $(s - x, t - y)$ to one man of type 2, and nothing to the rest. This allocation is worth an amount $A = v^\lambda(s, t)$ to the coalition. Now consider a second coalition, of composition (t, s) . A possible allocation is (y, x) to one man of type 1, $(t - y, s - x)$ to one man of type 2, and nothing to the rest. Since the utility functions are symmetric in the commodities, this must be worth the same amount, A , to the second coalition. Hence $v^\lambda(t, s) \geq A = v^\lambda(s, t)$. Repeating the argument, we obtain $v^\lambda(t, s) = v^\lambda(s, t)$, completing the proof of the lemma.

Let us now consider the value formula in its “orderings” version (see Appendix A.1). The λ -transfer value ϕ_i^λ to a typical trader of type 1 can be

¹⁴ The reason: He makes a positive λ -weighted contribution to any coalition in which at least one trader is of the opposite type and at least one trader has weight greater than 0.

expressed as a linear function of all the $v^\lambda(s, t)$, where s and t are integers ranging from 0 to n . Consider how the coefficient of a particular $v^\lambda(s, t)$ is formed. There will be positive contributions from coalitions containing i , and negative contributions from coalitions not containing i . For the positive part, we must count the number of orderings that put i in position " $s + t$ " and put exactly $s - 1$ other type 1 traders in positions preceding " $s + t$." This number is

$$\binom{n-1}{s-1} \binom{n}{t} (s+t-1)! (2n-s-t)! ,$$

or zero if $s = 0$. Similarly, for the negative part, we must have i in position " $s + t + 1$," and exactly s other type 1 traders in positions before " $s + t + 1$." The number of orderings that do this is

$$\binom{n-1}{s} \binom{n}{t} (s+t)! (2n-s-t-1)! ,$$

or zero if $s = n$. The desired coefficient of $v^\lambda(s, t)$ is the difference of these two numbers divided by the total number of orderings, which is $(2n)!$. This reduces to

$$(16) \quad \binom{n}{s} \binom{n}{t} \frac{(s+t-1)! (2n-s-t-1)! (s-t)}{(2n)!}$$

or

$$-\frac{1}{2n} \quad \text{if } s = t = 0 ,$$

or

$$\frac{1}{2n} \quad \text{if } s = t = n .$$

Since the coefficient (16) is *antisymmetric* in s and t , the lemma permits us to cancel the bulk of the terms in the value formula. It is this cancellation that makes the whole calculation manageable. We are left with

$$\begin{aligned} \phi_i^\lambda &= \frac{1}{2n} v^\lambda(n, n) + \sum_{s=1}^n \binom{n}{s} \frac{(s-1)! (2n-s-1)! s}{(2n)!} v^\lambda(s, 0) \\ &\quad - \sum_{t=1}^n \binom{n}{t} \frac{(t-1)! (2n-t-1)! t}{(2n)!} v^\lambda(0, t) - \frac{1}{2n} v^\lambda(0, 0) . \end{aligned}$$

Homogeneity of v^λ reduces this to

$$\phi_i^\lambda = \frac{1}{2} v^\lambda(1, 1) + \sum_{s=1}^n \binom{n}{s} \frac{s! (2n-s-1)! s}{(2n)!} [v^\lambda(1, 0) - v^\lambda(0, 1)] .$$

This, in turn, can be reduced to the very simple expression,¹⁵

¹⁵ The reduction:

$$\sum_{s=1}^n \binom{n}{s} \frac{s! (2n-s-1)! s}{(2n)!} = \frac{n! n!}{(2n)!} \sum_{s=1}^n \frac{(2n-s-1)! [n-(n-s)]}{(n-s)! n!}$$

(Continued on next page)

$$\phi_i^\lambda = \frac{1}{2} v^\lambda(1, 1) + \frac{1}{n+1} [v^\lambda(1, 0) - v^\lambda(0, 1)] .$$

A symmetrical argument yields

$$\phi_j^\lambda = \frac{1}{2} v^\lambda(1, 1) - \frac{1}{n+1} [v^\lambda(1, 0) - v^\lambda(0, 1)] ,$$

where j is a typical trader of type 2.

Thus far only the concavity, symmetry, and first-degree homogeneity of the utility functions have played a role. We now refer to the particular functions (13) and make the simple determinations

$$\begin{aligned} v^\lambda(1, 0) &= 0 , \\ v^\lambda(0, 1) &= \lambda_j , \\ v^\lambda(1, 1) &= \max(\lambda_i, 5\lambda_j) . \end{aligned}$$

Inserting these values, we obtain

$$(17) \quad \begin{aligned} \phi_i^\lambda &= \frac{1}{2} \max(\lambda_i, 5\lambda_j) - \frac{1}{n+1} \lambda_j && \text{(first type)} , \\ \phi_j^\lambda &= \frac{1}{2} \max(\lambda_i, 5\lambda_j) + \frac{1}{n+1} \lambda_j && \text{(second type)} . \end{aligned}$$

This is the λ -transfer value. The question is: For what choices of λ_i, λ_j is it feasible?

