Borel Determinacy

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Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots\rangle$ all of whose finite initial segments belong to Y. For each $A \subseteqq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0\rangle \in Y$, II picks y_1 with $\langle y_0, y_1\rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2\rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega\rangle \in A$. ($\omega =$ the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements of Y of even length such that always $\langle y_0, \cdots, y_{2n-1}, s(\langle y_0, \cdots, y_{2n-1}\rangle)\rangle \in Y$. A play $\langle y_0, y_1, \cdots\rangle$ of $\mathcal{G}(A, Y)$ is a *play according to* the strategy s if $(\forall n)(y_{2n} = s(\langle y_0, \cdots, y_{2n-1}\rangle))$. The notions of a *strategy for* II and a *play according to* a strategy for II are similarly defined. s is a winning strategy for I (II) if every play according to s is a win for I (II). $\mathcal{G}(A, Y)$ is *determined* if one of the players has a winning strategy.

The games $\mathcal{G}(A, Y)$ were introduced by Gale and Stewart [4], though several special cases had been studied earlier by Polish mathematicians.

We give $\mathcal{F}(Y)$ a topology by taking as a base the set of all subsets of $\mathcal{F}(Y)$ of the form $\{x : p \text{ is an initial segment of } x\}$ with $p \in Y$. $A \subseteqq \mathcal{F}(Y)$ is *Borel* if it belongs to the σ-algebra generated by the open subsets of $\mathcal{F}(Y)$. $\mathcal{G}(A, Y)$ is *Borel* just in case A is Borel. In this paper we prove that all Borel games are determined.

If $A \subseteqq \mathcal{F}(Y)$, $A \in \mathbf{\Sigma}^0_1 \Leftrightarrow A$ is open. For countable ordinals $\alpha \geqq 1$ we define inductively:

$A \in \mathbf{\Sigma}^0_\alpha \Leftrightarrow$ there is a sequence $\langle A_i : i \in \omega\rangle$ with $A = \bigcup_{i \in \omega} A_i$ and each $A_i \in \mathbf{\Pi}^0_{\beta_i}$ for some $\beta_i < \alpha$;

$$A \in \mathbf{\Pi}^0_\alpha \Longleftrightarrow \mathcal{F}(Y) - A \in \mathbf{\Sigma}^0_\alpha\ ;$$
$$A \in \mathbf{\Delta}^0_\alpha \Longleftrightarrow A \in \mathbf{\Sigma}^0_\alpha \quad\text{and}\quad A \in \mathbf{\Pi}^0_\alpha\ .$$

Every Borel set belongs to $\mathbf{\Sigma}^0_\alpha$ for some countable α. We say that $\mathcal{G}(A, Y)$ is

$\mathbf{\Sigma}^0_\alpha$ (open, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$, etc.) just in case A is.

Gale and Stewart [4] proved that all open games are determined. Wolfe [8] extended this result to $\mathbf{\Sigma}^0_2$ games, and Morton Davis [2] further extended it to $\mathbf{\Sigma}^0_3$ games. We showed [5] that, if a measurable cardinal number exists, then all Borel games (in fact all $\mathbf{\Sigma}^1_1$, or analytic, games) with Y countable are determined. If arbitrarily large measurable cardinals exist, then all Borel games are determined. J. Baumgartner reproved Davis' theorem ([2]) using the methods of [5] (but not assuming a measurable cardinal). Paris [7] extended Baumgartner's technique and proved that all $\mathbf{\Sigma}^0_4$ games are determined.

Friedman [3] showed that objects of higher type are needed to prove Borel determinacy. By *Zermelo set theory,* we mean the usual Zermelo-Fraenkel set theory with the axiom of replacement omitted (but keeping the axiom of *Aussonderung* or separation). In Zermelo set theory one has the power set operation P, but one cannot prove the existence of the union of the family $\{\omega, P(\omega), P(P(\omega)), \cdots\}$. Friedman showed that Borel determinacy, even for countable Y, is unprovable in Zermelo set theory. In fact Friedman's results (as slightly improved by us) show that, even for countable Y, $\mathbf{\Sigma}^0_4$ determinacy is not provable in the usual formal theory of ω and $P(\omega)$, $\mathbf{\Sigma}^0_5$ determinacy is not provable in the theory of ω, $P(\omega)$, and $P(P(\omega))$, and so on into the transfinite. Borel determinacy is probably then the first theorem whose statement does not blatantly involve the axiom of replacement but whose proof is known to require the axiom of replacement.

