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THE THEORY OF INFINITE GAMES[1]

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This paper develops in an abstract way the theory of infinite games. The kernel is replaced by an operator and the distributions by suitability chosen Banach spaces; the complete theory of the determinateness of a game is studied. Both weak and uniform upper and lower values are introduced and relations among them are obtained, (see Theorem 1 and lemma 7). A study of Bayes solutions is given and an analysis of the change of the value under perturbation of the operator is carried out. Some new examples of determinant games are presented to illustrate the general theory. A further discussion on non-linear games and games with constraints is given. Games invariant under groups of transformations are discussed. The usual terminology of the theory of games developed in the Annals of Mathematics Studies 24 is freely employed. A future paper on the applications of this theory to statistical decision functions is intended.

§1. Preliminaries and notation

Let ε and $\mathfrak{F}$ denote Banach spaces consisting of measures; more precisely, subspaces of abstract (L) spaces in the sense of Kakutani [1]. The reader may find it helpful to keep in mind any of the concrete examples given below. Let R and S be Banach spaces of functions; specifically, subspaces of abstract (M) spaces [2]. As usual, R^* and S^* will denote the conjugate Banach spaces. It is known that (L) spaces and (M) spaces are mutually conjugate spaces of one another although non-reflexive.

Let K denote a closed cone of elements in R, i.e., if $x, y \in K$, then $\lambda x + \mu y \in K$ for $\lambda, \mu \geqq 0$, and if $x, -x$ both belong to K, then $x = 0$. Similarly, P will denote a closed cone in ε. The dual cone K^* in R^* is defined as the set of all f in R^* with the property that $(x, f) \geqq 0$ for every x in K. [(x, f) is the value of the functional f acting on the element x. Throughout this paper the inner product notation will be employed.] Also, let Q be a closed cone contained in K^*. It is assumed that in S a closed cone L is given and that P is contained in L^*. We also suppose that K and L possess interior points u and v. This means that there exists a sphere about u and v lying entirely in K and L. This implies the existence of a number ρ such that for any $g \in K^*$ with $\| g \| = 1$ that $(u, g) \geqq \rho$. In view of this, nothing is lost by considering a cross section of K^*. More precisely, let K_u^* be the section of the cone K^* where $(u, g) = 1$. Q_u is defined analogously with the section taken relative to Q. Finally, P_v is constructed in a similar fashion.

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A schematic diagram of the chosen underlying spaces is given.

$$\begin{pmatrix} \varepsilon & R \\ S(\subset \varepsilon^*) & \mathfrak{F}(\subset R^*) \end{pmatrix} \qquad \begin{pmatrix} P & K \\ L(\subset P^*) & Q(\subset K^*) \end{pmatrix}$$

Space Array Cone Array

The corresponding entry in the cone array designates the cone chosen in the corresponding entry in the space array. For example, K is the cone selected in R. The generic elements of each of these spaces are indicated by $\begin{pmatrix} f & x \\ y & g \end{pmatrix}$, i.e., f, f' will denote points of ε, etc.

A few typical examples of such spaces and cones are in order.

Examples of ε or $\mathfrak{F}$ are:

(a) (V)—The space of functions of bounded variation on [0, 1].
(b) (L)—The space of integrable functions on [0, 1].
(c) (F)—The space of finitely additive set functions of bounded variation.
(d) (D)—The space of absolutely convergent series.

For the cone P in these respective spaces, we may take the set of positive measures.

Examples of R and S are:

(a′) (M)—Bounded Lebesgue measurable functions.
(b′) (C)—Continuous functions.
(c′) (C^n)—Continuously n times differentiable functions on the unit interval.

The cone K may be taken to be respectively (a″) all elements of (M) non-negative almost everywhere, (b″) non-negative elements of (C) and (c″) all elements of (C^n) for which $x^{(n)}(t) \geqq 0$ with $x^{(i)}(0) \geqq 0$ for $i = 1, \cdots, n - 1$.

Other examples will be considered later. It is to be remarked that with respect to any cone there can be defined a partial ordering. Specifically $x \geqq 0$ if and only if $x \in K$. Similarly, $y \geqq x$ shall mean $y - x \geqq 0$. Throughout the sequel, it is to be understood that any partial ordering used is relative to the prescribed cone in the space.

§2. Definitions

Let A and B denote bounded linear operators mapping ε into R. Suppose the conjugate mappings A^* and B^* take $R^* \supset \mathfrak{F}$ into S. Therefore, for f in P and g in Q, $(A^*g, f) = (g, Af)$ and $(B^*g, f) = (g, BF)$ are well defined. We also assume that for g in Q_u and f in P_v that $(g, Bf) \geqq \delta > 0$. This assumption is not really essential but is given for simplicity of exposition. In problems dealing directly with Infinite Games this requirement will always be fulfilled. In the theory of linear programming, this mild restriction can be dispensed of without difficulty.

A game is said to be defined whenever the operators A and B and the conical sections P_v and Q_u are given. The set of elements of P_v shall be referred to as the strategies of player I and similarly Q_u comprises the strategy set of player II.

Definition 1. Let $\bar{\Omega}_s(A, B, P_v)$ be the set of λ for which there exists an f in P_v such that $Af \geqq \lambda Bf$ ($Af - \lambda Bf$ is in K). Put $\bar{\lambda}_s(A, B, P) = \sup \lambda$ for λ in $\bar{\Omega}_s$. Note that $\bar{\lambda}_s$ depends only on P but not on Q.

Definition 2. $\underline{\lambda}_s$ and $\underline{\Omega}_s$ are defined in a similar manner with the above inequality reversed and sup replaced by inf.

Definition 3. Let $\bar{\Omega}_\omega = [\lambda \mid$ for any $g_1, \cdots g_n \in Q_u$ there exists an f in P_v with $(Af, g_i) \geqq \lambda(Bf, g_i)$ for $i = 1, \cdots, n]$. Clearly, f may depend upon g_i. Put $\bar{\lambda}_\omega(P, Q) = \sup \lambda$ for λ in $\bar{\Omega}_\omega$. Similarly, $\underline{\Omega}_\omega$ and $\underline{\lambda}_\omega$ are defined.

Definition 4. The set Q_u is said to be sufficient relative to P_v if whenever $x = Af - \lambda Bf$ for some f in P_v and $(g, x) \geqq 0$ for every g in Q_u, then x is in K. Similarly, P_v is sufficient if for any $y = A^*g - \lambda B^*g$ with g in Q_u, then $(y, f) \geqq 0$ for every f in P_v implies that y is in L. This must be verified for any real λ.

Definition 5. $\bar{\Omega}_s(A^*, B^*)$, $\underline{\Omega}_s(A^*, B^*)$ $\cdots$ and $\bar{\lambda}_s^* = \lambda_s(A^*, B^*)$ $\cdots$ are defined analogously in terms of the conjugate mappings.

An example of two sets sufficient with respect to K = the set of all continuous functions non-negative on the unit interval are: The space of all discrete measures and the space of all absolutely continuous measures. That is, if $\int x(t)\, dy(t) \geqq 0$ for all discrete measures $y(t)$, then clearly $x(t) \geqq 0$. Also, if $\int x(t)y(t)\, dt \geqq 0$ for all absolute continuous measures given in the form of integrable functions $y(t)$, then $x(t) \geqq 0$.

An interpretation of the set $\bar{\Omega}_s$ and $\bar{\Omega}_\omega$ are in order. Let one interpolate a payoff $(Af, g)/(Bf, g)$ with f ranging over P_v, the set of strategies of player I, and g over Q_u which represents the choices available to player II. If λ is in $\bar{\Omega}_s(A, B)$, then a strategy $f_0 \in P_v$ exists guaranteeing player I the amount λ, irrespective of any g that player II uses. Indeed, $Af_0 - \lambda Bf_0 \geqq 0$ (in K), and thus as $g \in Q_u \subset K^*$, we infer $(Af_0, g)/(Bf_0, g) \geqq \lambda$ for all g in Q_u. If Q_u is sufficient, then $\bar{\lambda}_s$ is the maximum amount which can be achieved within any ε by player I. In an analogous manner if λ in $\bar{\Omega}_\omega$, then player I can secure for himself the amount λ against any alternative prescribed in advance. In other words there exists an effective response against each g separately which secures the amount λ.

§3. Existence of a value

Throughout this section we assume that P_v and Q_u are sufficient, unless otherwise stated.

Lemma 1.

(a) $$\bar{\lambda}_s = \sup_{f \in P_v} \inf_{g \in Q_u} \frac{(g, Af)}{(g, Bf)},$$

(b) $$\underline{\lambda}_s^* = \inf_{g \in Q_u} \sup_{f \in P_v} \frac{(g, Af)}{(g, Bf)}.$$

Note that $(g, Bf) \geqq \delta > 0$ by assumption and hence the ratio is always defined.

Proof. Let $\lambda = \bar{\lambda}_s - \varepsilon$, then for some $f_0 \in P_v$, $Af_0 - \lambda Bf_0 \geqq 0$. But as $g \in Q_u \subset K^*$, this implies that $(Af_0 - \lambda Bf_0, g) \geqq 0$ for every g in Q_u. Or, $(Af_0, g)/(Bf_0, g) \geqq \bar{\lambda}_s - \varepsilon$ for every g. Hence, $\inf_{g \in Q_u} (Af_0, g)/(Bf_0, g) \geqq \bar{\lambda}_s - \varepsilon$ so that $\sup_{f \in P_v} \inf_{g \in Q_u} (Af, g)/(Bf, g) \geqq \bar{\lambda}_s - \varepsilon$. Now as the left-hand side is independent of ε, we deduce that $\sup_{f\epsilon P_v} \inf_{g\epsilon Q_u} (Af, g)/(Bf, g) \geqq \bar{\lambda}_s$. Let now $\lambda = \bar{\lambda}_s + \varepsilon$ then λ is not in $\bar{\Omega}_s(A, B)$ and hence for each f in P_v the inequality $Af - \lambda Bf \geqq 0$ fails. Thus by the sufficiency of Q_u there exists a g_f in Q_u depending on f such that $(Af, g_f) - \lambda(Bf, g_f) \leqq 0$, or $(Af, g_f)/(Bf, g_f) \leqq \lambda$. This yields that $\inf_{g\epsilon Q_u} (Af, g)/(Bf, g) \leqq \lambda$ for every f in P_v, whence $\sup_{f\epsilon P_v} \inf_{g\epsilon Q_u} (Af, g)/(Bf, g) \leqq \lambda = \bar{\lambda}_s + \varepsilon$. As ε is arbitrary, the proof of formula (a) of this lemma is complete. The proof of part (b) is similar.

REMARK. Without any assumption of sufficiency the following is true:

$$\bar{\lambda}_s \leqq \sup_{f\epsilon P_v} \inf_{g\epsilon Q_u} \frac{(g, Af)}{(g, Bf)} \leqq \inf_{g\epsilon Q_u} \sup_{f\epsilon P_v} \frac{(g, Af)}{(g, Bf)} \leqq \underline{\lambda}_s^*.$$

PROOF. The outer two inequalities have been established in the proof of the lemma. The inner inequality is standard but we prove it for completeness. In fact,

$$\frac{(g, Af)}{(g, Bf)} \leqq \sup_{f\epsilon P_v} \frac{(g, Af)}{(g, Bf)}$$

which yield

$$\inf_{g\epsilon Q_u} \frac{(g, Af)}{(g, Bf)} \leqq \inf_{g\epsilon Q_u} \sup_{f\epsilon P_v} \frac{(g, Af)}{(g, Bf)}.$$

This is valid for any f in P_v and thus the inequality remains true if $\sup_{f\epsilon P_v}$ is taken on the left-hand side.

LEMMA 2. *If P_v is sufficient, then $\underline{\lambda}_s^* \leqq \bar{\lambda}_\omega$.*

PROOF. Let $\lambda = \bar{\lambda}_\omega + \varepsilon$ which is clearly not in $\bar{\Omega}_\omega$. Thus for some finite set $g_1, \cdots, g_u \epsilon Q_u$ no f in P_v exists such that $(A^*g_i - \lambda B^*g_i, f) \geqq 0$ for every $i = 1, \cdots, n$. Consider the image M in n dimensional space of $f \epsilon (P_v)$

$$[f \to \{\xi_i\} = \{(A^*g_i - \lambda Bg_i, f)\}].$$

M is convex, contains no interior points of the positive cone I, hence we have a non zero functional $\eta = (\eta_i)$ for which $\eta(M) \leqq 0$ and $\eta(I) \geqq 0$ (thus $\eta_i \geqq 0$), and

$$0 \geqq \sum \eta_i(A^*g_i - \lambda B^*g_i, f) = (A^*\sum \eta_i g_i - \lambda B^*\sum \eta_i g_i, f)$$

for all f in P_v. Since $\eta_i \geqq 0$, $\sum \eta_i g_i$ is in Q and has a positive multiple in Q_u, and hence $\lambda \epsilon \underline{\Omega}_s$. Therefore, $\underline{\lambda}_s^* \leqq \lambda = \bar{\lambda}_\omega + \varepsilon$; as ε is arbitrarily small, the lemma is proved.

