GAFA, Geom. funct. anal.
Vol. 9 (1999) 256 – 282
1016-443X/99/020256-27 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

ON THE DIMENSION OF KAKEYA SETS AND RELATED MAXIMAL INEQUALITIES

J. Bourgain

1 Introduction

A *Kakeya set* in $\mathbb{R}^d$ is a compact set $E \subset \mathbb{R}^d$ containing a line segment in every direction, thus for all $\xi \in S_{d-1}$, there is $a \in \mathbb{R}^d$ so that

$$a + t\xi \in E \quad \text{for} \quad t \in [0,1]\,. \tag{1.1}$$

Such sets may be of zero measure. It seems reasonable to conjecture however that they are necessarily of full Hausdorff dimension, i.e.

$$\dim E = d\,. \tag{1.2}$$

This problem plays a major role in the theory of oscillatory integrals in harmonic analysis. It is also of relevance to questions related to the distribution of Dirichlet series (see [W2] for a survey). For $d = 2$, the conjecture is affirmative as shown by Davies in 1971 ([D]). Research for $d \geq 3$ is more recent. For $d = 3$, the best result to date is

$$\dim E \geq \tfrac{5}{2} \tag{1.3}$$

([W1]). In the same paper [W1], it is shown that in dimension d, one always has

$$\dim E \geq \tfrac{d}{2} + 1\,. \tag{1.4}$$

(1.4) is a small improvement on the "trivial" bound $\dim E \geq \frac{d+1}{2}$ and the gap between (1.2) and (1.4) for large d is obviously substantial. Particularly in this setting, it is tempting to try to improve further on (1.4).

We will prove here the following fact on the Hausdorff dimension H-dim.

Proposition 1.5. *If E is a Kakeya set in $\mathbb{R}^d$, then*

$$H\text{-}\dim E \geq \tfrac{13}{25}d + \tfrac{12}{25}\,. \tag{1.6}$$

Proposition 1.5 will be derived from the following statement related to Minkowski dimension M-dim.

Proposition 1.7. *Assume E a compact set in $\mathbb{R}^d$ such that for each $\xi \in S_{d-1}$ there are $a, b \in E$, $a \neq b$, satisfying*

$$a - b \,/\!\!/\, \xi \tag{1.8}$$

$$\tfrac{1}{2}(a+b) \in E\,. \tag{1.9}$$

Then

$$M\text{-dim } E \geq \tfrac{13}{25}(d-1)\,. \tag{1.10}$$

The trivial estimate here would be $\dim E \geq \frac{1}{2}(d-1)$. Proposition 1.7 has clearly the flavor of certain number theoretic results on sizes of sumsets and difference sets in the spirit of [Ru]. Of particular interest here is the following result of Balog and Szemerédi (cf. [BS], [N]): Let A, B be finite subsets of an Abelian group and $R \subset A \times B$ such that ($|A|$ denoting the cardinality of A)

$$|A| \leq N \tag{1.11}$$

$$|B| \leq N \tag{1.12}$$

$$\big|\{a+b \mid (a,b) \in R\}\big| \leq N \tag{1.13}$$

$$|R| > \delta N^2\,. \tag{1.14}$$

Then there is a proportional subset A' of A with small sumset $A'+A'$, i.e.

$$|A'| > c(\delta)N \tag{1.15}$$

$$|A'+A'| < C(\delta)N\,. \tag{1.16}$$

The proof of this fact as presented in [N] depends in particular on Szemerédi's uniformity lemma and leads therefore to very poor dependence of $c(\delta), C(\delta)$ on δ, making it useless for our purpose. Very recently, T. Gowers in his work on arithmetic progressions [G], give a new and very simple proof of the Balog-Szemerédi theorem with moreover a powerlike dependence of $c(\delta)$ and $C(\delta)$ on δ. This will be the main ingredient in our argument. We have tried to run it efficiently and produce the explicit factor $13/25 > 1/2$ in (1.6), (1.10). Very likely however, the same ideas may lead to better results.