Clearly, only the ratio between λ_i and λ_j is significant. The situation is illustrated in Figure 4 for the case $n = 1$. In general, let α be any feasible, symmetric imputation. Then $5\alpha_i + \alpha_j \leq 5$ (cf. (14)). But if α is to be a value of the game, we must have $\lambda_i \alpha_i = \phi_i^\lambda$ and $\lambda_j \alpha_j = \phi_j^\lambda$ for some choice of λ_i, λ_j . Hence the following inequality must be satisfied:

$$(18) \quad 5\lambda_j \phi_i^\lambda + \lambda_i \phi_j^\lambda \leq 5\lambda_i \lambda_j ,$$

with λ_i and λ_j nonnegative and not both zero. If we substitute (17) into (18), we find that $\lambda_i \leq 5\lambda_j$ implies

$$\left(\frac{5}{2} - \frac{1}{n+1} \right) (\lambda_i - 5\lambda_j) \geq 0 ,$$

while $\lambda_i \geq 5\lambda_j$ implies

$$\left(\frac{5}{2} \lambda_i + \frac{1}{n+1} \lambda_j \right) (\lambda_i - 5\lambda_j) \leq 0 .$$

It follows that the only solution is $\lambda_i = 5\lambda_j$; this conclusion is independent of

$$\begin{aligned} &= \frac{n!n!}{(2n)!} \left[\sum_{s=1}^n \binom{2n-s-1}{n-1} - \sum_{s=1}^{n-1} \binom{2n-s-1}{n} \right] \\ &= \frac{n!n!}{(2n)!} \left[\binom{2n-1}{n} - \binom{2n-1}{n+1} \right] = \frac{1}{n+1} , \end{aligned}$$

using the identity $\binom{r}{r} + \binom{r+1}{r} + \dots + \binom{R}{r} = \binom{R+1}{r+1}$.

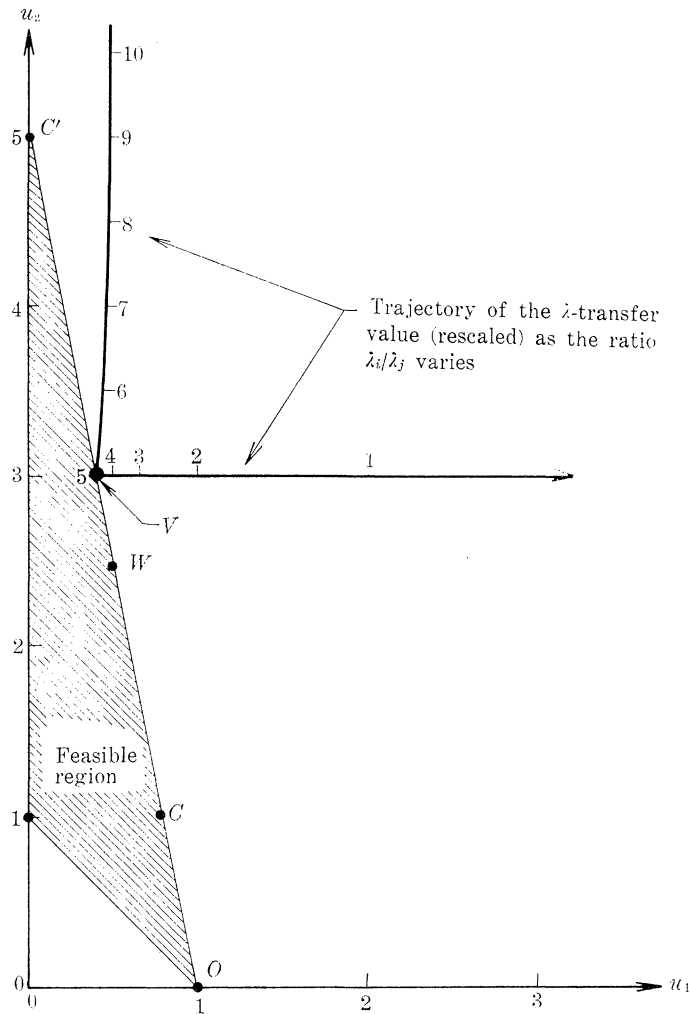


FIGURE 4
FEASIBILITY OF THE λ -TRANSFER VALUE ($n = 1$)

n .¹⁶ Hence the value of the game is unique and is given by

$$\phi_i = \frac{1}{2} - \frac{1}{5(n+1)} \quad (\text{first type}),$$

$$\phi_j = \frac{5}{2} + \frac{1}{n+1} \quad (\text{second type}).$$

Table 4 is intended for comparison with Table 3. In Figure 5 all three

¹⁶ This 5:1 ratio, valid for all n , could have been deduced several pages ago, by an argument similar to the one establishing symmetry of λ . However, the λ -transfer value computation is not helped (in this case) by knowing λ in advance.