LEMMA 3. *If Q_u is sufficient, then $\underline{\lambda}_\omega^* \leqq \bar{\lambda}_s$.* The proof is similar to that of Lemma 2. On combining Lemmas 1, 2, and 3 we get

THEOREM 1. *If P_v and Q_u are sufficient, then*

$$\bar{\lambda}_\omega \geqq \underline{\lambda}_s^* = \inf_{g\epsilon Q_u} \sup_{f\epsilon Pv} \frac{(Af, g)}{(Bf, g)} \geqq \sup_{f\epsilon P_v} \inf_{g\epsilon Q_u} \frac{(Af, g)}{(Bf, g)} = \bar{\lambda}_s \geqq \underline{\lambda}_\omega^*.$$

A more precise inequality will be given later.

DEFINITION 6. The game $G(A, B, P, Q)$ is said to be determined or to have a value if $\underline{\lambda}_s^* = \bar{\lambda}_s$. When Q_u and P_v are sufficient, this coincides with the usual meaning of value.

Some criteria will be now given to insure the existence of a value.

DEFINITION 7. The weak topology of a set H in a Banach space relative to a set G in the conjugate Banach space is defined as follows: A neighborhood $U(x_0 ; y_1, \cdots, y_n\varepsilon)$ is given as $[x \mid x \in H \text{ with } \mid (x, y_i) - (x_0, y_i) \mid < \varepsilon]$. This may not define a Hausdorff space unless there exist enough elements in G to distinguish points of H. A set H of elements is said to be weakly compact relative to G if under the weak topology on H induced by G, the space H constitutes a bicompact set.

LEMMA 4. *If Q_u is sufficient and the set H_λ of elements $Af - \lambda Bf$ for f in P_v is weakly compact relative to Q_u for any λ, then $\bar{\lambda}_s = \bar{\lambda}_\omega$.*

PROOF. It is evident from the definition that $\bar{\lambda}_s \leqq \bar{\lambda}_\omega$. Let $\lambda = \bar{\lambda}_\omega - \varepsilon$ and suppose that for no f in P_v is $Af - \lambda Bf$ in K. Since Q_u is sufficient we deduce for each f there is a g in Q_u with $(Af - \lambda Bf, g) < 0$. Let $G(g) = [Af - \lambda Bf \mid f$ in P_v and $(Af - \lambda Bf, g) < 0]$. The set $G(g)$ is an open set in the weak topology of H relative to Q_u. Furthermore, the set of all $G(g)$ cover H and hence there exists a finite number $G(g_1), G(g_2), \cdots, G(g_n)$ which cover H. This contradicts the definition of $\bar{\lambda}_\omega$ for this implies that for each f in P_v $(Af, g_i) \geqq \lambda(Bf, g_i)$ is violated by some g_i for $\bar{\lambda}_\omega - \varepsilon = \lambda$.

In a similar manner it is shown that

LEMMA 5. *If P_u is sufficient and the set for each λ of $A^*g - \lambda B^*g$ with g in Q_u is weakly compact relative to P_v, then $\underline{\lambda}_s^* = \underline{\lambda}_\omega^*$.*

Summing up, we have shown

THEOREM 2. *If Q_u and P_v are sufficient, then the value exists if either*

(a) *The set H_λ of elements $Af - \lambda Bf$ for f in P_v and any λ is weakly compact relative to Q_u or*

(b) *The set G_λ of elements $A^*g - \lambda B^*g$ for g in Q_u and any λ is weakly compact relative to P_v.*

REMARK. The stronger condition of weak bicompactness can be replaced by weak sequential compactness provided, however, that a weak separability condition is added. For a full discussion of this phenomenon see [3].

Another criterion very useful later is the notion of conditional compactness.

DEFINITION 8. A set H of elements is said to be conditionally compact relative to functionals of G if whenever the metric $\rho(x_1, x_2) = \sup_{f \in G} \mid (f, x_1) - (f, x_2) \mid$ is introduced for x_1, x_2 in H the space becomes conditionally compact. It is understood we have identified points for which $\rho(x_1, x_2) = 0$.

It is important to note that a covering of G by open sets U_α in this metric space does not imply the existence of a finite covering. This is due to the lack of completeness of the metric space.

LEMMA 6. *If P_v and Q_u are sufficient and M_λ = set of elements $Af - \lambda Bf$ for f in P_v is conditionally compact relative to Q_u for each λ, then $\bar{\lambda}_\omega = \bar{\lambda}_s$.* (The assumption $(Bf, g) \geqq \delta > 0$ is essential here.)

PROOF. The proof proceeds as in Lemma 4. Suppose to the contrary that $\lambda = \bar{\lambda}_\omega - \varepsilon > \bar{\lambda}_s$. Then for any f in P_v there exists a g with $(Af - (\bar{\lambda}_\omega - \varepsilon/2)Bf, g) < 0$. Let $G(g) = [h \mid h = Af - (\bar{\lambda} - \varepsilon/2)Bf$ and $(h, g) < 0]$. It is clear that the $G(g)$ are open sets in the metric space $M_{\lambda+\varepsilon/2}$ and cover $M_{\lambda+\varepsilon/2}$. Let $\bar{f}$ be in the completion of $M_{\lambda+\varepsilon/2}$ with respect to the metric. Thus there exists an f in P_v such that $| (\bar{f} - Af + (\bar{\lambda}_\omega - \varepsilon/2)Bf, g) | \leqq \eta$ uniformly for g in Q_u. Choose g_0 so that $(Af - (\bar{\lambda}_\omega - 3\varepsilon/4)f, g_0) < 0$ which is possible as $\bar{\lambda}_\omega - 3\varepsilon/4 > \bar{\lambda}_s$. As $(Bf, g) \geqq \delta$ for all f and g, we find that $\eta \leqq \varepsilon/4(Bf, g_0)$ for η chosen less than $\varepsilon\delta/4$. Thus, it follows that $(\bar{f}, g_0) \leqq \eta + (Af - (\bar{\lambda}_\omega - \varepsilon/2)Bf, g_0) \leqq (Af - (\bar{\lambda}_\omega - 3\varepsilon/4)Bf, g_0) < 0$. Hence, the $G(g)$ covers the completion as well which is now unconditionally compact and hence a finite subcovering must exist. Just as in the proof of Lemma 4 this is impossible.

Lemma 6 is a very useful result for applications. The corresponding result employing only weak sequential unconditional compactness does not necessarily imply the existence of a value.

We now obtain an expression for $\bar{\lambda}_\omega$ and $\underline{\lambda}^*_\omega$ resembling formulas (a) and (b) of Lemma 1. The following expressions will be established. If P_v and Q_u are sufficient, then

$$\underline{\lambda}^*_\omega = \sup_F \inf_g \sup_{1 \leqq i \leqq n} \frac{(Af_i, g)}{(Bf_i, g)}$$

and

$$\bar{\lambda}_\omega = \inf_G \sup_f \sup_{1 \leqq i \leqq n} \frac{(Af, g_i)}{(Bf, g_i)}$$

where F denotes the collection of all finite sets of P_v and G consists of the set of all finite sets of Q_u. In the above context $(f_1, \cdots f_n)$ represents a typical element of F and $(g_1, \cdots, g_m)$ denotes an element of G. The proof of the above facts is similar to Lemma 1. We present the proof only for $\underline{\lambda}^*_\omega$. To this end, let $\lambda = \underline{\lambda}^*_\omega + \varepsilon$. For any finite number $f_1, \cdots, f_n$ there exists a g depending on f_i with

$$\frac{(A^*g, f_i)}{(B^*g, f_i)} \leqq \lambda. \qquad \text{Consequently,} \qquad \sup_{f_1, \cdots, f_n} \frac{(A^*g, f_i)}{(B^*g, f_i)} \leqq \lambda.$$

Also, $\inf_g \sup_{f_1, \cdots, f_n} (A^*g, f_i)/(B^*g, f_i) \leqq \lambda$. Since this is valid for any finite collection $f_1, \cdots, f_n$ we obtain, $\sup_F \inf_g \sup_{f_1, \cdots, f_i} \leqq \underline{\lambda}^*_\omega + \varepsilon$. On the other hand if we let $\lambda = \underline{\lambda}^*_\omega - \varepsilon$, then there exists a set $(f_1, \cdots, f_n)$ for which $\sup_i (Af_i, g)/(Bf_i, g) \geqq \lambda$ for every g in Q_u. Since this holds for all g, we get $\inf_g \sup_i (Af_i, g)/(Bf_i, g) \geqq \lambda$. A fortiori, $\sup_F \inf_g \sup_i (Af_i, g)/(Bf_i, g) \geqq \underline{\lambda}^*_\omega - \varepsilon$. The proof is thus complete.

LEMMA 7. *If P_v and Q_u are sufficient, then* $\underline{\lambda}^*_\omega = \bar{\lambda}_s \leqq \underline{\lambda}^*_s = \bar{\lambda}_\omega$.

PROOF. The formula obtained above gives easily

$$\underset{=}{\lambda}^*_\omega = \sup_F \inf_g \sup_{F=(f_1,\cdots,f_n)} \frac{(Af_i, g)}{(Bf_i, g)} \geqq \sup_F \sup_{F=(f_1,\cdots,f_n)} \inf_g \frac{(Af_i, g)}{(Bf_i, g)}$$

$$= \sup_f \inf_g \frac{(Af, g)}{(Bf, g)} = \bar{\lambda}_s .$$

Similarly, $\underset{=}{\lambda}^*_s \geqq \bar{\lambda}_\omega$ invoking Theorem 1 furnishes thus the conclusion.

Some other conditions which are easily applied under which a value will exist are given in

THEOREM 3. *If P_v and Q_u are sufficient, then the value will exist if*

(a) *A and B are weakly compact operators (map bounded closed sets into weakly compact sets).*

(b) *A and B are completely continuous operators (map bounded closed sets into conditionally compact sets).*

PROOF. A direct application of Theorem 2 and Lemma 6.

A more detailed analysis of conditions of the type given in Theorem 3 are discussed in [3].

We now establish the proposition that every game can be imbedded in a game with a value by enlarging P and Q appropriately and extending A and B to the second conjugate operations.

THEOREM 4. *Every game $G(A, B, P, Q)$ can be extended to a game with a value.*

PROOF. Replace Q_u by K^*_u and P_v by L^*_v. Clearly K^*_u is contained in $R^* \supset \mathfrak{F}$ and L^*_v is contained in S^*. Also, replace A and B by A^{**} and B^{**} which map S^* into R^{**} and are the conjugate maps to A^* and B^*. The notion of positivity in R^{**} is defined by the cone K^{**} to which K^* is clearly sufficient as a result of the definition of the dual cone. Similarly, L is sufficient for L^*. Furthermore, as L^* is conjugate to L and thus L^*_u is a weak*compact set (compact in the weak topology viewed as functionals), the image $A^{**}f - \lambda B^{**}f$ for f in L^*_u is weakly compact relative to K^*. Thus Theorem 2 can be applied and the conclusion follows.

This result seems only of theoretical interest for it is extremely difficult to apply in any concrete situation.

It is easily established from the definitions and properties of

$$\bar{\lambda}_s = \sup_{f \in P_v} \inf_{g \in Q_u} \frac{(Af, g)}{(Bf, g)} \qquad \text{and} \qquad \underset{=}{\lambda}^*_s = \inf \sup \frac{(Af, g)}{(Bf, g)}$$

that the inf can be replaced by min and sup by max if $\underset{=}{\lambda}^*_s$ is in $\underset{=}{\Omega}_s$ or $\bar{\lambda}_s$ is in $\bar{\Omega}_s$ respectively. A criterion under which this will be so is given in the next lemma.

Strategies f in P_u for which $Af \geqq \bar{\lambda}_s Bf$ are called optimal strategies or solutions or minimax strategies. The same terminology applies to strategies g for which $A^*g \leqq \underset{=}{\lambda}^*_s B^*g$. It is immediate to verify that the set of optimal strategies for each player is convex and closed.

LEMMA 8. *If P_v is weakly compact relative to S, then $\bar{\lambda}_s$ is in $\bar{\Omega}_s$.*

PROOF. Consider the set A_n of all f in P_v with $A_n = \left[f \mid Af - \left(\bar{\lambda}_s - \frac{1}{n}\right)Bf \geqq 0 \right]$. It is easily verified that A_n is weakly closed non-void and $A_1 \supset A_2 \supset A_3 \cdots$. As P_v is weakly compact, we deduce that $\Gamma = \bigcap A_n \neq 0$. Let f_0 be in Γ, then it follows that $Af_0 - \bar{\lambda}_s Bf_0 \geqq 0$ which shows that $\bar{\lambda}_s$ is in $\bar{\Omega}_s$.