We will also consider certain maximal operators associated to the Kakeya problem (cf. [Bo1], [W2]). Fix $1 > \delta > 0$ and define for a measurable function f on $\mathbb{R}^d$, $\xi \in S_{d-1}$

$$f^*_\delta(\xi) = \sup_\tau \tfrac{1}{|\tau|}\int_\tau \big|f(x)\big|dx \tag{1.17}$$

where the spectrum is taken over all tubes τ of unit length, width δ and oriented in ξ-direction (there is the freedom of translation). Thus the measure $|\tau| \sim \delta^{d-1}$. One conjectures then the following inequality for $1 \leq p \leq d$

$$\|f^*_\delta\|_{L^p(S_{d-1})} \leq C_\varepsilon \left(\tfrac{1}{\delta}\right)^{\frac{d}{p}-1+\varepsilon} \|f\|_p \ \text{ for all } \ \varepsilon > 0\,. \tag{1.18}$$

This conjecture implies (1.2). In fact, if (1.18) holds for a certain $p \leq d$, then one may conclude that for any Kakeya set E in $\mathbb{R}^d$

$$\dim E \geq p\,. \tag{1.19}$$

For $d = 2$, (1.18) is again true for all $1 \le p \le 2$. For $d > 2$, the validity of (1.18) was established in [W1] in the range $p \le \frac{d}{2} + 1$, improving the author's result in [Bo1]. Based on Proposition 1.7, we obtain here for large dimension d a further improvement.

PROPOSITION 1.20. *There is some constant $c > 1/2$ (independent of d) such that* (1.18) *holds for $p \le cd$ in any dimension d.*

Applying Proposition 1.7 requires to generate triples in arithmetic progression. Standard results related to Roth's theorem [R] and further improvements ([He], [S]) again produce estimates that are too weak for our purpose. This forced us to investigate this issue independently considering our particular problem and produce ad hoc results of an independent interest we believe.

The author is grateful to T. Wolff for some discussions on the subject and to T. Gowers for making his preprint [G] available.

In section 2 of this paper, some combinatorial facts will be established related to the Balog-Szeremédi theorem. In section 3, we prove Proposition 1.7 and Proposition 1.5. Section 4 contains a new Roth-type result for the *reciprocals* of an arithmetic progression in sumsets. The proof of Proposition 1.20 appears in sections 5 and 6 of the paper.

We will use the notation $0 < c < C < \infty$ for various constants that may depend on the dimension d, except of course when they appear as exponents. We also will occasionally write "$< A^{\gamma+}$" (resp. "$> A^{\gamma-}$") for "$< C_\varepsilon A^{\gamma+\varepsilon}$ for all $\varepsilon > 0$" (resp. "$> c_\varepsilon A^{\gamma-\varepsilon}$ for all $\varepsilon > 0$").

2 Combinatorial Results

We will consider the setting of subsets of $\mathbb{Z}^d$ but our results most likely extend to arbitrary (torsion free) Abelian groups. We denote by $|A|$ the cardinality of A.

LEMMA 2.1. *Let A, B be finite subsets of $\mathbb{Z}^d$ and $\mathcal{G} \subset A \times B$ such that*

$$|A|, |B| \le N \tag{2.2}$$

$$|S| \le N \quad \text{where} \quad S = \big\{a+b \;\big|\; (a,b) \in \mathcal{G}\big\} \text{ is the } \mathcal{G}\text{-sumset} \tag{2.3}$$

$$|\mathcal{G}| > \alpha N^2 \,. \tag{2.4}$$

Then there exist $A' \subset A$, $B' \subset B$ satisfying the conditions

$$\big|(A' \times B') \cap \mathcal{G}\big| > \alpha^9 N^{2-} \tag{2.5}$$

$$|A' - B'| < N^{-1+} \alpha^{-13} \big|(A' \times B') \cap \mathcal{G}\big| \,. \tag{2.6}$$

Proof. It is a variant of [G, Proposition 1.2]

(i) Since, denoting $\|f\|_2$ the norm of $f \in L^2(\mathbb{T}^d)$, $\mathbb{T} = R/\mathbb{Z}$

$$|\mathcal{S}|^{1/2}\Big\|\Big(\sum_A e^{iax}\Big)\Big(\sum_B e^{-ibx}\Big)\Big\|_2$$
$$\geq \int \Big(\sum_A e^{iax}\Big)\Big(\sum_B e^{ibx}\Big)\Big(\sum_S e^{-inx}\Big) > |\mathcal{G}| > \alpha N^2 \quad (2.7)$$

by (2.4), we have by (2.3)

$$\sum_n r(n; A, B)^2 > \alpha^2 N^3 \quad (2.8)$$

where

$$r(n; A, B) = \big|\{(a, b) \in A \times B \mid a - b = n\}\big| . \quad (2.9)$$

