Math 151—Probability.

4:15 PM Section

Thursday, October 30, 2008

Name: Solutions.

Directions: No books or notes are allowed. You may use a calculator but can get full credit without one.

Question	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

1. A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are next to each other?

$$\frac{49 \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot 48!}{52!}$$

49 possible positions for 1st ace.

4! ways to order the 4 aces.

48! ways to order the remaining cards.

2. A drawer of socks contains seven black socks, eight blue socks and nine green socks. Two socks are chosen in the dark.
   (a) What is the probability that they match?
   (b) What is the probability of choosing a black pair?

a) $$\frac{\binom{7}{2} + \binom{8}{2} + \binom{9}{2}}{\binom{24}{2}}$$

b) $$\frac{\binom{7}{2}}{\binom{24}{2}}.$$

3. A box has three coins. One has two heads, one has two tails, and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped, and comes up heads.
   (a) What is the probability that the coin chosen is the two headed coin?
   (b) What is the probability that if the coin is thrown another time, it will come up heads?

a) Prob (2 headed | heads)

$$= \frac{P(\text{heads} \mid \text{2 headed})\, P(\text{2 headed})}{P(\text{heads} \mid \text{2 headed})\, P(\text{2 headed}) + P(\text{heads} \mid \text{fair})\, P(\text{fair}) + P(\text{heads} \mid \text{2 tailed}) \cdot P(\text{2 tail}}$$

$$= \frac{1/3}{1/3 + (1/2)(1/3) + 0 \cdot 1/3} = 2/3.$$

b) P(2nd toss heads | 1st toss heads)

$$= P(\text{2nd toss heads} \mid \text{2 headed})\, P(\text{2 headed} \mid \text{1st toss heads})$$
$$+ P(\text{2nd toss heads} \mid \text{fair})\, P(\text{fair} \mid \text{1st toss is heads}).$$

$$= (2/3)(1) + (1/3)(1/2) = 5/6$$

4. In a sequence of independent Bernoulli trials with probability p of success, what is the probability that there are exactly r successes before the the kth failure?

$$\binom{r+k-1}{r} p^r (1-p)^{k-1} (1-p)$$

last toss is a "failure".

5. Phone calls are received at a certain residence as a Poisson process with parameter $\lambda = 2$ per hour.
   (a) If the resident takes a 10 minute shower, what is the probability that the phone rings during that time?
   (b) How long can the shower be if the resident wishes the probability of receiving no phone calls to be at least .5?

a) $1 - e^{-\lambda/6} = 1 - e^{-1/3}$

b) $e^{-\lambda t} \geq .5$

$-\lambda t \geq \ln(1/2)$

$t \leq \frac{1}{\lambda} \ln(2) = 1/2 \ln(2)$ hours.

6. A list of n items is arranged in random order; to find a requested item, they are searched sequentially until the desired item is found. What is the expected number of items that must be searched through, assuming that each item is equally likely to be the one requested?

X = # of items searched

$P(X=k) = 1/n \qquad k = 1, \ldots, n$

$$E(X) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n}\left(\sum_{k=1}^{n} k\right) = \frac{1}{n}\,\frac{(n+1)(n)}{2}$$

$$= \frac{n+1}{2}.$$

7. Let X have the density

$$f(x) = \frac{1+\alpha x}{2}, \qquad -1 \le x \le 1, \qquad -1 \le \alpha \le 1$$

(α is fixed here)

(a) Verify that f is a density function and find the cdf of X
(b) Find $E(X)$.
(c) Find the variance of X.

a) $\frac{1+\alpha x}{2} \ge 0$ iff $1+\alpha x \ge 0$ iff $\alpha x \ge -1$.

but $\alpha x \ge -1$ since $-1 \le x \le 1$, $-1 \le \alpha \le 1$.

$$\int_{-1}^{1} \frac{1+\alpha x}{2}\,dx = \int_{-1}^{1} \frac{1}{2}\,dx + \frac{\alpha}{2}\int_{-1}^{1} x\,dx = \int_{-1}^{1} \frac{1}{2}\,dx = 1.$$

So f is a density.

$$F_X(x) = \int_{-1}^{x} \frac{1+\alpha u}{2}\,du \qquad -1 \le x \le 1$$

$$= 0 \qquad \text{else.}$$

for $-1 \le x \le 1$

$$F_X(x) = \frac{1}{2}\int_{-1}^{x} du + \frac{\alpha}{2}\int_{-1}^{x} u\,du = \frac{1}{2}(x+1) + \frac{\alpha}{2}\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

b) $$E(X) = \int_{-1}^{1} x\left(\frac{1+\alpha x}{2}\right)dx = \frac{\alpha}{2}\int_{-1}^{1} x^2\,dx = \alpha/3$$

c) $$E(X^2) = \int_{-1}^{1} x^2\left(\frac{1+\alpha x}{2}\right)dx = \int_{-1}^{1} \frac{x^2}{2}\,dx = 1/3$$

$$Var(X) = 1/3 - \left(\frac{\alpha}{3}\right)^2$$

8. Suppose that you collect coupons, that there are 11 different types of coupons and that you are equally likely to get a coupon of each of these types. How many trials would you expect to go through before you had a complete set of coupons?

$$1 + \frac{11}{10} + \frac{11}{9} + \frac{11}{8} + \frac{11}{7} + \frac{11}{6} + \frac{11}{5} + \frac{11}{4} + \frac{11}{3} + \frac{11}{2} + \frac{11}{1}$$

$$= 11 \left(\sum_{k=1}^{11} \frac{1}{k} \right)$$

(each term in the sum on the 1st line is the expected # of trials until the next new coupon is found).

9. If n men throw their hats into a pile and each man takes a hat at random, what is the expected number of matches? (Hint: express the number as a sum.)

Let $X_i = 1$ if i^{th} man gets the right hat.

$$X = \# \text{ of matches} = \sum_{i=1}^{n} X_i$$

$$E(X) = \sum_{i=1}^{n} E(X_i) = n\,E(X_1)$$

$$E(X_1) = \text{Prob}(X_1 = 1) = 1/n.$$

so $E(X) = n \cdot 1/n = 1$.

10. Consider 4 independent rolls of a 6 sided die. Let X be the number of 1's and Y be the number of 2's obtained.
    (a) Find the marginal distribution function (PMF) of Y
    (b) Find the conditional distribution of X given Y ($f_{(X|Y)}$).
    (c) Find the joint distribution of X and Y.

a) $\text{Prob}\{Y=j\} = \binom{4}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{4-j}$ $\quad j=0,1,2,3,4$

$= 0$ else.

b) $\text{Prob}\{X=k \mid Y=j\} = \binom{4-j}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{4-j-k}$

$j=0,1,2,3,4$

$0 \le k \le 4-j$

$= 0$ else.

c) $\text{Prob}\{X=k \mid Y=j\}$

$= \binom{4-j}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{4-j-k} \cdot \binom{4}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{4-j}$

$0 \le k+j \le 4$

$= 0$ else.