LEMMA 9. *If Q_u is weakly compact relative to R, then $\underline{\lambda}_s^*$ is in $\bar{\Omega}_s^*$.*

PROOF. Similar to Lemma 8.

§4. Completeness, admissibility and Bayes strategies

A strategy f in P_v is said to be admissible if there exists no other strategy f' such that $(Af, g) \leqq (Af', g)$ for all g in Q_u with inequality holding for at least one g. An element f_0 dominates f if for all g, we have $(Af_0, g) \geqq (Af, g)$. A set Γ of strategies is said to be complete if every strategy outside Γ is dominated by an element of Γ and every element of Γ is admissible.

THEOREM 5. *If the image Af of f in P_v is weakly compact relative to Q_u, then there exists a complete system.*

PROOF. We establish the existence of a complete system. To this end, let the set Q_u be well ordered $g_1, g_2, \cdots$. For any finite set of elements $g_{\alpha_1} \cdots g_{\alpha_n}$ the image of P_v into E^n under the map $f \rightarrow \{\xi_i\} = \{(Af, g_{\alpha_i})\}$ is a convex closed bounded set which possesses a complete undominated set M_α. A point x of a convex compact set in Euclidean space is undominated if there exists no other point of C with $y \geqq x$ (this inequality means that each component of x is not larger than the corresponding component of y). Let $\Gamma(g_{\alpha_1}, \cdots, g_{\alpha_n})$ be the weak closure of the inverse image of M_α in $A(P_v)$. Put $G(g_\alpha) = \bigcap_{\alpha_i < \alpha} \Gamma(g_{\alpha_1} \cdots g_{\alpha_k}, g_\alpha)$. Clearly $G(g_{\alpha_n}) \neq 0$, for otherwise the weak compactness gives

$$0 = \bigcap_{i=1} \Gamma(g_{\alpha_1(i)} \cdots g_{\alpha_{k_i}(i)} g_\alpha) = \Gamma(g_{\alpha_1 1}, g_{\alpha_1 2} \cdots g_{\alpha_{k_n}^n}, g_\alpha) \neq 0,$$

which is impossible. Also $G(g_\alpha)$ form a weakly closed decreasing collection of sets and hence $\Gamma = G(g_\alpha) \neq 0$ again by weak compactness. We assert that $\Gamma^* \subset \Gamma$ is complete. Let f_0 be not in Γ. Then for any finite number of $g_{\alpha_1} \cdots g_{\alpha_k}$ there exists a closed set $\Gamma'(g_{\alpha_1} \cdots g_{\alpha_k}) \subset \Gamma(g_{\alpha_1}, \cdots, g_{\alpha_k})$ such that for f' in $\Gamma'(g_{\alpha_1} \cdots, g_{\alpha_k})$, we have $(Af' g_{\alpha'_i}) \geqq (Af_0, g_{\alpha_i})$. Proceeding as above, we construct $G'(g_\alpha)$ and finally $\Gamma' \subset \Gamma$. Thus, we can exhibit an f' in Γ' with $(Af', g_\alpha) \geqq (Af_0, g_\alpha)$ for all g_α. As f_0 is not in Γ and a fortiori is not in Γ', it is thus not in some $G'(g_\alpha)$ and hence not in some $\Gamma'(g_{\alpha_k^0} \cdots g_{\alpha_k^0})$. Thus for some $g_{\alpha_i^0}$, we get $(Af' g_{\alpha_i^0}) > (Af_0, g_{\alpha_i})$. This shows that f^0 is dominated by an element of Γ. There may be duplication in Γ in the sense that there may exist f and f' in Γ such that $(Af, g) = (Af', g)$ for all g. This can be adjusted by considering equivalence classes and then choosing a representative from each class. The resulting set is called Γ^*. The definition of $\Gamma(g_{\alpha_1}, \cdots, g_{\alpha_n})$ implies that the elements of Γ^* are admissible and hence Γ^* is complete.

We remark that the theorem of Wald on minimal complete systems [4] can be slightly generalized by assuming weak separability in g and weak sequential compactness in f {see [3]}. Our result discards the assumption of separability and replaces the weak sequential compactness by weak compactness.

We next introduce the concept of Bayes solution. The set of f in P_v for which $\max_f (Af, g_0)$ is attained is said to constitute the Bayes solutions with respect to g_0. The set of f in P_v which reach within ε of the $\max_f (Af, g_0)$ are said to constitute the ε-Bayes solutions of g_0.

LEMMA 10. *The set of all admissible strategies is contained in the set of all ε-Bayes solutions, provided that the image AP_v is weakly compact.*

PROOF. Let f_0 be admissible and we assume the contrary that it is not ε-Bayes for any g. Thus for any g there exists an f_g such that $(Af_g, g) \geqq (Af_0, g) + \varepsilon$. Proceeding in the usual way (see Lemma 2), we show that for any finite number $g_1, \cdots, g_n$ there exists an f such that $(Af, g_i) \geqq (Af_0, g_i) + \varepsilon$ for $i = 1, \cdots, n$. Again, in a similar manner to the proof of Theorem 5 with the aid of weak compactness of AP_v, we get the existence of an f such that $(Af, g) \geqq (Af_0, g) + \varepsilon$ for all g. This contradicts the admissibility of f_0.

LEMMA 11. (Wald) *The set of all admissible strategies coincides with a complete system.*

PROOF. This is evident from the definition of admissible strategy and complete system.

LEMMA 12. *If the image AP_v is weakly compact and the image A^*Q_u is sequentially weakly compact, then any admissible f_0 is Bayes for some g.*

PROOF. In view of Lemma 10 for any ε there exists a g_ε such that $\varepsilon + (f_0, A^*g_\varepsilon) \geqq (f, A^*g_\varepsilon)$ for all f. Let $\varepsilon_n = 1/n$ and apply the weak sequential compactness of A^*Q_u. We obtain a g_0 such that $(f_0, A^*g_0) \geqq (f, A^*g_0)$ for all f in P_v.

We now employ a device introduced by Wald [4] to describe Bayes solutions in terms of games. The modified operator A' is defined as follows: $A'f = Af - (v, f)Af_0$. It is now assumed that f_0 is Bayes for g_0.

LEMMA 13. *The set of all minimax strategies for player* I *is contained in the set of all Bayes strategies relative to g_0. Moreover, f_0 is minimax for player* I.

PROOF. We consider $(A'f, g_0) = (Af, g_0) - (Af_0, g_0) \leqq 0$ $(Bf, g_0) = 0$ for f in P_v as f_0 is Bayes for g_0. Thus the value is less than or equal to zero. However, f_0 yields $A'f_0 \equiv 0$ and thus the value is zero. Hence for any minimax f' we get $(Af', g_0) = (Af_0, g_0) = \max_f (Af, g_0)$ and this establishes the proposition.

It is clear generally that not every Bayes strategy is admissible; however, the following simple criterion is often useful. If f_0 is Bayes relative to g_0 where g_0 is interior to Q_u (interior here means that for any $x \neq 0$ in K, we have $(x, g_0) > 0$), then f_0 is admissible. Indeed, if f_0 were not admissible, then there exists an f' such that $Af_0 \leqq Af'$ where it is assumed that Q_u is sufficient. But this gives $(Af_0, g_0) < (Af', g_0)$ since g_0 is interior which contradicts the Bayes property of f_0 relative to g_0.

The converse is not true in general. However, if the space Q_u of g is spanned by a finite number of g and if AP_v is weakly closed, then if f_0 is admissible it is a limit of f_n which are Bayes strategies relative to interior points g in Q_u. This result was first achieved by Arrow and Blackwell, but we present a very simple geometrical proof. In view of the assumption, the set of g can be looked upon as an n dimensional simplex Q_u. We construct the game A' of Lemma 13 for which f_0 is minimax and $A'f_0 = 0$ and the value is thus zero. The simplex Q_u can be placed in E^{n+1} so as to be represented as the points where $r_i \geqq 0$ $\sum r_i = 1$.

Let T denote the quadrant in Euclidean $n + 1$ space consisting of those points where $t \in R$ implies $t_i \geqq 0$ for $i = 1, \cdots, n + 1$. This is the dual cone to the simplex Q_u; that is the set of all points in E^{n+1} where $\sum_{i=1}^{n+1} t_i r_i \geqq 0$ for every r in Q_u. If we plot the points $V = \{(A'f, g_i)\}$ in E^{n+1} for each f in P_v, then the set of all such points is convex closed and does not overlap with T but touches T at points which are the inverse images of optimal f strategies for the game A'. (For a complete description of this method of analyzing finite games see [5].) It is asserted now that the set V touches T only at the origin. Let f be so that $(A'f, g_i) \geqq 0 = (A'f_0, g_i)$; but since f_0 was admissible, we deduce that $(A'f, g_i) \equiv 0$. In particular, if the space P_v is also spanned by a finite number of strategies, then V is polyhedral and touches T only at the origin. Consequently, V and T can be separated by a plane which intersects T only at the origin. This plane corresponds to an optimal g strategy for player II which is interior to the simplex Q_u. Whence, we conclude that f_0 is also a Bayes strategy for an interior g strategy. In the general case, where V is no longer assumed to be polyhedral, we replace Q_u by $Q_u^{(m)}$ where $Q_u^m = (g \mid g \in Q_u$ and $g = \sum_{i=1}^{n+1} \eta_i g_i$ and $\eta_i \geqq \frac{1}{m})$. Let T_m denote the dual cone. As $Q_u^{(m)} \to Q_u$ and $T_m \to T$ the game corresponding to (P_v, T_u^m) tends to the game described by (Q_u, T). In the game (P_v, T_u^m), we secure the solutions by moving T_m along the 45° line until T_m touches V. The points of contact come from the optimal f strategies and the separating planes of T_m and V provide the optimal g strategies [5]. Since the contact point of V and T is unique, it can be verified easily that the origin is the only limit point as $m \to \infty$ of any sequence of contact points between T_m and V. For m sufficiently large let $\bar{f}$ correspond to a contact point of T_m and V near f_0, i.e. $|(A'\bar{f}, g_i) - A'f_0, g_i)| \leqq \varepsilon$. This exists by virtue of the preceding remark. Since any g strategy for (P_v, T_u^m) is interior to Q_u, we get that $\bar{f}$ is a Bayes strategy relative to an interior g strategy of Q_u. We have thus shown

THEOREM 6. *If the set Q_u of strategies for player* II *is spanned by a finite number of g_i and the image of Af is weakly closed, then any admissible f_0 is uniformly approximable by an f which is Bayes relative to an interior point of Q_u.*

§5. Perturbation of value

This section is devoted to studying the effect of change of the operator and the change of the cone upon the value of the game.

LEMMA 14. *If A and B are held fixed and $P_v^{(1)} \subset P_v^{(2)}$ and $Q_u^{(1)} \subset Q_u^{(2)}$ while K and L are unchanged, then $\bar{\lambda}_s^{(1)} \leqq \bar{\lambda}_s^{(2)} \leqq \underline{\lambda}_s^{*(2)} \leqq \underline{\lambda}_s^{*(1)}$.*

PROOF. This is immediate for the condition $Af - \lambda Bf \geqq 0$ is independent of Q_u and depends only on the cone P_v. Use is made here of Corollary 1 to Lemma 1.

REMARK. It is to be noted that for purposes of generality one could also vary K and L appropriately. However, in most applications only the case treated in Lemma 14 will be used. Results dealing with the effects of changes of K and L on the value are analogous.

The existence of a value is preserved when the cones P and Q are enlarged, but may be destroyed when the cones are decreased. For practical purposes it is desirable to decrease the cones as far as maintaining the value allows.

DEFINITION 9. (i) A set of operators A_n converges uniformly to A with respect to P_v if $A_nf \Rightarrow Af$ uniformly for f in P_v, i.e. (A_nf, g) converges uniformly to (Af, g) for all f in P_v and g in Q_u.

(ii) A set of operators A_n converges strongly to A if for each f in P_v, then $A_nf \rightarrow Af$, i.e., (A_nf, g) converges to (Af, g) uniformly with respect to g in Q_u for each f in P_v.

(iii) A set of operators A_n converges weakly to A relative to P_v and Q_u if for each f in P_v and g in Q_u, then $(A_nf, g) \rightarrow (Af, g)$.

LEMMA 15. *If A_n converges strongly to A_0, and if $\bar{\lambda}_n$ is the value $\bar{\lambda}_s$ of $G(A_n, B, P, Q)$, then* $\lim \bar{\lambda}_n \geqq \bar{\lambda}_0$. *We assume here that Q is sufficient.*

PROOF. Let $\bar{\lambda}_0 - \varepsilon = \lambda$. Then there exists an f_0 with $A_0f_0 \geqq \lambda Bf_0$. As B is strictly positive and A_n converges strongly to A_0, we secure for $n \geqq n_0$ that $(A_0 - A_n)f_0 \leqq \varepsilon Bf_0$. Whence, on combination, we have $A_nf_0 \geqq \lambda Bf_0 - (A_0 - A_n)f_0 \geqq (\bar{\lambda}_0 - 2\varepsilon)Bf_0$. Thus we conclude that $\bar{\lambda}_n \geqq \bar{\lambda}_0 - 2\varepsilon$ for $n \geqq n_0$, or $\lim \bar{\lambda}_n \geqq \bar{\lambda}_0$.