From (2.8), there is clearly some

$$\alpha^2 < \rho_1 < 1 \quad (2.10)$$

such that if we let

$$D = \big\{n \mid r(n; A, B) \sim \rho_1 N\big\} \text{ (popular differences)} . \quad (2.11)$$

then

$$|D| > \alpha^2 \rho_1^{-2} N^{1-} . \quad (2.12)$$

(ii) Consider the graph $R \subset A \times B$ defined by

$$(a, b) \in R \Leftrightarrow a - b \in D . \quad (2.13)$$

Since by (2.11), (2.12)

$$\sum_{b\in B} |R_b| = \sum_{n\in D} \big|\{(a, b) \in A\times B \mid a-b=n\}\big| = \sum_{n\in D} r(n; A, B) > \alpha^2 \rho_1^{-1} N^{2-} \quad (2.14)$$

the expectation when averaging b_0 over B

$$\mathbb{E}_{b_0}[|R_{b_0}|] > \alpha^2 \rho_1^{-1} \tfrac{N^{2-}}{|B|} . \quad (2.15)$$

Choose $0 < \rho_2 < 1$ and define

$$Y = \big\{(a, a') \in A \times A \;\big|\; |R_a \cap R_{a'}| < \rho_2 N\big\} . \quad (2.16)$$

Then

$$\mathbb{E}_{b_0}\big[|R_{b_0}^2 \cap Y|\big] = \tfrac{1}{|B|} \sum_{b\in B} \sum_{(a,a')\in Y} \chi_{R_b}(a)\chi_{R_b}(a')$$
$$= \tfrac{1}{|B|} \sum_{(a,a')\in Y} |R_a \cap R_{a'}| < \rho_2 \tfrac{N^3}{|B|} . \quad (2.17)$$

Define for some choice of

$$0 < \rho_3 < 1 \tag{2.18}$$

$$A' = A'_{b_0} = \Big\{a \in R_{b_0} \Bigm| |\{a' \in R_{b_0} \mid (a, a') \in Y\}| < \rho_3 N\Big\} . \tag{2.19}$$

Then, by (2.17), (2.19), one gets clearly

$$\mathbb{E}_{b_0}\big[|R_{b_0} \backslash A'|\big] < (\rho_3 N)^{-1}\mathbb{E}\big[|R^2_{b_0} \cap Y|\big] < \rho_2\rho_3^{-1}\tfrac{N^2}{|B|} . \tag{2.20}$$

To ensure that, for some $b_0 \in B$

$$|R_{b_0} \backslash A'| = o(|A'|) \tag{2.21}$$

take thus, from (2.15), (2.20)

$$\alpha^2\rho_1^{-1} > \rho_2\rho_3^{-1}N^{0+} \tag{2.22}$$

hence

$$\rho_3 > \frac{\rho_1\rho_2}{\alpha^2}N^{0+} . \tag{2.23}$$

In particular

$$|A'| \sim |R_{b_0}| > \tfrac{\alpha^2}{\rho_1}N^{1-} \tag{2.24}$$

by (2.15).

Iteration of the construction permits us then to obtain $A' \subset A$ and ρ_1 such that

$$|A'| \sim \tfrac{\alpha^2}{\rho_1}N^{1-} \text{ and } \big|\mathcal{G} \cap (A' \times B)\big| > \tfrac{\alpha^2}{\rho_1}N^{-0}|\mathcal{G}| = \alpha^3\rho_1^{-1}N^{2-} . \tag{2.25}$$

(iii) Repeat the construction with B. Thus by (2.25), (2.3)

$$|S|^{1/2}\Big\|\Big(\sum_{A'} e^{iax}\Big)\Big(\sum_B e^{-ibx}\Big)\Big\|_2 \geq \int\Big(\sum_{A'} e^{iax}\Big)\Big(\sum_B e^{ibx}\Big)\Big(\sum_S e^{-inx}\Big)$$

$$> \alpha^3\rho_1^{-1}N^{2-} \tag{2.36}$$

$$\sum_n r(n; A', B)^2 > \alpha^6\rho_1^{-2}N^{3-} . \tag{2.27}$$

There is thus, by (2.25)

$$\tfrac{\alpha^4}{\rho_1}N^{0-} < \rho_4 < \tfrac{|A'|}{N} = \tfrac{\alpha^2}{\rho_1} \tag{2.28}$$