TABLE 4
THE VALUE

n	V	
1	0.400	3.000
2	0.433	2.833
3	0.450	2.750
4	0.460	2.700
5	0.467	2.667
10	0.482	2.591
as $n \rightarrow \infty$	$\frac{1}{2} - \frac{1}{5n}$	$\frac{5}{2} + \frac{1}{n}$

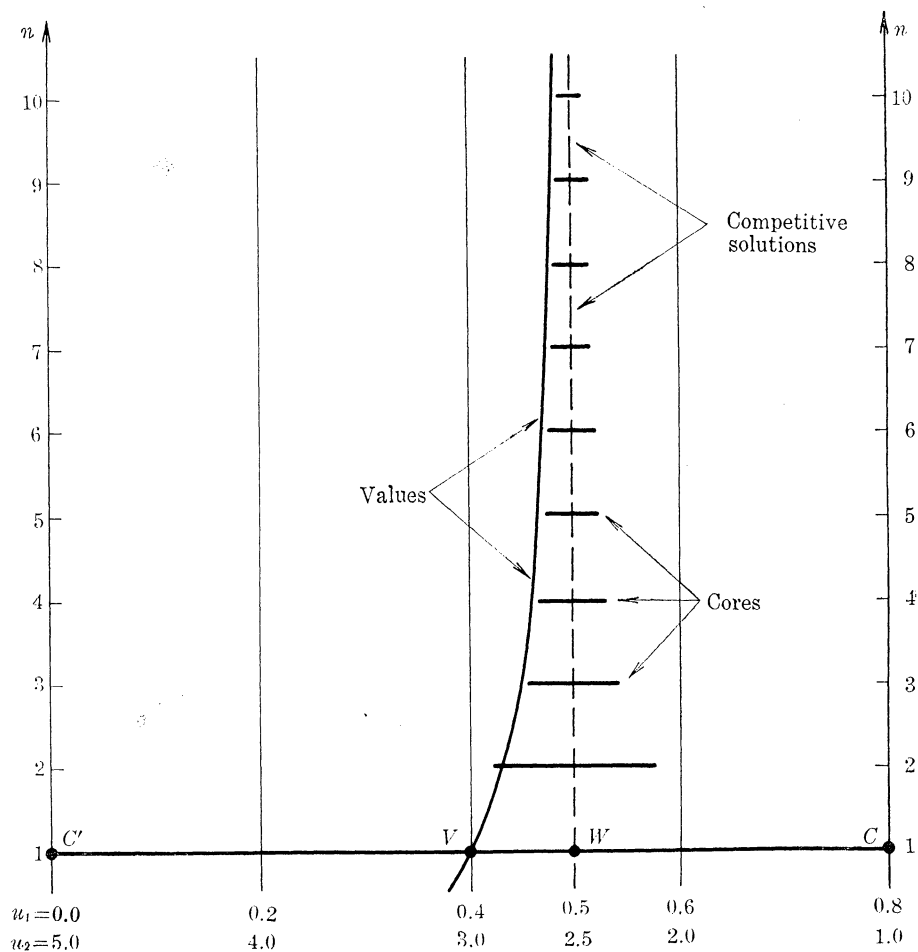


FIGURE 5
CONVERGENCE OF SOLUTIONS

kinds of solutions are shown together, as functions of market size. We see that the value is always more favorable to traders of type 2 than the competitive solution, and that it begins inside the core but moves outside at $n = 3$ because of its somewhat slower rate of convergence. These comparisons should not be interpreted too broadly, however, since as indicated at the end of Section 2 the no-transfer value is essentially a cardinal concept, while the other solutions are not. If we tamper with our example, applying nonlinear transformations to the traders' utilities, we can alter all such qualitative features. Indeed, for any fixed n , a pair of differentiable order-preserving concavity-preserving transformations for (13) can be found that place the value point V at any designated spot in the interior of the contract curve CC' , without affecting the other solutions.

The RAND Corporation and Yale University, U.S.A.

APPENDIX

A.1. C-GAMES WITH SIDE PAYMENTS¹

N : the set of *players*.

v : the *characteristic function*: $v(S)$ for each $S \subseteq N$ is a real number representing the monetary worth of the coalition S . Ordinarily v is *superadditive*, in the sense that if $S \cap T = \emptyset$ then $v(S) + v(T) \leq v(S \cup T)$.