LEMMA 16. *If A_n^* converges strongly as A^*, then* $\overline{\lim}\ \underline{\lambda}_n^* \leqq \underline{\lambda}_0^*$.

PROOF. Similar to the above proof.

REMARK. Even if both A_n and A_n^* converge strongly to A_0 and A_0^* and each A_n possesses a value, it does not follow that A_0 will have a value. This will be shown later by use of the classical counter example of Ville.

LEMMA 17. *If $A_n \rightarrow A$ uniformly with respect to P_v, then $\bar{\lambda}_n \rightarrow \bar{\lambda}_0$, provided that Q_u is sufficient.*

PROOF. Let $\lambda_n = \bar{\lambda}_n - \varepsilon$ and f_n be such that $A_nf_n \geqq (\bar{\lambda}_n - \varepsilon)Bf_n$. The uniform convergence and sufficiency guarantees an n_0 such that for $n \geqq n_0$, $(A - A_n)f_n \geqq -\varepsilon Bf_n$. This gives

$$Af_n = (A - A_n)f_n + A_nf_n \geqq -\varepsilon Bf_n + (\bar{\lambda}_n - \varepsilon)Bf_n,$$

whence $\bar{\lambda}_0 \geqq \bar{\lambda}_n - 2\varepsilon$ or $\bar{\lambda}_0 \geqq \overline{\lim}\ \bar{\lambda}_n$. Combining this with the conclusion of Lemma 18, we conclude that $\lim \bar{\lambda}_n = \bar{\lambda}_0$. We assume that P and Q are sufficient.

THEOREM 7. *If A_n and A_n^*, converge uniformly to A and A^* with respect to P_v and Q_u respectively, then if each $G(A_n, B, P, Q)$ has a value, then $G(A, B, P, Q)$ has a value.*

PROOF. This follows easily by applying Lemma 17 and a similar result obtained from Lemma 17 by replacing A_n, A and P_v by A_n^*, A^* and Q_u and the conclusion by $\lim \underline{\lambda}_n^* = \underline{\lambda}_0^*$.

It is to be understood in the future that $A_1 \geqq A_2$ shall mean that for f in P_v, then $A_1f \geqq A_2f$.

THEOREM 8. *Let*

(a) A_n *and* A_n^* *converge strongly to* A_0 *and* A_0^* ;

(b) $A_n \geqq A_{n+1}$ *for each* n;

(c) *Each* $G(A_n, B, P, Q)$ *has a value and* P_v *and* Q_u *are sufficient*;

(d) *Each* A_n *with* $n \neq 0$ *maps* ε *into* R *and* ε *is the conjugate space to the space containing the image of* $A_n^* g$, *then* A_0 *has a value.*

PROOF. It follows trivially that

$$\bar{\lambda}_1 \geqq \bar{\lambda}_2 \geqq \cdots \bar{\lambda}_n \cdots \geqq \bar{\lambda}_0$$

or $\lim \bar{\lambda}_n = \mu \geqq \bar{\lambda}_0$. Consider $\mu - \varepsilon$ which is in $\bar{\Omega}_n$ for each n. Let $\Gamma_n = [f \mid A_n f \geqq (\mu - \varepsilon)Bf, f \text{ in } P_v]$. As $A_n \geqq A_{n+1}$, we obtain that $\Gamma_n \supset \Gamma_{n+1}$ and each Γ_n is weak-closed in ε as Q_u is sufficient. Hence $\bigcap \Gamma_n \neq 0$ as ε is a conjugate space by assumption (d) and therefore each Γ_n is weak*compact. Thus $A_n f_0 \geqq (\mu - \varepsilon)Bf_0$ for each n with $f_0 \in \bigcap \Gamma_n$ and hence by (a), $Af_0 \geqq (\mu - \varepsilon) Bf_0$. Thus $\bar{\lambda}_0 \geqq \mu - \varepsilon$ which yields that $\bar{\lambda}_0 = \mu$. Also $A_0^* g_n = A_n^* g_n + (A_0^* - A_n^*) g_n \leqq A_n^* g_n$ since $A_0^* \leqq A_n^*$ which is a consequence of $A_0 \leqq A_n$ and the sufficiency of P_v. Let g_n be such that $A_n^* g_n \leqq (\underline{\lambda}_n^* + \varepsilon)B^* g_n$, then we get $A_0^* g_n \leqq (\underline{\lambda}_n^* + \varepsilon)B^* g_n$. Therefore $\underline{\lambda}_0^* \leqq \underline{\lambda}_n^* + \varepsilon$ or $\underline{\lambda}_0^* \leqq \underline{\lambda}_n^*$. Consequently $\underline{\lambda}_0^* \leqq \underline{\lambda}_n^* = \bar{\lambda}_n \to \bar{\lambda}_0$. As the opposite inequality is evident, we conclude $\bar{\lambda}_0 = \underline{\lambda}_0^*$.

COROLLARY 1. *The same hypothesis as Theorem* 8 *except that* (a) *is replaced by weak convergence instead of strong convergence, then* A *has a value.*

The proof is similar with $\bar{\lambda}_\omega$ and $\underline{\lambda}_\omega^*$ used instead of $\bar{\lambda}_s$ and $\underline{\lambda}_s^*$ and also Lemma 7 employed.

COROLLARY 2. *Under the same hypothesis as Theorem* 8, *then* $\bar{\lambda}_0$ *is in* $\bar{\Omega}(A_0, B, P, Q)$.

PROOF. As $\bar{\lambda}_0 \leqq \bar{\lambda}_n$ for n and the weak*compactness yields that $\bar{\lambda}_0$ is in $\bar{\Omega}(A_n, B, P, Q)$ for each n. We apply the analysis with $\mu - \varepsilon$ replaced by $\bar{\lambda}_0$ and the first part of the argument of Theorem 8 shows that $\bar{\lambda}_0$ is in $\bar{\Omega}(A_0, B, P, Q)$.

COROLLARY 3. *Let*

(a) A_n *and* A_n^* *converge strongly to* A_0 *and* A_0^* ;

(b) $A_n^* \leqq A_{n+1}^*$ *for each* n;

(c) *Each* $G(A_n, B, P, Q)$ *have a value and* Q_u *is sufficient*;

(d) *Each* A_n^* *maps* $\mathfrak{F}$ *into* S *and* $\mathfrak{F}$ *is the conjugate space to the space containing* $A_i f$,

then A_0 *has a value.*

Applications of Theorem 7 and Theorem 8 will be given in the next section. We close this section with some remarks concerning equivalence of games.

DEFINITION 9a. Let A, B, K and L be fixed, then a cone or game $(P_v^{(1)}, Q_u^{(1)})$ is said to be equivalent to $(P_v^{(2)}, Q_u^{(2)})$ if $\bar{\lambda}_s^{(1)} = \bar{\lambda}_s^{(2)}$ and $\underline{\lambda}_s^{*(1)} = \underline{\lambda}_s^{*(2)}$.

One could also define one-sided equivalence concepts.

LEMMA 18. *If the value of* $G(A, B, P^{(1)}, Q^{(1)})$ *exists with* $P_v^{(1)} \subset P_v^{(2)}$ *and* $Q_u^{(1)} \subset Q_u^{(2)}$, *then* $(P_v^{(1)}, Q_u^{(1)})$ *is equivalent to* $(P_v^{(2)}, Q_u^{(2)})$.

PROOF. This follows easily from Lemma 14.

LEMMA 19. *Let the image* $M_\lambda = A - \lambda B(P_v)$ *be weakly compact for each* λ

relative to $Q_u^{(1)}$. *Let* $Q_u^{(2)} \subset Q_u^{(1)}$ *and* P_v *be sufficient, then the games* $G(A, B, P, Q^{(1)})$ *and* $G(A, B, P, Q^{(2)})$ *are equivalent.*

PROOF. An easy application of Theorem 2 and Lemma 18.

DEFINITION. We say that Q_u is sufficient in the strong sense relative to P_v if Q_u is sufficient and whenever $\Gamma = (A - \lambda B)(P_v)$ does not intersect K there exists an element of Q_u which separates Γ and K.

LEMMA 20. *Let* P_v *and* Q_u *be sufficient and let* Q_u *be sufficient in the strong sense, then* $\bar{\lambda}_s = \underline{\lambda}_s^*$.

PROOF. Let $\lambda = \bar{\lambda}_s + \varepsilon$, then for no f in P_v is it true that $Af - \lambda Bf$ is in K. Thus $(A - \lambda B)P_v$ does not intersect K and hence there exists a g in Q_u such that $(Af - \lambda Bf, g) \leqq c$ for f in P_v, $(x, g) \geqq c$ for x in K. It easily follows that $c = 0$ and $(f, A^*g - \lambda B^*g) \leqq 0$ for all f in P_v. As P_v is sufficient, this yields that $A^*g - \lambda B^*g \leqq 0$. Hence, $\underline{\lambda}_s^* \leqq \lambda = \bar{\lambda}_s + \varepsilon$.

The proof is thus complete since $\bar{\lambda}_s \leqq \underline{\lambda}_s^*$.

THEOREM 9. *Let* $G(A, B, P, Q^{(1)})$ *have a value and* P_v *and* $Q_u^{(2)} \subset Q_u^{(1)}$ *are sufficient. If* $Q_u^{(2)}$ *is sufficient in the strong sense relative to* P_v, *then* $G(A, B, P, Q^{(2)})$ *is equivalent to* $G(A, B, P, Q^{(1)})$.

These ideas will be used in later context.

§6. Non-linear Games

In this section we indicate how the entire preceding theory can be carried over to games generated by non-linear operators. Let $\phi(f, g)$ be a bounded real valued function convex in g for each f and concave in f for each g. Let $\psi(f, g)$ be a function of the same type as ϕ except that ψ is convex in the first variable and concave in the second variable. We assume for definiteness that ϕ and ψ are positive valued and $\psi(f, g) \geqq \delta > 0$. The elements f and g traverse the sections P_v and Q_u, respectively.

DEFINITION 10. Let $\bar{\lambda}_s(\phi, \psi)$ be the supremum of all λ for which there exists an f in P_v with $\phi(f, g) \geqq \lambda\psi(f, g)$ uniformly for g in Q_u. Analogously, $\bar{\lambda}_\omega(\phi, \psi)$, $\underline{\lambda}_s^*(\phi, \psi)$ and $\bar{\lambda}_\omega^*(\phi, \psi)$ are defined.

It is to be remarked that no sufficiency considerations need be mentioned in view of the manner in which the λ has been defined. This applies even to the case where ϕ and ψ arise from linear operators. However, the important distinction now is that $\bar{\lambda}_s$ depends on both cones P and Q. Yet most of the important theorems are valid for $\bar{\lambda}_s$, etc. It is assumed that ϕ satisfies the following continuity requirement, namely: if $f_\alpha \to f_0$ weakly, then $\underline{\lim}\, \phi(f_\alpha, g) \leqq \phi(f_0, g)$ for each g and that if $g_\alpha \to g_0$ then $\overline{\lim}\, \phi(f, g_\alpha) \geqq \phi(f, g_0)$ and an analogous continuity condition is put on $\psi(f, g)$. Actually, in establishing the determinancy of the value, only one of these continuity conditions need be imposed for ϕ and ψ. It is now asserted that most of the theorems remain valid for this setup with little change in the method of proof. In particular, Theorem 1, Theorem 2, Lemma 6, Lemma 7, Theorem 3, Lemma 8, Lemma 9, Theorem 7, and Theorem 8 are all valid. These results will be used without further justification. An example of a lemma not valid for this definition of $\bar{\lambda}_s(P, Q)$ is Lemma 14, furthermore the equivalence theory does not carry over. The concepts of Bayes solution and admissibility and their results extend over under this generalized non-linear context.

We now establish a proposition about the variation of the cones.

THEOREM 10. *Let ϕ and ψ be held fixed. Let K and L be fixed. Let $P_1 \subset P_2 \subset P_3 \cdots \to P_0$ (strongly) and $Q_1 \subset Q_2 \subset \cdots \to Q_0$ (strongly). If $G(A, B, P_n, Q_n)$ has a value, then $G(A, B, P_0, Q_0)$ has a value.*

REMARK. The cone P_n tends strongly to P_0 means here that for any f in P_0 and any ε there exists an n dependent only on ε, and an f' in P_n such that $\phi(f' - f, g) \leqq \varepsilon$ and $\psi(f' - f, g) \leqq \varepsilon$ for all g in Q_0. A corresponding meaning is attached to $Q_n \to Q_0$.