such that if we define

$$D' = \big\{n \bigm| r(n; A', B) > \rho_4 N\big\} \tag{2.29}$$

then

$$|D'| > \rho_1^{-2}\rho_4^{-2}\alpha^6N^{1-} . \tag{2.30}$$

Consider the graph

$$R' = \big\{(a, b) \in A' \times B \bigm| a - b \in D'\big\} . \tag{2.31}$$

Hence

$$\sum_{b\in B} |R'_b| = |R'| = \sum_{n\in D'} r(n; A', B) > \rho_1^{-2}\rho_4^{-1}\alpha^6 N^{2-} \tag{2.32}$$

and there exists

$$\frac{\alpha^2}{\rho_1} > \rho_5 > \frac{\alpha^6}{\rho_1^2\rho_4} N^{0-} \tag{2.33}$$

and $B' \subset B$ satisfying

$$|R'_b| \sim \rho_5 N \quad \text{for} \quad b \in B' \tag{2.34}$$

$$|B'| > \frac{\alpha^6}{\rho_1^2\rho_4\rho_5} N^{1-}\,. \tag{2.35}$$

Our aim is to ensure moreover that if $a \in A'$, $b \in B'$, then

$$\big|\{a' \in R'_b \mid (a, a') \notin Y\}\big| > \tfrac{1}{2}\rho_5 N\,. \tag{2.36}$$

The left member of (2.36) is at least

$$\begin{aligned} &|R'_b| - \big|\{a' \in A' | (a, a') \in Y\}\big| \\ &> \rho_5 N - \rho_3 N \end{aligned} \tag{2.37}$$

by definition of A', cf. (2.19). Thus for (2.36) to hold it suffices that

$$\rho_5 > 10\rho_3 \tag{2.38}$$

hence, by (2.33)

$$\frac{\alpha^6}{\rho_1^2\rho_4} N^{0-} > \rho_3 \tag{2.39}$$

and, from (2.29)

$$\tfrac{\alpha^6}{\rho_1^2} N^{0-} > \rho_3 \tfrac{\alpha^2}{\rho_1}\,. \tag{2.40}$$

Choose thus

$$\rho_3 \sim \tfrac{\alpha^4}{\rho_1} N^{0-} \tag{2.41}$$

and in order to fulfill (2.23)

$$\rho_2 \sim \tfrac{\alpha^6}{\rho_1^2} N^{0-}\,. \tag{2.42}$$

Again by iteration, we get some $B' \subset B$ and ρ_4, ρ_5 such that (2.34), (2.35) hold and, taking (2.25), (2.35) into account

$$\big|\mathcal{G} \cap (A' \times B')\big| > \frac{\alpha^9}{\rho_1^3\rho_4\rho_5} N^{2-} > \alpha^9 N^{2-} \tag{2.43}$$

which is (2.5).

(iv) Finally, we evaluate $|A' - B'|$.

Take $a \in A'$, $b \in B'$. By (2.36)

$$C = \{a' \in R'_b \mid (a,a') \notin Y\} \tag{2.44}$$

satisfies

$$|C| > \tfrac{1}{2}\rho_5 N\,. \tag{2.45}$$

By definition of Y,

$$|R_a \cap R_{a'}| \geq \rho_2 N \quad \text{for} \quad a' \in C\,. \tag{2.46}$$

Write for $a' \in C$, $b' \in R_a \cap R_{a'}$

$$a - b = (a - b') - (a' - b') + a' - b\,. \tag{2.47}$$

The number of representations of $a-b'$ in $A-B$ is at least $\rho_1 N$ (by definition of R). Idem for $a' - b'$.