Using the arguments of this paper and the proof of Morton Davis [2], it can be seen that the improved Friedman theorem is essentially best possible: $\mathbf{\Sigma}^0_4$ determinacy for Y countable is provable in the theory of ω, $P(\omega)$, and $P(P(\omega))$; and so on. Similar sharp positive and negative results can be proved concerning $\mathbf{\Delta}^0_\alpha$ determinacy. Details will be given in our forthcoming monograph [6].

In [5] we introduced a basic technique for proving that a game $\mathcal{G}(A, Y)$ is determined. One associates with $\mathcal{G}(A, Y)$ a game $\mathcal{G}(A^*, Y^*)$ over a large set Y^*. A^* is topologically similar to A (usually A^* is open). One then proves that $\mathcal{G}(A^*, Y^*)$ is equivalent to $\mathcal{G}(A, Y)$: I (II) has a winning strategy for $\mathcal{G}(A^*, Y^*)$ if and only if I (II) has a winning strategy for $\mathcal{G}(A, Y)$. In [5] and in Paris [7], Y^* is a set of sequences of ordinal numbers. To prove $\mathcal{G}(A^*, Y^*)$ equivalent to $\mathcal{G}(A, Y)$, combinatorial properties of sets of ordinal numbers are used. Unless one makes assumptions unprovable in Zermelo-Fraenkel set theory, such as the existence of a measurable cardinal, the combinatorial problems become unmanageable when A is more complicated than Σ_4^0. In [1] Blass proved the equivalence of two very strong forms of determinacy by associating with a game $\mathcal{G}$ a game $\mathcal{G}^*$, such that among the *moves* of $\mathcal{G}^*$ are fragments of *strategies* for $\mathcal{G}$. (J. Mycielski discovered this proof independently.) This provided us with a clue as to the general sort of $\mathcal{G}(A^*, Y^*)$ needed to prove Borel determinacy.

The author wishes to thank Robert Solovay for suggesting many ways to improve an earlier version of this paper and William Mitchell for suggesting a change in our original construction which makes the reduction more elegant.

1. Open games and I-imposed subgames

Suppose $A \subseteq \mathcal{F}(Y)$ and II does not have a winning strategy for $\mathcal{G}(A, Y)$. There is a first move y_0 by I such that II does not have a winning strategy for $\mathcal{G}(A, Y)$ starting at the position $\langle y_0 \rangle$ (in the obvious sense), for otherwise II has a winning strategy for $\mathcal{G}(A, Y)$. If II has no winning strategy for $\mathcal{G}(A, Y)$ starting at the position $\langle y_0, \cdots, y_{2n} \rangle$, then for every y_{2n+1} there is a y_{2n+2} such that II has no winning strategy for $\mathcal{G}(A, Y)$ starting at $\langle y_0, \cdots, y_{2n+2} \rangle$. By choosing one such y_{2n+2} for each such $\langle y_0, \cdots, y_{2n+1} \rangle \in Y$, we get a strategy for I. If A is closed, this is a winning strategy, since

$$\langle y_i : i \in \omega \rangle \notin A \Longrightarrow (\exists n)(\langle y_0, \cdots, y_n, y'_{n+1}, \cdots \rangle \notin A \text{ for every } y'_{n+1}, y'_{n+2}, \cdots) .$$

In other words, since I's strategy cannot lose at any finite position, it must win. We have thus reproved the Gale-Stewart theorem that all closed games are determined.

X is a *subgame* of Y if $X \subseteq Y$ and X satisfies the conditions put on Y in the first sentence of this paper. Following Davis [3], we call a subgame X of Y a I-*imposed* subgame if

$$\langle y_0, \cdots, y_{2n} \rangle \in X \text{ and } \langle y_0, \cdots, y_{2n}, y_{2n+1} \rangle \in Y \Longrightarrow \langle y_0, \cdots, y_{2n}, y_{2n+1} \rangle \in X .$$

In other words, any move legal for II in Y is legal for II in X.