PROOF. Let $\bar{\lambda}_n = \bar{\lambda}_s$ for $G(A, B, P_n, Q_n)$ and define $\underline{\lambda}_n^*$ similarly. Let $\bar{\lambda}_0 - \varepsilon = \lambda$, then there exists an f in P_0 such that $\phi(f, g) \geqq \lambda\psi(f, g)$ for all g in Q_0. The hypothesis furnishes an f_n in P_n such that $\phi(f_n, g) \geqq (\bar{\lambda}_0 - 3\varepsilon)\psi(f_n, g)$ for g in Q_0 and a fortiori for g in Q. Thus $\bar{\lambda}_n \geqq \lambda_0 - 3\varepsilon$ and hence $\underline{\lim}\ \bar{\lambda}_n \geqq \bar{\lambda}_0$. For every f in P_0 there exists a g_f in Q_0 such that $\phi(f, g_f) \leqq (\bar{\lambda}_0 + \varepsilon)\ \psi(f, g_f)$. Applying the hypothesis for Q_i and ε replaced by $\varepsilon/6$ shows that there exists a g_f in Q_n such that $\phi(f, g_f) \leqq (\bar{\lambda}_0 + 2\varepsilon/3)\psi(f, g_f)$. This implies $\bar{\lambda}_n \leqq \bar{\lambda}_0 + \frac{2}{3}\varepsilon$ and hence $\overline{\lim}\ \bar{\lambda}_n \leqq \bar{\lambda}_0$. This gives $\lim \bar{\lambda}_n = \bar{\lambda}_0$. In a similar manner, we conclude $\lim \underline{\lambda}_n^* = \underline{\lambda}_0^*$. as $\underline{\lambda}_n^* = \bar{\lambda}_n$, the assertion $\bar{\lambda}_0 = \underline{\lambda}_0^*$ follows.

COROLLARY. *Let the hypothesis be as before except that the game $G(A, B, P_n, Q_n)$ is not necessarily determined, then* $\lim \bar{\lambda}_n = \bar{\lambda}_0$ and $\lim \underline{\lambda}_u^* = \underline{\lambda}_0^*$.

7. Applications

The most typical application is that where $\varepsilon = \mathfrak{F} = (V)$ and $R = S = (C)$. The cones chosen are those indicated on § 1. Let $u = v = 1$ the function identically one. Finally, put $Af = \int K(x, y)\, df(x)$, with $K(x, y)$ continuous for $0 \leqq x, y \leqq 1$, and $Bf = (u, f)u$. The operator B in this case is one-dimensional, mapping P_v into the function u. The conditions of Theorem 2 are satisfied as (V) is the conjugate space to (C). Actually, A and B are completely continuous operators (see [3]). Therefore, the value exists with P_u taken as the set of all completely additive distributions on the unit interval. The mapping A^* can be taken in this case to be $A^*g = \int K(x, y)\, dg(y)$ which is the relevant contraction of the true conjugate operator.

It is important to emphasize that the value in the abstract sense for this example and for the other cases treated before agree completely with the usual meaning of value as appear in the literature. Theorem **11** investigates corresponding games where the cones of permissible strategies are of different types.

THEOREM 11. *The games $G(A, B, V, V)$, $G(A, B, D, D)$ and $G(A, B, L, L)$ are equivalent.* (See § 1 for definitions of V, D, and L.)

PROOF. This follows by an easy double application of Lemma 19, as D and L are clearly sufficient in this circumstance.

The meaning of Theorem 11 is that for a game generated by a continuous kernel on the unit square, one can find an effective optimal strategy whose yield is uniformly within ε of the value amongst the absolutely continuous distributions (L) or the discrete distributions (D).

Even more, if $K(x, y)$ is a continuous function for $0 \leqq x, y \leqq 1$, then if P is any sufficient set in (V), the value of $G(A, B, P, P)$ exists and is equal to the value of the game $G(A, B, V, V)$. This also follows from Lemma 19.

A set T of positive integrable functions is said to be equi-integrable with respect to a fixed measure μ if for any ε there exists a $\eta(\varepsilon)$ such that if $\mu(S) \leqq \eta$, then $\int_S f(t)\, d\mu(t) \leqq \varepsilon$ uniformly for all f in T.

A few examples of equi-integrable absolutely continuous sets of measures are

(a) All positive measurable functions bounded by a fixed constant C. This is immediate as $\int_S f\, dt \leqq Cm(S)$.

(b) All absolutely continuous measures = integrable functions $\left(\int f^p\right)^{1/p} \leqq C$ with $p > 1$. Indeed, $\int_S f \leqq [m(S)]^{1/p'}C$ by the Holder inequality.

(c) All measures for which $\int f \log^+ f \leqq C$. This follows from a theorem of Hardy.

In the next lemma (M) is chosen for R and S with the usual choice of K in (M) (see § 1).

Lemma 21. *If $K(x, y)$ is measurable and bounded on the unit square and P is an equi-integrable set of absolutely continuous distributions, then there exists a sequence of continuous kernels which as operators converge uniformly to $K(x, y)$ relative to P.*

Proof. Extend $K(x, y)$ to a larger region by putting $K(x, y) = 0$ outside of the unit square. Consider

$$K_n(x,y) = n^2 \int_y^{y+1/n} \int_x^{x+1/n} K(t, u)\, dt\, du.$$

It follows that $K_n(x, y)$ are continuous and converge almost everywhere to $K(x, y)$. We now verify that A_n defined by K_n converge uniformly to A determined by K relative to the given cone P. By the theorem of Egoroff, $K_n(x, y) \to K(x, y)$ uniformly except for a set S of a small measure η. Let $S_x = [y \mid (x, y) \text{ in } S]$ and let $T = [x \mid m(S_x) > 0]$. If ω_s denotes the characteristic function of S, we obtain by use of the Fubini theorem

$$\eta \geqq \iint \omega_s(x,y)\, dx\, dy = \int_T m(S_x)\, dx.$$

Let

$$T_\eta = [x \mid m(S_x) > \sqrt{\eta}],$$

then

$$m(T_n) \leqq \sqrt{\eta}.$$

Now

$$\left| \iint [K_n(x, y) - K(x, y)]f(x)g(y)\, dx\, dy \right|$$

$$\leqq \int_{S^c} \int | K_n - K | fg + 2c \int_{T_\eta} f \int_{S_x'} g + 2c \int_{T-T_\eta} f \int_{S_x''} g$$

where S^c denotes the complement of S and where S_x' corresponds to those S_x where $m(S_x) > \sqrt{\eta}$ and S_x'' to those S_x where $m(S_x) \leqq \sqrt{\eta}$. The hypothesis implies $\int_{T_\eta} f \leqq \varepsilon$ and $\int_{S_x''} g \leqq \varepsilon$ for η chosen sufficiently small. A simple estimate combining these facts yields

$$\left| \iint (K_n - K)\, fg\, dx\, dy \right| \leqq \varepsilon + 4C\varepsilon$$

with this estimate independent of f and g in P.

THEOREM 12. *If K is bounded and measurable, then the game $G(A, B, P, P)$ has a value where P is given as in Lemma* 21.

REMARK. The meaning of the value is taken as in definition 10 and therefore the sufficiency criteria is automatically satisfied. The value does not exist in the abstract sense, definition (6), since the cones are not sufficient.

PROOF. This is a simple consequence of Theorem 7, Lemma 21, and Theorem 3.

COROLLARY 1. *If K is bounded and measurable, then the game $G(A, B, L_c, L_c)$ is determined, which means that the value exists, where L_c consists of all absolutely continuous distributions with bound C.*

THEOREM 13. *If $K(x, y)$ is lower semi-continuous or upper semi-continuous, then $G(A, B, V, V)$ has a value.*

PROOF. We treat only the case of $K(x, y)$ lower semi-continuous. It is well known that there exists a sequence of continuous kernels $K_n(x, y) \geqq K_{n+1}(x, y) \to K(x, y)$ everywhere. The hypothesis of Theorem 8 can be verified and the conclusion is obtained by applying Theorem 8.

This result was independently obtained by I. Glicksberg.

One can show now by a counterexample that pure convergence is not sufficient to guarantee a value for the limit kernel. The well-known counterexample of Ville, namely

$$K(x, y) = \begin{cases} 1 & 1 = x \geqq y \geqq 0 \\ 1 & 1 > x > y \geqq 0 \\ 0 & x = y \\ -1 & 1 > y > x \geqq 0 \\ 1 & 1 = y > x \geqq 0 \end{cases}$$

is a kernel with the value not existing for $G(A, B, V, V)$, but $K(x, y)$ is a function of Baire class 1 and it can therefore be approximated strongly by kernels $K_n(x, y)$ continuous for each x and y.

§8. Games with constraints

As a special application of the method exploited in the preceding section, we now indicate how an analysis patterned after Lagrange multipliers can be used to study games with constraints. To best illustrate the fundamental ideas we begin with a special case. Let $\varepsilon = \mathfrak{F} = (V)$ and $R = S = (C)$ with the usual cones chosen as described on § 1. Let $Af = \int_0^1 K(x, y)\, df(x)$ where K is continuous and $Bf = (u, f)u \equiv 1$ (since $v = u = 1$) for f a distribution (i.e. f in P_v). Of course, $A^*g = \int K(x, y)\, dg(y)$. Suppose in addition that we have restricted P_u by outside constraints of the form $(f, k) = \alpha$ and $(h, g) = \beta$. Specifically, if k corresponds to the continuous function $k(x)$, then we consider only those distributions for which $(f, k) = \int k(x)\, df(x) = \alpha$ and a similar statement applies to the second constraint. Geometrically, this means that we have replaced our original conical section P_u of all distributions on the unit interval by a smaller linear convex section. It shall now be demonstrated that if (α, β) are interior points to the set Λ of all points in Euclidean 2 space E^2 obtained by the mapping $\begin{pmatrix} f \text{ in } P_u \\ g \text{ in } Q_u \end{pmatrix} \to [(f, k), (h, g)]$, then a Lagrangian procedure is valid. Precisely,

LEMMA 22. *Under the conditions stated above there exists constants a and b so that the game given by $G(\bar{A}, B, P_u, Q_u)$ with $\bar{A}f = Af + a(f, k)u + b(f, u)h$ possesses solutions f_0 and g_0 for which $(f_0, k) = \alpha$ and $(h, g_0) = \beta$ with $\bar{A}f_0 \geqq \lambda_0 Bf_0$ and $\bar{A}^*g_0 \leqq \lambda_0 B^*g_0$.*

REMARK. Lemma 22 yields directly that $(Af_0, g) \geqq \lambda(Bf_0, g)$ and $(A^*g_0, f) \leqq \lambda(B^*g_0, f)$ where $\lambda = \lambda_0 - a\alpha - b\beta$ for all f and g satisfying $(f, k) = \alpha$ and $(g, h) = \beta$.

PROOF. Let $\varepsilon = \mathfrak{F} =$ direct product of $(V) \otimes E^2$. Choose $R = S = (C) \otimes E^2$. Let $\bar{P}_u$ and $\bar{Q}_v$ be the direct product of the usual positive cones in these spaces. For the normalizing function in R, we select $(\frac{1}{3}u, \frac{1}{3}, \frac{1}{3})$ where $u = u(x) = 1$.