The number of representations of $a' - b$ in $A' - B$ is at least $\rho_4 N$ (by definition of R'). Thus the number of representations of $a - b$ as

$$a - b = (a_1 - b_1) - (a_2 - b_2) + a_3 - b_3 \tag{2.48}$$

where

$$a_1, a_2 \in A\,,\ a_3 \in A'\,;\ b_1, b_2, b_3 \in B \tag{2.49}$$

is at least

$$\left(\tfrac{1}{2}\rho_5 N\right)(\rho_2 N)(\rho_1 N)^2 \rho_4 N\,. \tag{2.50}$$

This implies that

$$|A' - B'| \leq \frac{2|A|^2|A'|\,|B|^3}{\rho_1^2\rho_2\rho_4\rho_5 N^5} \overset{\text{by (2.25)}}{<} \alpha^2\rho_1^{-3}\rho_2^{-1}\rho_4^{-1}\rho_5^{-1}N \tag{2.51}$$

$$\overset{\text{by (2.43)}}{<} \alpha^{-7}\rho_2^{-1}N^{-1+}\big|\mathcal{G} \cap (A' \times B')\big| \tag{2.52}$$

$$\overset{\text{by (2.42)}}{<} \alpha^{-13}N^{-1+}\big|\mathcal{G} \cap (A' \times B')\big| \tag{2.53}$$

which is (2.6).

This proves Lemma 2.1.

Lemma 2.54. *Let A, B be finite subsets of $\mathbb{Z}^d$ and $\mathcal{G} \subset A \times B$ such that*

$$|A|, |B| \leq N \tag{2.55}$$

$$|S| \leq N \quad \textit{where } S = \{a + b \mid (a,b) \in \mathcal{G}\}\,. \tag{2.56}$$

Then

$$|D| < N^{2-\frac{1}{13}+} \quad \textit{where} \quad D = \{a - b \mid (a,b) \in \mathcal{G}\}\,. \tag{2.57}$$

Proof. Take $\mathcal{G}' \subset \mathcal{G}$ such that

$$D = \{a - b \mid (a - b) \in \mathcal{G}'\} \tag{2.58}$$

and the map $\mathcal{G}' \to D : (a,b) \to a-b$ is one to one.

Writing

$$|\mathcal{G}'| = |D| = \alpha N^2 \tag{2.59}$$

it follows from Lemma 2.1 (condition (2.3) remains valid replacing $\mathcal{G}$ by $\mathcal{G}'$) and (2.6), that

$$\big|\mathcal{G}' \cap (A' \times B')\big| \leq |A' - B'| < N^{-1+}\alpha^{-13}\big|\mathcal{G}' \cap (A' \times B')\big|\,. \tag{2.60}$$

Hence

$$\alpha < N^{-\frac{1}{13}+} \tag{2.61}$$

and (2.59), (2.61) imply (2.57).

Remark. Consider the special case $\mathcal{G} = A \times B$. Under the assumption

$$|A|, |B|, |A+B| \leq N \tag{2.62}$$

one may prove

$$|A-B| < CN^{3/2}\,. \tag{2.63}$$

On the other hand, for all $\varepsilon > 0$, there are arbitrary large values of N and sets $A, B \subset \mathbb{Z}$ such that

$$|A| = N\,, \quad |A+B| < (1+\varepsilon)N \tag{2.64}$$

while

$$|A-B| > c(\varepsilon)N^{1+\alpha}\,, \quad \alpha = \frac{\log 7/6}{\log 7} \tag{2.65}$$

(see [Ru] for these results).

Lemma 2.66. *Let A, B be finite subsets of $\mathbb{Z}^d$, $\mathcal{G} \subset A \times B$ and*

$$|A|, |B| \leq N \tag{2.67}$$

$$|S| \leq N \text{ where } S = \big\{a+b \;\big|\; (a,b) \in \mathcal{G}\big\} \tag{2.68}$$

$$|\mathcal{G}| = \alpha N^2\,. \tag{2.69}$$

Then there exists $A' \subset A$, $B' \subset B$ such that

$$|A' + A'| < \alpha^{-33}N^{1+} \tag{2.70}$$

$$B' \text{ is contained in a translate of } A' \tag{2.71}$$

$$\big|\mathcal{G} \cap (A' \times B')\big| > \alpha^{30}N^{2-}\,. \tag{2.72}$$

Proof. Let $A_1 \subset A$, $B_1 \subset B$ be the sets obtained in Lemma 2.1. Thus from (2.5), (2.6)

$$\big|(A_1 \times B_1) \cap \mathcal{G}\big| > \alpha^9 N^{2-} \tag{2.73}$$

$$|A_1 - B_1| < \alpha^{-12}N^{1+} \tag{2.74}$$

Let $A' = A_1$. One has the general inequality, cf. [Ru].

$$|A' + A'| < \frac{|A' - B_1|^2}{|B_1|} \tag{2.75}$$

and hence, from (2.73), (2.74).