Example. Suppose II has no winning strategy for $\mathcal{G}(A, Y)$. Let X be the set of elements p of Y such that II has no winning strategy for $\mathcal{G}(A, Y)$ starting at any initial segment p' of p. Arguments we have already given show that X is a I-imposed subgame of Y. If A is closed, every play in $\mathcal{F}(X)$ is a win for I. The subgame is I's *winning subgame for* $\mathcal{G}(A, Y)$.

II-*imposed subgames* and II's *winning subgame for* an open game are similarly defined.

2. Determinacy for finite levels of the Borel hierarchy

Our basic plan for proving Borel determinacy is as follows. Let $A_0, A_1, \cdots$

be closed subsets of $\mathcal{F}(Y)$. We produce another set Y^* such that any element of $\mathcal{F}(Y^*)$ can be thought of as a pair $\langle x, x' \rangle$ with $x \in \mathcal{F}(Y)$. $\{\langle x, x' \rangle: x \in A_i\}$ will be *clopen* (closed and open) in $\mathcal{F}(Y^*)$ for each i. If $A \in \Sigma^0_{n+1}$, n finite, generated as Σ^0_{n+1} by the A_i, then $A^* = \{\langle x, x' \rangle: x \in A\}$ will be Σ^0_n in $\mathcal{F}(Y^*)$. Furthermore, we show that, if $\mathcal{G}(A^*, Y^*)$ is determined, then $\mathcal{G}(A, Y)$ is determined. By iterating this reduction into the transfinite, we can associate with each Borel game an equivalent open game.

Fix $\mathcal{G} = \mathcal{G}(A, Y)$. Let $A_0, A_1, \cdots$ be closed subsets of $\mathcal{F}(Y)$. Let $f: \omega \to \{-1, 0\}$ be such that $f^{-1}(\{-1\})$ and $f^{-1}(\{0\})$ are both infinite.

We define a game $\mathcal{G}^* = \mathcal{G}(A^*, Y^*)$. With each position $p \in Y^*$ of even length will be associated a position $p_0 \in Y$ of even length. If p extends p', then p_0 extends p'_0. With every position p in $\mathcal{G}^*$ will also be associated a subgame X_p of Y, with $p_0 \in X_p$, and a number i_p.

If p is the empty position, p_0 is empty, $X_p = Y$, and $i_p = 0$.

If the length of p is $2n$ and $f(n) = -1$, play continues at p by I's choosing a move allowable in X_p at p_0 and then II's choosing a move allowable in X_p at the position chosen by I. If p' is the new position in $\mathcal{G}^*$, p'_0 is the position in $\mathcal{G}$ resulting from II's move, $X_{p'} = X_p$, and $i_{p'} = i_p$.

If the length of p is $2n$ and $f(n) = 0$, play continues by I's choosing a I-imposed subgame X^{i_p} of X_p with $p_0 \in X^{i_p}$. II now has two options:

(i) II may extend p_0 to a position in X^{i_p} of even length avoiding A_{i_p} (i.e., a position $\tilde{p}$ such that every $x \in \mathcal{F}(Y)$ extending $\tilde{p}$ is not in A_{i_p}). In this case, if p' is the new position in $\mathcal{G}^*$, p'_0 is the position chosen by II, $X_{p'} = X_p$, and $i_{p'} = i_p + 1$.

(ii) Option (ii) is open to II only if II has a winning strategy starting at p_0 for $\mathcal{G}(\mathcal{F}(X^{i_p}) - A_{i_p}, X^{i_p})$. If this holds and II takes option (ii), then, if p' is the new position in $\mathcal{G}^*$, $p'_0 = p_0$, $X_{p'}$ is II's winning subgame for $\mathcal{G}(\mathcal{F}(X^{i_p}) - A_{i_p}, X^{i_p})$ starting at p_0, and $i_{p'} = i_p + 1$. (II's winning subgame starting at p_0 is defined in the obvious way allowing all moves at positions not extending p_0.)

Declaring which option he takes is to be construed as part of II's move.