The operator A'_n applied to vector elements of the form $f = (\xi f(x), \xi', \xi'')$ where $\xi, \xi', \xi'' \geqq 0$ and $\xi + \xi' + \xi'' = 1$ and $f(x)$ in P_u produces the following vector element in R.

$$A'_n f = \begin{Bmatrix} \xi Af - \xi'[\beta_n u - h] - \xi''[h - \beta u] \\ \xi[\alpha_n - (f, k)] \\ \xi[(f, k) - \alpha] \end{Bmatrix}$$

where $\alpha_n = \alpha + 1/n$ and $\beta_n = \beta + 1/n$ where n is sufficiently large. Also

$$B'f = \begin{Bmatrix} \xi Bf \\ 0 \\ 0 \end{Bmatrix}$$

The conjugate operator A'^*_n is easily seen to give when applied to an element $\bar{g}$

$$A'^*_n\bar{g} = \begin{Bmatrix} \eta A^*g + \eta'[\alpha_n u - k] + \eta''[k - \alpha u] \\ -\eta[\beta_n - (h, g)] \\ -\eta[(h, g) - \beta] \end{Bmatrix}$$

while

$$B'^*_n\bar{g} = \begin{Bmatrix} \eta B^*g \\ 0 \\ 0 \end{Bmatrix}.$$

In this lemma the conical sections we are dealing with are bicompact, and in fact weak (*) sequentially compact. Although the operator B is not strictly positive for $\bar{P}_u$ and $\bar{Q}_v$, it follows immediately that Theorem 2 applies and consequently the value and optimal strategies $\bar{f}_n = (\xi(n)f_n, \xi'(n), \xi''(n))$ and $\bar{g}_n = (\eta(n)g_n, \eta'(n), \eta''(n))$ exist (see Lemmas 8 and 9). $\xi(n)$ and $\eta(n)$ are not zero, for if $\xi(n) = 0$, say, then from the first component

$$-\xi'(n)(\beta_n - h) - \xi''(n)(h - \beta) \geqq 0$$

so that

$$-\xi'(n)(\beta_n - (h, g)) - \xi''(n)((h, g) - \beta) \geqq 0$$

for all g in Q_u, contradicting the fact that (α, β) and (α_n, β_n) for n sufficiently large are interior points of the image Λ. Consequently, from the second and third components we have $\alpha_n \geqq (f_n, k) \geqq \alpha$ and $\beta_n \geqq (g_n, h) \geqq \beta$. If we expand $(A'_n\bar{f}, \bar{g}) - \lambda'_n(B'_n\bar{f}, \bar{g})$, we obtain

$$\xi(n)\eta(n)(Af, g) + a_n(f, k) + b_n(k, g) - \lambda'_n\xi(n)\eta(n)(Bf, g)$$

where $\bar{f}$ now is any element of the form $(\xi(n)f, \xi'(n), \xi''(n))$ with f varying over P_u and $\bar{g}$ represents the analogous type element. The existence of a value yields.

$$(*) \qquad (A'_n\bar{f}_n, \bar{g}) \geqq \lambda'_n(B'_n\bar{f}_n, \bar{g})$$

$$(**) \qquad (A'_n\bar{f}, \bar{g}_n) \leqq \lambda'_n(B'_n\bar{f}, \bar{g}_n)$$

As n goes to infinite $a_n/\xi(n)\eta(n)$ is bounded. Otherwise, when n is large $a_n(f, k)/\xi(n)\eta(n)$ becomes the dominant term of those involving f in the expansion of (*) and thus the optimal strategy in f for (*) is achieved for n large with f_n such that (f_n, k) concentrates near the boundary values of the image set $\Gamma = [(f, k)]$. This is incompatible with the inequality $\alpha \leqq (f_n, k) \leqq \alpha_n$ with α interior of Γ. Similarly, it follows that $b_n/\xi(n)\eta(n)$ is bounded. We select limit points $a, b, g_0, f_0, \lambda_0$ of $a_n/\xi(n)\eta(n)$, $b_n/\xi_0\eta_0$, g_n, f_n and λ'_n which exist on account of the weak (*) compactness of P_u and Q_u. It is clear that $\beta = (h, g_0)$, $\alpha = (f_0, k)$ with (*) and (**) reducing to $(\bar{A}f_0, g) \geqq \lambda_0(Bf_0, g)$ and $(\bar{A}f, g_0) \leqq \lambda_0(Bf, g_0)$. The proof of Lemma 22 is hereby complete.

The notation is the same as in Lemma 22.

THEOREM 14. *If f ranges over P_u subject to the constraints $(f, k_i) = \alpha_i$ for $i = 1, \cdots, n$ while g traverses Q_u and $(h_j, g) = \beta_j$ for $j = 1, \cdots, m$, then there exist* *constants* (a_i, b_j) $i = 1, \cdots, n$; $j = 1, \cdots, m$ *such that the modified game* $\bar{A}f = Af + \sum_{i=1}^{n} a_i(f, k_i)u + \sum_{j=1}^{m} b_j(f, u)h_j$ *and* $\bar{B}f = Bf$ *with* f *in* P_u *and* g *in* Q_u *and no other requirements possesses optimal strategies satisfying the constraints provided the restriction imposed on* α_i, β_j *given below hold.*

REMARK. The set $S = \{(f, k_i), (h_j, g)\}$ in E^{n+m} is convex as f traverses P_u and g ranges over Q_u and we assume for Theorem 14 that $\{(\alpha_i), (\beta_j)\}$ is interior to S. The conclusion of Theorem 13 is not valid when (α_i, β_j) is on the boundary of S. We leave it to the reader to furnish simple examples of this fact.

The proof of Theorem 13 is similar to Lemma 22 with the natural adaptations. In this case the usual sections in $P_u \otimes E^{2n}$ and $Q_u \otimes E^{2m}$ become the strategy sets with the appropriate choice of A' and B'. The details are straightforward and we do not present them.

It is worth remarking that an analogous treatment is valid for the case when the contraints are of the form of linear inequalities.

Finally it is important to realize that there may exist other solutions to the game $G(\bar{A}, B, P_u, Q_u)$ which do not satisfy the constraints. The Theorem only produces at least one solution of the type described there.

We now proceed to analyze this general constraint problem from a different point of view. For each set of constants $\bar{a} = (a_i)$ and $\bar{b} = (b_j)$ we can consider the game $G(\bar{A}_{\bar{a},\bar{b}}, B, P_u, Q_u)$. We denote the vector $(\bar{a}, \bar{b})$ by $\bar{c}$. Let the sets of ε effective strategies where ε is fixed be denoted by $\Lambda \otimes \Gamma = \Omega_{\bar{c}}$. It is trivial to verify that $\Omega_{\bar{c}}$ is convex. We consider the image set T_c of $\Omega_{\bar{c}}$ in E^{n+m} by the napping ϕ of a point $\{f, g\}$ in $\Omega_{\bar{c}}$ into $\{(f, k_i), (h_j, g)\}$. The image set $T_{\bar{c}}$ is thus contained in S. This mapping is a point to convex set mapping. We now investigate some of its properties.

LEMMA 23. *The set* W *of all* $\bar{c} = (\bar{a}, \bar{b})$ *for which* $T_{\bar{c}}$ *covers a fixed value* $\gamma = (\alpha, \beta) = (\alpha_i, \beta_j) i = 1, \cdots, n$ *and* $j = 1, \cdots, m$ *constitutes a convex set.*

PROOF. Let $\bar{c}_1 = (\bar{a}_1, \bar{b}_1)$ and $\bar{c}_2 = (\bar{a}_2, \bar{b}_2)$ be in W. Let $\bar{c}_0 = t\bar{c}_1 + (1 - t)\bar{c}_2$ with $0 < t < 1$. Let (f_1, g_1) and (f_2, g_2) be optimal ε effective strategies which correspond respectively to $\bar{c}_1, \bar{c}_2$ and cover γ. Put $f_0 = tf_1 + (1 - t)f_2$ and $g_0 = tg_1 + (1 - t)g_2$.

A simple calculation shows that

$$(\bar{A}_{\bar{a},\bar{b}}, f_0, g) = t(\bar{A}_{\bar{a},\bar{b}}f_1, g) + (1 - t)(\bar{A}_{\bar{a},\bar{b}}f_2g) = t(\bar{A}_{\bar{a}_1, \bar{b}_1}f_1, g) + (1 - t)(\bar{A}_{\bar{a}_2, \bar{b}_2}f_2, g) \geqq tv_1 + (1 - t)v_2 - \varepsilon$$

where v_i denotes the value of the game $G(\bar{A}_{\bar{a}_i, \bar{b}_i}, B, P_u, Q_u)$. In a similar manner, it follows that $(\bar{A}_{\bar{a},\bar{b}}f, g_0) \leqq tv_1 + (1 - \varepsilon)v_2 + \varepsilon$. By choosing optimal strategies instead of ε — effective strategies, one finds the value equal to $v_0 = tv_1 + (1 - t)v_2$. Since f_0, g_0 is therefore ε effective and maps evidently into the point (α, β) the conclusion of the lemma is hereby established.

We now consider the game where the operator A is taken to be $\underline{A}f = \sum_{i=1}^{n} a_i(f, k_i)u + \sum_{j=1}^{m} b_j(f, u)h_j$. In this case, optimal strategies for player I are those which maximize $\sum_{i=1}^{n} a_i(f, k_i)$ while player II searches for distributions g which minimize $\sum b_j(h_j, g)$. We assume in what follows that $k_i \geqq 0$ ane $h_j \geqq 0$. The cone X of all $\bar{a} = (a_i)$ for which $\sum_{i=1}^{n} a_i(f, k_i) \geqq 0$ for every f in P_u is called the dual to the section S_1 which consists of all points in E^n of the form (f, k_i) with f in P_u. It is easy to show that as $\bar{a}$ traverses the boundary of X, then the maximum of $\sum_{i=1}^{n} a_i(f, k_i)$ is attained for points of S_1 which also range over the complete boundary of S_1. A similar analysis applies to the form $\sum_{j=1}^{m} b_j(h_j, g)$ where we denote the dual cone to (g, h_j) with g in Q_u by Y.

Furthermore, the game $A_\lambda = A + \lambda \underline{A}$ with λ sufficiently large possesses optimal strategies for which the solutions of $\underline{A}$ are ε effective. Consequently, as $(\bar{a}, \bar{b})$ traverses the boundary of $X \otimes Y$ the mapping ϕ covers the boundary of S. It is now asserted that the mapping ϕ covers also the full interior of S. This is a consequence of the following lemma which possesses independent interest.

LEMMA 24. *Let θ be a point set mapping of a bounded convex polyhedral set R in E^n into a convex bounded set R' in E^n possessing the following properties:*

(a) *The image of any point in R is a closed convex set in R' which is continuous in the classical sense of such mappings. (If z_n is in $\theta(x_n)$ converges to z_0 and $x_n \to x_0$ then z_0 is in $\theta(x)$),*

(b) *The inverse image of any point in R' is a closed convex set,*

(c) *The boundary of R covers by the mapping θ the boundary of R', then θ covers the interior of R'.*

PROOF. Let us suppose the contrary. Then there exists a sphere about a point p_0 in R' which is not covered. We consider a sufficiently fine barycentric subdivision M of diameter $1/n$ of R and we construct the simplicial mapping θ_n which takes the vertices into any chosen points of their image sets and the remainder of the mapping is completed simplicially. It follows easily that for n sufficiently large θ_n does not cover p_0. Let ψ denote the mapping of R' into itself which maps every point p in R' into the first point $\bar{p}$ beyond p_0 on the line segment connecting p to p_0, which is covered by θ_n. On account of condition (c) for n chosen sufficiently large such points $\bar{p}$ will exist. Since θ_n is continuous, it follows that ψ is continuous. Finally, we consider the point-set mapping ϕ of R into itself of the form for r in R, $\phi(r) = \theta_n^{-1}\{\psi[\theta_n(r)]\}$. In view of condition (b) and the form of θ_n, we obtain that ϕ maps points of R into closed non-void convex sets in R. This mapping is continuous in the sense as described in (a). Indeed, let $r_m \to r_0$ and suppose s_m in $\phi(r_m)$ converges to s_0. Therefore, $\psi[\theta_n(r_m)] \to \psi(\theta_n(r_0))$ and $\theta_n(s_m)$ tends to $\theta_n(s_0)$. As $\psi(\theta_n(r_m)) = \theta_n(s_m)$, we infer that $\psi(\theta_n(r_0) = \theta_n(s_0)$ which is the desired conclusion. The classical Kakutani fixed point theorem [11] applied to the mapping ϕ provides a point r with r in $\phi(r)$. However, θ_n maps points r and points from the set $\phi(r)$ into opposite sides of a line segment through p_0. Consequently, we deduce a contradiction and the proof of the lemma is complete.

Lemma 24 enables us to give a new proof of Theorem 13. In fact, the discussion preceding Lemma 24 including Lemmas 23 and 24 show the existence of constants $\bar{c} = (a_i, b_j)$ so that optimal ε solutions of the game generated by $\bar{A}_{\bar{a},\bar{b}}$ satisfy the constraints $\gamma = (\alpha_i, \beta_j)$ for any preassigned γ. As in the proof of Theorem 13 since γ is interior, we can conclude that as a function of ε the constants $\bar{c}$ are bounded. A simple compactness argument allows us to remove the qualitying ε.

Further special cases of this constraint theory will be treated later in the paper.

§9. Reduction theory

This section illustrates the use of the abstract theory as a guide to solve certain types of games. The idea is to use the nature of the kernel in choosing the spaces and partial orderings.