$$|A' + A'| \le \frac{|A_1|\,|A_1 - B_1|^2}{|(A_1 \times B_1) \cap \mathcal{G}|} < \alpha^{-33} N^{1+} \,. \tag{2.76}$$

We construct B' by averaging. Define

$$B' = B'_{n_0} = B_1 \cap (A_1 + n_0) \tag{2.77}$$

with thus $n_0 \in B_1 - A_1$. Then, by (2.73)

$$\begin{aligned} \sum_{n_0 \in B_1 - A_1} \big|(A' \times B') \cap \mathcal{G}\big| &= \sum_{n_0} \sum_{(a,b) \in \mathcal{G}} \chi_{A_1}(a) \chi_{B_1}(b) \chi_{b - A_1}(n_0) \\ &= |A_1| \big|\mathcal{G} \cap (A_1 \times B_1)\big| > \alpha^{18} N^{3-} \,. \end{aligned} \tag{2.78}$$

Thus, by (2.74), there is some n_0 such that the set B' will satisfy

$$\big|(A' \times B') \cap \mathcal{G}\big| > \frac{\alpha^{18} N^{3-}}{|A_1 - B_1|} > \alpha^{30} N^{2-} \,. \tag{2.79}$$

This proves Lemma 2.66.

Remark. If $A \subset \mathbb{Z}^d$ satisfies

$$|A| = N \,, \quad |A + A| < CN \tag{2.80}$$

then, cf. [Ru], a general estimate asserts that for all $h = 2, 3, \dots$

$$|\underbrace{A \pm A \pm A \cdots \pm A}_{h \text{ terms}}| < C^{2h} N \,. \tag{2.81}$$

Hence, in Lemma 2.66, we also have

$$|\underbrace{A' \pm A' \pm \cdots \pm A'}_{h \text{ terms}}| \le \alpha^{-C} N^{1+} \tag{2.82}$$

for all h.

We will also need to translate our lattice results to statements about entropy numbers of subsets of $\mathbb{R}^d$. If $A \subset \mathbb{R}^d$, denote for $\delta > 0$ by $\mathcal{N}_\delta(A)$ the metrical entropy numbers of A, i.e. the minimum number of δ-balls in $\mathbb{R}^d$ needed to cover A.

Lemma 2.83. *Let A, B be bounded subsets of $\mathbb{R}^d$ and $\mathcal{G} \subset A \times B$ such that*

$$\mathcal{N}_\delta(A), \mathcal{N}_\delta(B) \le N \tag{2.84}$$

$$\mathcal{N}_\delta(S) \le N \text{ where } S = \big\{a + b \bigm| (a,b) \in \mathcal{G}\big\} \,. \tag{2.85}$$

Then

$$\mathcal{N}_\delta(D) < c N^{2 - \frac{1}{13}+} \text{ where } D = \big\{a - b \bigm| (a,b) \in \mathcal{G}\big\} \,. \tag{2.86}$$

Proof. Define

$$\tilde{A} = \{a \in \mathbb{Z}^d \mid (\delta a + B_\delta) \cap A \neq \phi\} \tag{2.87}$$

$$\tilde{B} = \{b \in \mathbb{Z}^d \mid (\delta b + B_\delta) \cap B \neq \phi\} \tag{2.88}$$

$$\tilde{\mathcal{G}} = \{(a,b) \in \mathbb{Z}^{2d} \mid ((\delta a + B_\delta) \times (\delta b + B_\delta)) \cap \mathcal{G} \neq \phi\}\,. \tag{2.89}$$

Then $\tilde{\mathcal{G}} \subset \tilde{A} \times \tilde{B}$ and from (2.84)–(2.85)

$$|\tilde{A}| \leq C\mathcal{N}_\delta(A) \leq CN \tag{2.90}$$

$$|\tilde{B}| \leq C\mathcal{N}_\delta(B) \leq CN \tag{2.91}$$

$$|\tilde{S}| \leq C\mathcal{N}_{2\delta}(S) \leq CN \text{ where } \tilde{S} = \{a + b \mid (a,b) \in \tilde{\mathcal{G}}\} \tag{2 .92}$$

(the constant C depends on dimension d).