If x is a play of $\mathcal{G}^*$, x_0 is the obvious *associated play* of $\mathcal{G}$. I wins $\mathcal{G}^*$ just in case $x_0 \in A$

LEMMA. *If $\mathcal{G}^*$ is determined, then $\mathcal{G}$ is determined.*

Proof. Suppose first that I has a winning strategy s^* for $\mathcal{G}^*$. Let p be a position in $\mathcal{G}^*$ of length $2n$ which is consistent with s^*. We describe a strategy t_p for I in X_p starting at p_0. Any play according to t_p will reach a position $\tilde{p}$ with which we shall associate a certain position p' in $\mathcal{G}^*$. p' extends p, length $(p') = 2n + 2$, $(p')_0 = \tilde{p}$, and p' is consistent with s^*.

If $f(n) = -1$, I simply moves at p according to s^* and any reply by II legal in X_p will produce $\tilde{p}$, p' with the desired properties.

Suppose $f(n) = 0$. Let $X^{i_p} = s^*(p)$. If I has a winning strategy for $\mathcal{G}' = \mathcal{G}(\mathcal{F}(X^{i_p}) - A_{i_p}, X^{i_p})$ starting at p_0, let t_p be such a strategy. If I plays according to t_p, a position $\tilde{p}$ of even length avoiding A_{i_p} is reached. Let p' be obtained from p by I's playing X^{i_p} and II's taking option (i) and producing $\tilde{p}$. If II has a winning strategy for $\mathcal{G}'$ starting at p_0, p' is obtained by letting I play X^{i_p} and II take option (ii).

These considerations allow us to obtain a strategy s for I for $\mathcal{G}$. I pretends he is playing $\mathcal{G}^*$ according to s^*. As long as II cooperates, I associates with the position p in $\mathcal{G}$ of even length a position p^* in $\mathcal{G}^*$ of even length consistent with s^* such that $(p^*)_0 = p$. I's strategy is obtained by playing according to the associated strategy t_{p^*} until he reaches the position $\widetilde{p^*}$, to which he associates $p^{*\prime}$.

In the course of playing according to t_{p^*}, II may make a move producing a position not in X_{p^*}. When this happens, there is an initial segment p^1 of p^* such that II has departed from his winning subgame for $\mathcal{G}'' = \mathcal{G}(\mathcal{F}(X^{i_{p^1}}) - A_{i_{p^1}}, X^{i_{p^1}})$. Choose the shortest such p^1. I begins playing a winning strategy t for $\mathcal{G}''$. If the position in $\mathcal{G}$ remains in $X^{i_{p^1}}$, a position r in $\mathcal{G}$ is obtained such that for some 2-move extension q of p^1, $(q)_0 = r$ and q is consistent with s^*. I associates q with r and continues as before. If q is not obtained, II must depart from X_{p^1}. I treats this departure just as he treated the departure from X_{p^*}. Since the sequence $p^*, p^1, \cdots$ is decreasing, I eventually associates a position in $\mathcal{G}^*$ with the position in $\mathcal{G}$.

I's guess as to II's response to X^0 can be revised at most once. After this, his guess concerning X^1 can be revised at most once, and so on. Thus we have produced a strategy s for I for $\mathcal{G}$ such that any play x of $\mathcal{G}$ according to s determines a play x^* of $\mathcal{G}^*$ according to s^* such that $(x^*)_0 = x$. Since x^* is a win for I, so is x. In other words, s is a winning strategy.

Now suppose that II has a winning strategy s^* for $\mathcal{G}^*$. Let p be a position in $\mathcal{G}^*$ of length $2n$ consistent with s^*. We describe a strategy t_p for II in X_p starting at p_0. Every play according to t_p will reach a position $\tilde{p}$ with which we shall associate a certain position p' in $\mathcal{G}^*$. p' extends p, length $(p') = 2n + 2$, $(p')_0 = \tilde{p}$, and p' is consistent with s^*.

The case $f(n) = -1$ is handled as before.