EXAMPLE. Let $M(x, y)$ satisfy $M_{yy}(x, y) \geqq 0$ with $M_{yy}(x, y)$ continuous in both variables for $0 \leqq x, y \leqq 1$. In other words $M(x, y)$ is convex in y for each x. Let R consist of the space of all functions $h(y)$ twice differentiable and let $K_{y_0}(0 < y_0 < 1)$ consist of the cone of functions for which $h(y_0) \geqq 0$, $h'(y_0) = 0$ and $h''(y) \geqq 0$ all y. The operator B is the same as before and the operator A is taken as $(Af)(y) = h(y) = \int M(x, y)\, df(x)$. We investigate first the solution to the problem $Af - \lambda Bf \geqq 0$ for a fixed λ. This is equivalent to the following three inequalities:

$$\text{(a)} \qquad \int_0^1 M_{yy}(x, y)\, df(x) \geqq 0.$$

$$\text{(b)} \qquad \int_0^1 M_y(x, y_0)\, df(x) = 0.$$

$$\text{(c)} \qquad \int_0^1 M(x, y_0)\, df(x) \geqq \lambda.$$

If $f(x)$ is a distribution, then the hypothesis implies that the first inequality gives no restriction. There exists a convex combination of two pure distributions which fulfill (b) and (c) for a fixed λ. Indeed, consider the mapping of f into the two dimensional space

$$\left(\int_0^1 M_y(x, y_0)\, df(x), \qquad \int_0^1 M(x, y_0)\, df(x)\right).$$

The extreme points of this convex set lie among the images of the pure distributions I_x the set of which is connected. Thus any f satisfying (b) and (c) can be represented as a convex combination of two pure strategies. A similar analysis can be applied to the cones $K_0\{h(0) \geqq 0, h'(0) \geqq 0, h''(y) \geqq 0\}$ and $K_1\{h(1) \geqq 0, h'(1) \leqq 0, h''(y) \geqq 0\}$. Furthermore, if a λ exists satisfying (a) — (c), then by Taylor's expansion

$$\int M(x, y)\, df(x) = \int M(x, y_0)\, df(x) + (y - y_0) \int M_y(x, y_0)\, df(x)$$
$$+ \frac{(y - y_0)^2}{2!} \int_0^1 M_{yy}(x, \zeta)\, df(x) \geqq \lambda \text{ for } 0 < y_0 < 1.$$

A similar statement applies to $y_0 = 0$ or 1. Conversely if $f_0(x)$ is optimal for player I, then $\int M(x, y)\, df_0(x) \geqq v$ for all y. As $\int M(x, y)\, df(x)$ is convex a minimum occurs at y_0, say with $0 < y_0 < 1$, whence $\int M_y(x, y_0)\, df(x) = 0$. It follows that f_0 belongs to K_{y_0} and satisfies (a) – (c) with $\lambda = v$; thus f_0 is represented as indicated above. Again, the case $y_0 = 0$ or 1 is handled similarly. It follows immediately from this that player II has a pure optimal strategy.

The essential idea of the proof has been to reduce the game to a different problem where the nature of the solution is evident. The special properties of the kernel have been employed in choosing the partial orderings or equivalently the cones.

The above analysis carries over to the case where y ranges over an n dimensional convex set and $M(x, y)$ is convex in y for each x. The n dimensional Taylor expansion serves in this case, and yields the result that there exists an optimal strategy for player I which is representable as a convex combination of $n + 1$ pure strategies.

In the specific instance where $M(x, y)$ is convex in both variables, then it follows easily that player I concentrates at $x = 0$ and $x = 1$ while player II must restrict himself to any pure strategies y where $M(O, y) = M(1, y)$ [12].

We now combine this fact with the theory of games with constraints to resolve the game $M(x, y)$ with the above convexity properties subject to the constraint $\int_0^1 x\, df(x) = \alpha$ and $\int_0^1 y\, dg(y) = \beta$ where $0 < \alpha, \beta < 1$. As a consequence of Lemma 22, we know that there exist constants λ and μ such that optimal strategies of the game $M'(x, y) = M(x, y) + \lambda x + \mu y$ exist satisfying the constraints. Clearly $M'(x, y)$ is also convex in both variables and thus the optimal strategy for y is located at any point where $M(O, y) + \mu y = M(1, y) + \lambda + \mu y$. Consequently, we may take $\lambda = M(O, \alpha) - M(1, \alpha)$. A simple calculation shows $-\mu = (1 - \beta)M_y(0, \alpha) + \beta M_y(1, \alpha)$. Thus an explicit answer is given to the constraint problem in this case which exhibits how to compute the Lagrange multipliers. We remark in passing that such a scheme in general can be developed for any game with linear constraints.

Further results using these ideas for analytic kernels shall appear in a subsequent paper.

The next two sections invstigate the relation of certain classes of games and groups of transformations. Some results in this direction were introduced earlier in connection with statistical applications [6, 7]. In a later paper, we shall also extend these ideas to the study of statistical inference problems left invariant under a given group of operations. However, in this discussion we are concerned only with game theoretic problems where the strategy space for both players consists of distributions.

§10. Convolution games

This section is devoted to studying games whose kernel is defined on a topological group G. Precisely, the kernel is of the convolution type. Consider

$$(*) \qquad (Af, g) = \iint K(t^{-1}u)\, df(t)\, dg(u)$$

and the operator B is given as before and where t and u traverse the group G.

A few examples of such games are:

(a) Let G = real line, then (Af, g) takes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t-u)\, df(t)\, dg(u)$$

(b) Let G = unit circle and $K(t)$ a periodic function, then

$$(Af, g) = \int_0^{2\pi}\int_0^{2\pi} K(t-u)\, df(t)\, dg(u).$$

(c) Let G = set of integers, then

$$(Af, g) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} a_{n-k}\xi_k\eta_n$$

where

$$\xi_k \geqq 0 \sum \xi_k = 1 \quad \text{and} \quad \eta_n \geqq 0, \qquad \sum \eta_n = 1$$

with a_r uniformly bounded.

All the above examples cited are abelian; non abelian cases can be given, although they are considerably more complicated. A very important such class useful in applications consists of the group of all affine transformations of the real line into itself.

It is supposed that the reader is familiar with the elements of harmonic analysis on a locally compact group [8], [9].

Case 1. The group G is compact. We may assume that the Haar measure dt of G is of total measure 1. Also in this case we suppose that $K(t)$ is bounded real and measurable, so that $K(t^{-1}u)$ is a measurable function in the product space.

LEMMA 25. *An optimal strategy for each player is the Haar measure.*

PROOF. Performing a change of variable $t^{-1}u = u'$ and using the invariance of the Haar measure, we obtain

$$\int K(t^{-1}u)\, du = \int K(u')\, du' = v.$$

A symmetric calculation applies to the other player.

We now extend a result of L. J. Savage.

LEMMA 26. *If G is compact Abelian, then a necessary and sufficient condition that the solution be unique is that $\int K(t)\mathfrak{X}(t)\, dt \neq 0$ for any character $\mathfrak{X} \neq I$ (identity character).*

Necessity. Suppose the contrary, that the Fourier transform $\int K(t)\mathfrak{X}_0(t) = 0$ for some character $\mathfrak{X}_0 \neq I$. Let

$$f(t)\,dt = \left(1 + \frac{\mathfrak{X}_0(t) + \bar{\mathfrak{X}}_0(t)}{2}\right) dt.$$

We verify now that $f(t)\,dt$ is optimal. Clearly $f(t) \geqq 0$ as $|\mathfrak{X}(t)| = 1$. Also $\int f(t)\,dt = \int 1\,dt = 1$ as $\mathfrak{X}_0(t)$ is orthogonal to I. Furthermore, we have

$$\int K(t^{-1}u)f(u)\,du = \int K(u)f(ut)\,du$$

$$= \int K(u)\,du + \frac{\mathfrak{X}_0(t)\int K(u)\mathfrak{X}_0(u)\,du + \bar{\mathfrak{X}}_0(t)\int K(u)\bar{\mathfrak{X}}_0(u)}{2} = \int K(u)\,du = v.$$

We have used the fact that $\mathfrak{X}(ts) = \mathfrak{X}(t)\mathfrak{X}(s)$. This furnishes the needed contradiction of the uniqueness.

Sufficiency. Since the Haar measure has positive measure on any open set it follows that any optimal strategy must yield a return of v for every open set. In particular if μ_0 is another solution, then $\int K(t^{-1}u)(dt - d\mu_0(t)) \equiv 0$. This states that if $dw(t) = dt - d\mu_0(t)$ then we must show that $dw(t) \equiv 0$. It is no loss of generality in assuming that the value of the game is non zero. As $\int K(t)\mathfrak{X}(t)\,dt \neq 0$ for $\mathfrak{X} \neq I$ and the value is non zero implies $\int \mathfrak{X}(t)K(t)\,dt \neq 0$ for every character $\mathfrak{X}$. The Fourier transform of $\int K(t^{-1}u)\,dw(t) = 0$ yields $\int \mathfrak{X}(t)K(t)\,dt \int \mathfrak{X}(t)\,dw(t) = 0$. This gives $\int \mathfrak{X}(t)\,dw(t) = 0$ for every character. The classical uniqueness theorem of Harmonic Analysis on a locally compact Abelian group yields that $dw(t) \equiv 0$. This establishes the conclusion as dt and $d\mu_0$ are both normalized.

REMARK. In the case of the compact Abelian Group, a statement can be made as to the dimension of the optimal strategies. The dimension of the optimal strategies is essentially equal to the dimension of the linear space spanned by the characters orthogonal to $K(t)$. This is a consequence of Godement's theory [9]. Godement shows that if σ_K = linear space spanned by the characters orthogonal to $K(t)$ and $\mu(t)$ is any continuous function for which $\int K(t^{-1}u)\mu(t)\,dt = 0$, then $\mu(t)$ is in σ_K. This is not true for any locally compact Abelian group, however, for the particular cases of compact groups where the structure of the ideals of the group algebra is known it is valid. This fact yields the equality of the dimension of the optimal strategies and the character space indicated above. We now generalize the result of Lemma 26 to the general compact group which is not necessarily Abelian.

Let $M(t)$ denote a typical unitary matrix representation of G.

THEOREM 15. *The optimal solution to (*) is unique and equal to the Haar measure if and only if* $\int K(t)M(t)\,dt$ = *non singular for any unitary representation* $M(t) \neq I(t)$ *identity representation).*

PROOF. *Sufficiency. If* $\mu(t)$ is any other optimal strategy, then let $dv(t) = dt - d\mu(t)$ and we obtain $\int K(t^{-1}u)\,dv(t) \equiv 0$. Applying the non commutative Fourier representation theorem, we obtain

$$0 = \int \left[\int K(t^{-1}u)\,dv(t) \right] M(u)\,du = M(K)M(v) \quad [8],$$

where $M(K) = \int K(t)M(t)\,dt$ and $M(v) = \int M(t)\,dv(t)$. As $M(K)$ is non-singular, we get that $M(v) = 0$ for every M where we have assumed that the value is non-zero. By the use of the completeness theorem (Peter Weyl Theorem [8]), we have that $dv = 0$.

Necessity. Let $\int K(t)M_0(t) = M_0(K)$ be singular, with $M_0 \neq I$, then there exists a non-zero vector x_0 such that $M_0(K)x_0 = 0$. Therefore, for any vector y the inner product $(M_0(K)x_0, y)$ vanishes. As $M_0(t)$ is unitary, we have that $(M_0(t)x_0, y_0) \neq 0$ for some y_0. Let

$$f(t)\,dt = \left\{ 1 + \frac{\varepsilon}{2}\left[(M_0(t)x_0, y_0) + \overline{(M_0(t)x_0, y_0)}\right] \right\} dt.$$

Clearly $f(t)\,dt \geqq 0$ for ε chosen sufficiently small and one verifies as in the proof of Lemma 26 that $f(t)\,dt$ is an optimal strategy. Use is made here of the fact that

$$M_0(tu) = M_0(t)M_0(u) \text{ and } (M_0(K)x_0, M_0^*(u)y) = (M_0(K)x_0, y_1) = 0$$

for every u in G. (M_0^* is the transpose conjugate of M_0.) The contradiction of the hypothesis of uniqueness is now evident and the proof of the necessity is thereby complete.

Case 2. The group G is locally compact but not compact. As an indication of the scope of the ideas, we treat only the special case of the real line. It is supposed that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K(t)\,dt$$

converges. Furthermore, we assume $K(t)$ is bounded and continuous. It follows easily that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+a}^{T+a} K(t)\,dt$$

converges to a value independent of a uniformly, for any finite interval of values a. Let μ_T denote a measure generated ty the density equal to $1/2T$ on the interval $(-T, T)$ and zero outside this interval. We obtain for such μ that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} K(t - u)\, d\mu_T(t) = \lim \frac{1}{2T} \int_{-T+u}^{T+u} K(t) = v.$$

By an application of bounded convergence, we get that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} dv(u) \int_{-\infty}^{\infty} K(t - u)\, d\mu_T(u) = v$$

for any distribution $dv(u)$. This does not necessarily imply the game has a value but that $\underline{\lambda}_s^* \geqq v \geqq \bar{\lambda}_s$. If the convergence

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} K(t - u)\, d\mu_T(t)$$

is uniform for any set of u, then it can be shown that v is the value of the extended game guaranteed by Theorem 4.