Since clearly

$$\mathcal{G} \subset \delta\tilde{\mathcal{G}} + (B_\delta \times B_\delta) \tag{2.93}$$

we have

$$D \subset \delta\{a - b \mid (a,b) \in \tilde{\mathcal{G}}\} + B_{2\delta} \tag{2.94}$$

and thus from Lemma 2.54

$$\mathcal{N}_\delta(D) \leq C\big|\{a - b \mid (a,b) \in \tilde{\mathcal{G}}\}\big| < CN^{2-\frac{1}{13}+}\,. \tag{2.95}$$

Lemma 2.96. *Let A, B be bounded subsets of $\mathbb{R}^d$, $\mathcal{G} \subset A \times B$ such that*

$$\mathcal{N}_\delta(A), \mathcal{N}_\delta(B) \leq N \tag{2.97}$$

$$\mathcal{N}_\delta(S) \leq N \text{ where } S = \{a + b \mid (a,b) \in \mathcal{G}\} \tag{2.98}$$

$$\mathcal{N}_\delta(\mathcal{G}) > \alpha N^2\,. \tag{2.99}$$

Then there exist $A' \subset A + B_{C\delta}$, $B' \subset B + B_{C\delta}$ satisfying

$$\mathcal{N}_\delta(A' + A') < C.\alpha^{-33} N^{1+} \tag{2.100}$$

$$B' \text{ is contained in a translate of } A' \tag{2.101}$$

$$\mathcal{N}_\delta\big(\mathcal{G} \cap (A' \times B')\big) > \alpha^{30} N^{2-}\,. \tag{2.102}$$

Proof. Define $\tilde{A}, \tilde{B} \subset \mathbb{Z}^d$ and $\tilde{\mathcal{G}}$ as in the proof of Lemma 2.83. Thus, by (2.93), (2.99)

$$|\tilde{\mathcal{G}}| > c\alpha N^2\,. \tag{2.103}$$

Apply lemma 2.66 to get $A_1 \subset \tilde{A}$, $B_1 \subset \tilde{B}$ such that

$$|A_1 + A_1| < C\alpha^{-33} N^{1+} \tag{2.104}$$

$$B_1 \text{ is contained in a translate of } A_1 \tag{2.105}$$

$$\big|\tilde{\mathcal{G}} \cap (A_1 \times B_1)\big| > c\alpha^{30} N^{2-}\,. \tag{2.106}$$

From (2.89)

$$\delta\tilde{\mathcal{G}} \subset \mathcal{G} + (B_\delta \times B_\delta) \tag{2.107}$$

hence, by (2.106)

$$\mathcal{N}_\delta\big((\mathcal{G} + (B_\delta \times B_\delta)) \cap (\delta A_1 \times \delta B_1)\big) > c\alpha^{30} N^{2-} . \tag{2.108}$$

Define

$$A' = \delta A_1 + B_\delta \subset \delta \tilde{A} + B_\delta \subset A + B_{2\delta} \tag{2.109}$$

$$B' = \delta B_1 + B_\delta \subset B + B_{2\delta} . \tag{2.110}$$

Thus, by (2.104)

$$\mathcal{N}_\delta(A' + A') \leq C|A_1 + A_1| < C\alpha^{-33} N^{1+} \tag{2.111}$$

(2.105) clearly implies (2.101) and from (2.108)–(2.110)

$$\mathcal{N}_\delta\big(\mathcal{G} \cap (A' \times B')\big) > c\alpha^{30} N^{2-} . \tag{2.112}$$

This proves the lemma.

For later use, we also need the following more technical version of Lemma 2.96.

Lemma 2.113. *Let A, B be bounded subsets of $\mathbb{R}^d$, $\mathcal{G} \subset A \times B$ such that*

$$\mathcal{N}_\delta(A), \mathcal{N}_\delta(B) \leq N \tag{2.114}$$

$$\mathcal{N}_\delta(\mathcal{G}) > \alpha N^2 . \tag{2.115}$$

Let ζ_ℓ, η_ℓ ($\ell = 1, \dots, \ell_0$) be positive numbers satisfying

$$\delta^\tau < \zeta_\ell, \eta_\ell < \delta^{-\tau} \qquad (1 \leq \ell \leq \ell_0) \tag{2.116}$$

for some (small) $\tau > 0$. Here ℓ_0 is fixed ($\ell_0 = 6$ in the application).