If $f(n) = 0$, let $\mathcal{G}'$ be as follows: $\mathcal{G}'$ is played over X_p starting at p_0. II wins $\mathcal{G}'$ if a position $\tilde{p}$ is obtained such that, for some I imposed subgame

X^{i_p} of X_p, if I plays X^{i_p} at p, then s^* calls for II to take option (i) and produce $\tilde{p}$. If II has a winning strategy for $\mathcal{G}'$, we let t_p be such a winning strategy, and we get $\tilde{p}$, p' in the obvious way. If I has a winning strategy for $\mathcal{G}'$, p' is obtained by letting I play his winning subgame for $\mathcal{G}'$ and II move according to s^*. Note that II must take option (ii).

The construction of a winning strategy for II for $\mathcal{G}$ is now like the earlier construction of a winning strategy for I, and we omit it.

We wish to extract a little more information from the proof of the lemma. For each n let $\sigma(n)$ be the number of $i < n$ such that $f(i) = -1$. Suppose s^{*n} is a fragment of a strategy for $\mathcal{G}^*$ for one of the players, defined on all positions (with the right player to move) of length $<2n$. Then the proof of the lemma gives us an operation for producing a fragment s^n of a strategy for the same player for $\mathcal{G}$, defined on all positions of length $<2\sigma(n)$. Furthermore, if $n > 0$ and s^{*n-1} is the restriction of s^{*n} to positions of length $<2(n-1)$, then the s^n derived from s^{*n} agrees with the s^{n-1} derived from s^{*n-1}. Finally, if s^* is a strategy for $\mathcal{G}^*$ and s is the strategy for $\mathcal{G}$ gotten in this way, then, for each play x of $\mathcal{G}$ according to s there is a play x^* of $\mathcal{G}^*$ according to s^* with $x^*_0 = x$. These facts will be used in Section 3.

Recall also the fact that if n_0 is the least number with $f(n_0) = 0$, every position in $\mathcal{G}$ of length $\leq 2n_0$ is a position in $\mathcal{G}^*$, and vice versa. This trivial fact is extremely important in Section 3.

THEOREM. *For every finite k, every Σ^0_k game is determined.*

Proof. All Σ^0_1 (open) games are determined. Assume all Σ^0_k games are determined and let $A \in \Sigma^0_{k+1}$. Assume for definiteness that k is odd. Then

$$A = \bigcup_{n_1} \bigcap_{n_2} \cdots \bigcup_{n_k} A_{n_1 \cdots n_k}$$

with each $A_{n_1 \cdots n_k}$ closed. Let $A_0, A_1, \cdots$ be an enumeration of all the $A_{n_1 \cdots n_k}$. Let $\mathcal{G}^*$ be defined as before. $\mathcal{G}^*$ is Σ^0_k, since I wins $\mathcal{G}^*$ just in case

$$(\exists n_1)(\forall n_2) \cdots (\exists n_k) \text{ (II takes option (ii) on } A_{n_1 \cdots n_k}) \,.$$

By the lemma, $\mathcal{G}$ is determined.

Remark. Our proof has used the axiom of choice. If Y can be well-ordered, the axiom of choice is not needed to prove that Σ^0_k games over Y are determined. We may assume Y is a set of sequences of ordinal numbers. Let $g(n_1 \cdots n_k)$ code $A_{n_1 \cdots n_k}$. In $L[g, Y]$ the axiom of choice holds, so there is a winning strategy for one of the players for $\mathcal{G}(A, Y)$. By absoluteness, this is really a winning strategy.

In the case of full Borel determinacy, we need in addition to know that each Borel set can be analyzed as in the second paragraph of Section 3.

3. Borel determinacy

The operation $\mathcal{G} \longrightarrow \mathcal{G}^*$ can be iterated. This allows us to associate with every $\mathbf{\Sigma}^0_k$ game an equivalent open game. By taking limits at limit ordinals, we are able to iterate $\mathcal{G} \longrightarrow \mathcal{G}^*$ into the transfinite and associate with every Borel game an equivalent open game.