Finally, we close this section with an additional observation which converts the convolution game into an equivalent game for the case of an Abelian group. If we consider $(Af, g) = \int K(t^{-1}u)\, df(u)\, dg(t)$ and apply the Parseval relation, we obtain

$$(Af, g) = \int \hat{K}(\hat{u})\hat{f}(\hat{u})\hat{g}(\hat{u})\, d\hat{u} \qquad (*)$$

where $\hat{K}$, $\hat{f}$ and $\hat{g}$ denote the Fourier transform of K, f, g which are defined on the dual group. The validity of (*) can be insured under appropriately strong hypothesis placed on $K(t)$. If the group possesses a suitable notion of symmetry, then $\hat{K}$, $\hat{f}$ and $\hat{g}$ can be restricted to real transforms. The solution to the game can thus be achieved by solving in that case

$$\inf_{\hat{g}} \sup_{\hat{f}} \int \hat{K}(\hat{u})\hat{f}(\hat{u})\hat{g}(\hat{u})\, d\hat{u} = \sup_{\hat{f}} \inf_{\hat{g}} \int \hat{K}(\hat{u})\hat{f}(\hat{u})\hat{g}(\hat{u})\, d\hat{u}$$

where g and f are transforms of distributions (positive definite functions with $g(e) = 1$ and $f(e) = 1$). In certain instances this problem is simpler to deal with than the original formulation. Specific applications of the results of this section will be given in a later section.

The Relations of Game Theory and Groups of Transformations

We deal here first with concrete kernels and develop afterwards an abstract theory. Let $K(x, y)$ denote a kernel defined on the space $X \otimes Y$ and let T denote a group of homeomorphisms of X onto itself. It is assumed that measurable sets are transformed into measurable sets.

A kernel K is said to be invariant with respect to T if there exists an induced set of transformations $\overline{T}$ which transform Y onto itself such that for any t in T there exists a $\bar{t}$ in $\overline{T}$ such that $K(tx, y) = K(x, \bar{t}y)$. A game is said to be invariant with respect to T if the kernel is invariant. It is furthermore assumed that the elements of T generate linear bounded transformations of norm 1 on the space of measures into itself which map distributions into distributions. We now establish the following lemma fundamental in all that follows. It is trivial to show that $\bar{t}^{-1} = \overline{t^{-1}}$. The game generated by $K(x, y)$ is assumed to be invariant with respect to T.

LEMMA 27. *If $f_0(x)$ is an optimal strategy for player* I, *then $f_0(tx)$ for each t in T is also an optimal strategy for player* I.

PROOF. Since $f_0(x)$ is optimal, we get

$$\iint K(x, y)\, df_0(x)\, dg(y) \geqq v$$

for all distributions $g(y)$. Using the invariance, we obtain

$$\int_Y \int_X K(x, y)\, df_0(tx)\, dg(y) = \int_Y \int_{t^{-1}X} K(t^{-1}x, y)\, df_0(x)\, dg(y)$$

$$= \int_Y \int_{t^{-1}X} K(x, \bar{t}^{-1}y)\, df_0(x)\, dg(y) = \iint_{t^{-1}X\,\bar{t}Y} K(x, y)\, df_0(x)\, dg(\bar{t}y)$$

$$= \int_X \int_Y K(x, y)\, df_0(x)\, dg(\bar{t}y) \geqq v$$

which shows that $f_0(tx)$ is optimal.

Lemma 27 establishes the fact that the elements of T transform the convex set of optimal strategies into itself. In what follows tf will denote the distribution $f(tx)$.

THEOREM 16. *If the space of distributions is weakly compact with respect to the cone K of R and the group T is Abelian, then there exists an optimal f_0 such that $tf_0 = f_0$ for every t in T. That is, there exists an invariant optimal strategy with respect to the transformations of T.*

REMARK. The equality of $tf_0 = f_0$ is to be understood as the equality of two elements of the Banach space ε.

PROOF. Due to the hypothesis of weak compactness, it follows by Lemma 5 and Lemma 8 that the game is determined and that the set Γ of optimal strategies for player I is non empty. Let f_0 be an optimal strategy. Consider for a fixed t_0 in T the distribution

$$f_n = \frac{t_0 f_0 + \cdots t_0^n f_0}{n}$$

which is also an optimal strategy as a consequence of the convexity of the set of optimal strategies and Lemma 27. Let $\bar{f}$ be a weak limit point whose existence is guaranteed by the weak compactness; it follows easily that for any given x in R

$$|(t_0 f_n - f_n, x)| = \left| \frac{(t_0^{n+1} f_0 - t_0 f_0, x)}{n} \right| \leqq \frac{2}{n} \to 0.$$

Thus we conclude that $t_0\bar{f} - \bar{f} = 0$. Restricting ourselves to the set of fixed points Γ_{t_0} of Γ under t_0 which is non empty and convex, it is shown as above that for t_1 in T there exists in Γ_{t_0} an f with $t_1 f = f$. The Abelian character of T is used here in establishing the fact that t_1 transforms Γ_{t_0} into itself. Similarly for any finite number of t , $\cdots$, t_n , there exists a common invariant f in Γ. Let $G(t_\alpha) = [f \mid t_\alpha f = f \text{ with } f \text{ in } \Gamma]$. Now $S = \bigcap_t G(t_\alpha) \neq 0$ since otherwise as Γ is also a weakly closed subset of ε and hence weakly compact, there exists a finite number of sets $G(t_i)$ with $\bigcap_{i=1}^n G(t_i) = 0$. This was shown to be false. Thus any element of S furnishes the required conclusion of the theorem.

We now remove the hypothesis that T is Abelian.

THEOREM 17. *Under the same hypothesis as Theorem* 16, *except that the group T constitutes a solvable group, then there exists an optimal f_0 such that $tf_0 = f_0$ for every t in T.*

PROOF. Since T is a solvable group, there exists a finite chain $T \supset T_1 \supset \cdots \supset T_n$ with T_{i+1} a maximal normal subgroup of T_i , T_i/T_{i+1} Abelian, and finally T_n is a simple Abelian group. Since T_n is Abelian there is a non void set Γ of optimal strategies f for which $tf = f$ for all t in T_n (see proof of Theorem 16).

Let S be the set of all t in T_{n-1} for which $tf_0 = f_0$ for all f_0 in Γ. One verifies easily that this constitutes a normal subgroup of T and hence by the simplicity and the maximal character of T_n relative to T_{n-1} either coincides with T_n or T_{n-1} . Suppose $S = T_n$. We observe that

$$tsf_0 = t[f_0(sx)] = f_0(sx) = sf_0$$

and hence any typical element of the co-set ts for t in T_n and s in T_{n-1} yields the same effect acting upon the elements of Γ. Thus we can consider T_{n-1}/T_n as transforming Γ into itself. As T_{n-1}/T_n is Abelian, we apply Theorem 16 and deduce the existence of an f in Γ for which $t^*f = f$ with $t^*\varepsilon\ (T_{n-1}/T_n)$. As T_n leaves Γ pointwise unchanged, we deduce for f that $tf = f$ for t in T_{n-1} . Proceeding in this way produces an f which T_{n-2} leaves invariant, etc., and finally an f which is invariant under the transformations of the total group t in T.

It is important to note that no topological structure was imposed on the group T to ascertain the validity of Theorems 16 and 17. However, additional results can be obtained if a suitable topological artifice is placed on T. In the next theorem questions of measurability and the validity of the use of Fubini theorem are not discussed. In any specific applications these matters are easily handled. The following proofs suppose that any procedures involving such considerations are valid.

THEOREM 18. *If T is compact and the set of optimal strategies for player* I *is non empty, then there exists an invariant optimal strategy.*

PROOF. Let f_0 be an optimal strategy, then consider

$$\bar{f}(E) = \int_T f_0(tE)\, dt$$

with E any measurable set in X and where dt denotes the Haar measure of T of total measure 1. One verifies immediately that $f(x)$ is a distribution. Indeed,

$$\int d\bar{f}(x) = \iint df_0(tx)\, dt = \int dt \int df_0(tx) = 1$$

where

$$\int df_0(tx) = 1,$$

as t transforms distributions into distributions. Also

$$\int K(x, y)\, d\bar{f}(x) = \iint K(x, y)\, df_0(tx)\, dx\, dt = \iint K(tx, \bar{t}y)\, df_0(tx)\, dt.$$

Replacing x by $t^{-1}x$ yields that

$$\iint K(x, \bar{t}y)\, df_0(x)\, dt = \int \left(\int K(x, \bar{t}y)\, df_0(x) \right) dt \geqq v.$$

This establishes that $\bar{f}$ is optimal. Finally,

$$t_0 f = \bar{f}(t_0 E) = \int f_0(t_0 t E)\, dt = \int f_0(t'E)\, dt' = \bar{f}(E)$$

where the substitution $t_0 t = t'$ and the invariance of the Haar measure have been used.

We now formulate an abstract version of this invariance theory. Let T denote a group of operators transforming ε into itself and such that each t in T maps the conical section P_v into P_v. Let $\{T'\}$ denote a group of operators which map R into itself. The game $G(A, B, P, Q)$ is said to be invariant with respect to $\{T\}$ and $\{T'\}$ if for any $t \in T$ there exists a t' in T' such that $t'A = At$ and $t'B = Bt$.

We assume the cones P_v and Q_u are sufficient.

LEMMA 28. *If f_0 is such that $Af_0 \geqq \lambda Bf_0$, then tf_0 has the same property.*

PROOF. Consider for any g in Q_u

$$\begin{aligned}(Atf_0, g) = (t'Af_0, g) &= (Af_0, t'^*g) \geqq \lambda(Bf_0, t'^*g) \\ &= (t'Bf_0, g) = \lambda(Btf_0, g).\end{aligned}$$

As Q_u is sufficient, this implies that $Atf_0 \geqq \lambda Btf_0$.

The role of Lemma 28 is similar to that of Lemma 27. We now remark without elaborating on the details that the analogous statement to Theorem 17 is valid for this abstract setup.

Applications

It is worthwhile to give several specific applications of the above theory.

Example I. Let X and Y be a finite set of n points. The game is equivalent to a matrix game. A convolution game corresponds to a circular matrix where $a_{ij} = a_K$ with $K = i - j$ modulo n. The remark following Lemma 23 applies here and the full description of the optimal strategies are:

Let r_{1_0} be the real part of the complex vector

$$1, \omega_n^{1_0}, \omega_n^{21_0}, \cdots, \omega_n^{(n-1)1_0}$$

where ω_n is a primitive n^{th} root of unity with $\sum a_K \omega_n^{k1} = 0$. This last equality expresses the vanishing of the Fourier transform. The set of all optimal strategies are given by the form $(1/n, \cdots, 1/n) + r_{1_0}$ and cyclic permutations of r_{1_0}.

Example II. Let X and Y be the unit circle and let $K(t)$ be any function defined on the circle or in terms of the unit interval a periodic functions of period one. The kernel given by $K(t - u)$ defines a convolution game acting on the compact circle group. The condition of uniqueness expressed in Lemma 26 becomes the requirement of the non-vanishing of all of the Fourier coefficient of $K(t)$ except for the zero$^{\text{th}}$ coefficient. This example was also considered by L. J. Savage. The Parseval relationship expresses the game in terms of the Fourier coefficients. Explicitly,

$$\iint K(t - u)\, df(t)\, dg(u) = \sum b_n a_n c_n,$$

where b_n, a_n, and c_n are the Fourier coefficients of f, g and K respectively. The remark following Lemma 26 also applies here and a full description of optimal strategies can be given.

Example III. Let $K(t - u)$ be the kernel defined with $-\infty < t < \infty$ and $-\infty < u < \infty$, then this defines a convolution game on the group consisting of the real line. The condition of uniqueness is the non-vanishing everywhere of the Fourier transform of $K(t)$.

Example IV. Let X and Y be the unit intervals. Consider the group of transformation T acting on the unit interval which consists of the identity and the transformation which takes x into $1 - x$. Let the induced group $\overline{T}$ be the same group transforming the y interval into itself. Let $K(x, y) = K(1 - x, 1 - y)$ with K continuous, then as a consequence of Theorem 15 one can find an optimal strategy for both players which is invariant with respect to the group of transformations T.

Example V. We generalize the results of the previous section in the following manner. There exists a group of transformation T operating on the product space $X \otimes Y$ which transform $X \otimes Y$ into itself. The kernel $K(x, y)$ is invariant with respect to T if $K(t(x, y)) = K(x, y)$. The induced transformation on the measure space becomes

$$t[g(x)f(y)] = t\mu(x, y) = \mu(t(x, y)).$$

We consider the special example where T consists of the identity and a reflection about the line $y = x$. The measure $g(x)f(y)$ goes over into the measure $f(x)g(y)$.

If $K(x, y) = -K(y, x)$, then it can be shown as a consequence of Theorem 15 that if $g(x)f(y)$ is optimal for both players, then so is $t(g(x)f(y))$ optimal for both. The negative sign is needed to establish the conclusion of Lemma 27 for this example. This is the well known fact that for a symmetric game the set of optimal strategies for both players coincide.

In general, the applications of the above invariance theory is much more abundant to problems of statistical inference.

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