A : the set of *imputations* α (feasible utility vectors), characterized by

$$\sum_{i \in N} \alpha_i \leq v(N).$$

C : the *core*, defined as the set of imputations $\alpha \in A$ that satisfy every coalition:

$$\sum_{i \in S} \alpha_i \geq v(S), \quad \text{all } S \in N.$$

ϕ : the *value*, an imputation defined by

$$\phi_i = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})],$$

where s, n denote the number of elements in S, N , respectively. An equivalent definition, often easier to work with, is

$$\phi_i = \frac{1}{n!} \sum_{\omega \in \Omega} [v(P_{\omega,i} \cup \{i\}) - v(P_{\omega,i})],$$

where Ω is the set of all orderings of N and $P_{\omega,i}$ is the set of predecessors of player i in the ordering ω .

A.2. AN EXCHANGE ECONOMY WITH MONEY²

N : the set of *traders*.

¹ See [17]. The term "c-game" is short for "game that is adequately represented by its characteristic function."

² See [20, 23].

x : an allocation of the m goods among the n traders:

$$x = \{x^i \mid i \in N\}, \quad \text{where } x^i = (x_1^i, \dots, x_m^i).$$

Thus, x_j^i is the amount of good j held by trader i .

U^i : the utility function of trader i , assumed to be of the form:

$$U^i(x^i, \xi^i) \equiv u^i(x^i) + \xi^i,$$

where u^i is concave and differentiable and ξ^i represents the net change from the initial money position.

a : the initial allocation. We assume every good to be present: $\sum_N a^i > 0$.

b : an optimal allocation, characterized by

$$\sum_N u^i(b^i) = \max_x \sum_N u^i(x^i),$$

the maximization subject to $x \geq 0$ and $\sum_N x^i = \sum_N a^i$. It is unique if the u^i are strictly concave.

π : the competitive price vector, defined by

$$\pi_j \begin{cases} = \\ \geq \end{cases} \partial u^i(b^i) / \partial x_j^i \quad \text{if } b_j^i \begin{cases} > \\ = \end{cases} 0,$$

where b is any optimal allocation.

ω : the competitive imputation, defined by

$$\omega_i = u^i(b^i) + \pi \cdot (a^i - b^i),$$

where b is any optimal allocation. Uniqueness of π and ω follows from the differentiability of the u^i .

v : the characteristic function, defined by

$$v(S) = \max_x \sum_S u^i(x^i),$$

the maximization subject to $x \geq 0$ and $\sum_S x^i = \sum_S a^i$. The core and value can now be defined as in Appendix A.1.

Replication: Let k identical economies be regarded as a single economy, having kn traders of n different types. The competitive price vector of the enlarged market is just π , while the competitive imputation is just the kn -dimensional vector (ω, \dots, ω) (k times). The characteristic function of the enlarged market is homogeneous of the first degree, in the sense that $v(hS) = hv(S)$, where hS denotes a coalition having exactly h times as many traders of each type as S . Unlike the competitive solution, the core and the value depend on k . However, both converge to the competitive imputation as $k \rightarrow \infty$.

A.3. C-GAMES WITHOUT SIDE PAYMENTS³

N : The set of players.

V : the generalized characteristic function: $V(S)$ is a compact, convex, nonempty set of s -dimensional vectors, representing the feasible utility vectors for the coalition $S \subseteq N$. Ordinarily V is superadditive,

³ See [3].

in the sense that if $S \cap T = \emptyset$ and if $\alpha \in V(S), \beta \in V(T)$, then the $s + t$ -dimensional vector (α, β) (with coordinates properly identified) is in $V(S \cup T)$.

A : the set of *imputations*: $A = V(N)$.

C : the *core*, defined as the set of imputations $\alpha \in A$ that satisfy every coalition, i.e., it is never the case that $\beta \in V(S)$ and $\beta_i > \alpha_i$, all $i \in S$.

λ : a *scaling vector* $(\lambda_i | i \in N)$. The components must be nonnegative and not all zero.

v^λ : the λ -*transfer characteristic function*, defined by

$$v^\lambda(S) = \max_{\alpha \in V(S)} \sum_{i \in S} \lambda_i \alpha_i.$$

ϕ^λ : the λ -*transfer value*, defined by

$$\phi_i^\lambda = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} [v^\lambda(S) - v^\lambda(S - \{i\})].$$

\emptyset : the set of *values* of the game, comprising all $\phi \in A$ for which there exists a scaling vector λ such that $\lambda_i \phi_i = \phi_i^\lambda$, all $i \in N$.

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