Let $A \subseteqq \mathcal{F}(Y)$ and $A \in \mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$. If $\alpha \neq 1$, represent A as $\bigcup_{i \in \omega} A^i$, with each $A^i \in \mathbf{\Pi}^0_{\beta_i}$ for some $\beta_i < \alpha$. If $\beta_i \neq 1$, represent A^i as $\bigcap_{j \in \omega} A^{ij}$, with each $A^{ij} \in \mathbf{\Sigma}^0_{\gamma_{ij}}$ for some $\gamma_{ij} < \beta_i$. Continue in this manner. The *components of* A are the A^i, the complements of the A^{ij}, etc. If $\alpha > \beta \geqq \omega$, the components of A represented as $\mathbf{\Pi}^0_\beta$ are the *components of order* β. If $0 \leqq k < \omega$, the *components of order* k are the components of A represented as $\mathbf{\Pi}^0_{k+1}$. Let

$$A_0^\beta,\ A_1^\beta,\ \cdots$$

be an enumeration of all the components of order β, for each $\beta < \alpha$. (If there are only finitely many components of order β, let one of them be repeated infinitely often in the enumeration; it is easy to check there are only countably many components of order β.)

Let $g\colon \omega \longrightarrow \alpha \cup \{-1\}$ be such that $g^{-1}(\{\beta\})$ is infinite for each $\beta < \alpha$ and $g^{-1}(\{-1\})$ is infinite. Fix $\beta \leqq \alpha$. Let $n_0 < n_1 < \cdots$ be all numbers n such that $g(n) < \beta$. Define

$$g_\beta(i) = g(n_i)\ .$$

Now define for each $\beta < \alpha$,

$$f_\beta(n) = \begin{cases} 0 & \text{if } g_{\beta+1}(n) = \beta\ ; \\ -1 & \text{otherwise}\ . \end{cases}$$

We shall define, for each $\beta \leqq \alpha$, a game $\mathcal{G}_\beta = \mathcal{G}(A_\beta, Y_\beta)$. We also define a sequence $A_i^{*\beta}$ of closed subsets of $\mathcal{F}(Y_\beta)$. We let $\mathcal{G}_0 = \mathcal{G}(A, Y)$. Our definition will have the following properties:

For each $\beta \leqq \gamma \leqq \alpha$, with each position p of even length in $\mathcal{G}_\gamma$ will be associated a position p_β in $\mathcal{G}_\beta$ of even length such that

(i) $p_\gamma = p$.

(ii) If p extends p' then p_β extends p'_β.

(iii) $\beta < \delta < \gamma \Longrightarrow (p_\delta)_\beta = p_\beta$.

(iv) If x is a play of $\mathcal{G}_\gamma$, then the set of p_β for p occurring in x determines a unique play x_β of $\mathcal{G}_\beta$.

Suppose p is a position in $\mathcal{G}_\beta$ of length $2n$. If $g_\beta(n) = -1$, play of $\mathcal{G}_\beta$ continues by I and then II extending (subject to certain constraints) p_0. If $g_\beta(n) = \gamma \geqq 0$, play of $\mathcal{G}_\beta$ continues by I choosing (subject to certain constraints) a I-imposed subgame X of Y_γ, and II either

(i) extending (subject to certain constraints) p_γ to a position in X of even length avoiding an $A_i^{*\gamma}$ determined by p, or

(ii) restricting himself to a certain II-imposed subgame of X, every play of which is in $A_i^{*\gamma}$.

Each $A_i^{*\beta}$ will arise at some position p in any play of $\mathcal{G}_{\beta+1}$.

We define $A_i^{*\beta}$ as follows:

$$x \in A_i^{*\beta} \Longleftrightarrow x_0 \in A_i^\beta \, .$$

If our construction is as described above, each $A_i^{*\beta}$ is closed. To see this, note that, if for each i, $A_i^{*\beta}$ is closed, the set of plays x of $\mathcal{G}_{\beta+1}$ with $x_\beta \in A_i^{*\beta}$ is clopen; hence each $A_i^{*\beta+1}$ is closed. For β a limit ordinal, each $A_i^{*\beta}$ is an intersection of clopen sets, and so is closed.

We now give our formal definition of the $\mathcal{G}_\beta$. Get $\mathcal{G} = \mathcal{G}(A, Y)$. If $\mathcal{G}_\beta$ is defined, $\mathcal{G}_{\beta+1}$ is defined from $\mathcal{G}_\beta$ just as $\mathcal{G}^*$ was defined from $\mathcal{G}$ in Section 2, with f_β replacing f and $A_i^{*\beta}$ replacing A_i. If p is a position in $\mathcal{G}_{\beta+1}$, p_β is defined just as p_0 was defined in Section 2. If $\delta < \beta$, set $p_\delta = (p_\beta)_\delta$. Suppose β is a limit ordinal and $\mathcal{G}_\gamma$ is defined for $\gamma < \beta$. Let n_0 be the least n such that $g_\beta(n) \geqq 0$. Let n_{i+1} be the least n such that $g_\beta(n) > g_\beta(n_i)$. To play $\mathcal{G}_\beta$, first play $2n_0$ moves of $\mathcal{G}_{g_\beta(n_0)}$, then $2(n_1 - n_0)$ more moves of $\mathcal{G}_{g_\beta(n_1)}$, etc. Note that the positions in $\mathcal{G}_{g_\beta(n_i)}$ of length $\leqq 2n_i$ are exactly the same as the positions in $\mathcal{G}_{\beta'}$ of length $\leqq 2n$, for any $\beta' < \beta$ with $g_\beta(n_i) \leqq \beta'$. For p a position in $\mathcal{G}_\beta$ of length $\leqq 2n_i$, $p_\gamma = p$ for $\gamma \geqq \beta_i$; $p_\gamma = (p_{\beta_i})_\gamma$ for $\gamma < \beta_i$.

The reader can easily check that our construction has the properties described earlier.

A play x of $\mathcal{G}_\beta$ is a win for I if $x_0 \in A$. Notice that $\mathcal{G}_\alpha$ is open, since I wins $\mathcal{G}_\alpha$ just in case

$$(\exists i) \big(\text{II takes option (ii) on } A^i\big) \, .$$

Now suppose that s_α is a winning strategy for one of the players for $\mathcal{G}_\alpha$. For each $\beta \leqq \alpha$ set

$$\sigma_\beta(n) = \text{the number of } i < n \text{ such that } g(i) < \beta \, .$$

Let s_α^n be the strategy s_α restricted to positions of length $< 2n$. For each $\beta < \alpha$ we define a fragment s_β^n of a strategy for $\mathcal{G}_\beta$ (for the same player as s_α). s_β^n will be defined on all positions of length $< 2\sigma_\beta(n)$.

Let $\beta_1^n > \beta_2^n > \cdots > \beta_{k_n}^n$ be all values $\geqq 0$ taken by g on arguments $< n$. Set

$$s_\beta^n = s_\alpha^n \qquad \text{for } \beta > \beta_1^n \, .$$

(We can do this since Y_α and Y_β agree on positions of length $\leqq 2n$.) $s_{\beta_i^n}^n$ is obtained from $s^n_{\beta^n_i+1}$ by our standard operations as in Section 2. We set

$$\begin{aligned} s^n_\beta &= s^n_{\beta^n_i} && \text{for } \beta^n_{i+1} < \beta \leqq \beta^n_i \ , \\ s^n_\beta &= s^n_{\beta^n_{k_n}} && \text{for } \beta \leqq \beta^n_{k_n} \ . \end{aligned}$$

We assert that s^{n+1}_β agrees with s^n_β. If $\beta > \beta^{n+1}_1$,

$$s^{n+1}_\beta = s^{n+1}_\alpha \ , \quad \text{which agrees with } s^n_\alpha = s^n_\beta \ .$$

If $s^{n+1}_{\beta^{n+1}_i+1}$ agrees with $s^n_{\beta^{n+1}_i+1}$ then $s^{n+1}_{\beta^{n+1}_i}$ agrees with $s^n_{\beta^{n+1}_i}$ by Section 2. The other cases are similar to the first.

Suppose that x^0 is a play of $\mathcal{G}_0$ according to s_0. By induction on $\beta \leqq \alpha$, we define a play x^β of $\mathcal{G}_\beta$ according to s_β, in such a way that $\beta_1 < \beta_2 \Longrightarrow (x^{\beta_2})_{\beta_1} = x^{\beta_1}$. Successor steps are handled exactly as in Section 2. If β is a limit ordinal, the $x^{\beta'}$, $\beta' < \beta$ converge to a play x^β according to s_β.

It follows as in Section 2 that s_0 is a winning strategy for $\mathcal{G}_0 = \mathcal{G}(A, Y)$, and so we have proved

THEOREM. *All Borel games are determined.*

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