Assume for each $\ell = 1, \dots, \ell_0$

$$\mathcal{N}_\delta\big(\{\zeta_\ell a + \eta_\ell b \mid (a,b) \in \mathcal{G}\}\big) \leq N . \tag{2.117}$$

Then there are $A' \subset A + B_{C\delta^{1-3\tau}}$, $B' \subset B + B_{C\delta^{1-\tau}}$ satisfying

$$\mathcal{N}_\delta(A' + A') < \alpha^{-C} \delta^{-Cd\tau} N \tag{2.118}$$

$$\eta_\ell B' \text{ is contained in a translate of } \zeta_\ell A' \ (1 \leq \ell \leq \ell_0) \tag{2.119}$$

$$\mathcal{N}\big(\mathcal{G} \cap (A' \times B')\big) > \alpha^C \delta^{Cd\tau} N^2 \tag{2.120}$$

(the constant C in the exponent depends only on ℓ_0.)

Proof. Take $\ell = 1$ and define

$$\begin{aligned} A_1 = \zeta_1 A , \quad B_1 = \eta_1 B , \\ \mathcal{G}_1 = \big\{(\zeta_1 a, \eta_1 b) \bigm| (a,b) \in \mathcal{G}\big\} \subset A_1 \times B_1 . \end{aligned} \tag{2.121}$$

Thus, by (2.114), 2.115), (2.116)

$$\mathcal{N}_\delta(A_1) < C\delta^{-d\tau} \mathcal{N}_\delta(A) < C\delta^{-d\tau} N \tag{2.122}$$

$$\mathcal{N}_\delta(B_1) < C\delta^{-d\tau} N \tag{2.123}$$

$$\mathcal{N}_\delta(\mathcal{G}_1) > c\delta^{2d\tau}\alpha N^2\,. \tag{2.124}$$

Apply Lemma 2.96 with N replaced by $C\delta^{-d\tau}N$ and α by $c\delta^{d\tau}\alpha$. This gives $A_1'' \subset A_1 + B_{C\delta}$, $B_1'' \subset B_1 + B_{C\delta}$ satisfying

$$\mathcal{N}_\delta(A_1'' + A_1'') < C(\alpha\delta^{d\tau})^{-C} N \tag{2.125}$$

$$B_1'' \text{ is contained in a translate of } A_1'' \tag{2.126}$$

$$\mathcal{N}_\delta\big(\mathcal{G}_1 \cap (A_1'' \times B_1'')\big) > c(\alpha\delta^{d\tau})^C N^2\,. \tag{2.127}$$

Put

$$A_1' = \zeta_1^{-1} A_1'' \subset A + B_{C\zeta_1^{-1}\delta} \subset A + B_{C\delta^{1-\tau}} \tag{2.128}$$

$$B_1' = \eta_1^{-1} B_1'' \subset B + B_{C\delta^{1-\tau}}\,. \tag{2.129}$$

Thus, from (2.125), (2.116)

$$\mathcal{N}_\delta\,(A_1' + A_1') < C\delta^{-d\tau}\mathcal{N}_\delta(A_1'' + A_1'') < C(\alpha\delta^{d\tau})^{-C} N\,. \tag{2.130}$$

From (2.116)

$$\eta_1 B_1' \text{ is contained in a translate of } \zeta_1 A_1'\,. \tag{2.131}$$

From (2.127)

$$\mathcal{N}_\delta\big(\mathcal{G} \cap (A_1' \times B_1')\big) > c(\alpha\delta^{d\tau})^C N^2\,. \tag{2.132}$$

Replace next $A, B, \mathcal{G}$ by $A_1', B_1', \mathcal{G}_1 = \mathcal{G} \cap (A_1' \times B_1')$. Then (2.114), (2.115) still hold with N replaced by $C\delta^{-d\tau}N$ and α by $(\alpha\delta^{d\tau})^C$. From (2.117) for $\ell = 2$, one gets by the preceding $B_2' \subset B_1' + B_{C\delta^{1-\tau}} \subset B + B_{C\delta^{1-\tau}}$ such that

$$\eta_2 B_2' \text{ is contained in a translate of } \zeta_2 A_1' \tag{2.133}$$

$$\mathcal{N}_\delta\big(\mathcal{G} \cap (A_1' \times B_2')\big) > c(\alpha\delta^{d\tau})^C N^2\,. \tag{2.134}$$

Also $\eta_1 B_2' \subset \eta_1 B_1' + B_{C\eta_1\delta^{1-\tau}}$ is contained in a translate of $\zeta_1(A_1' + B_{C\zeta_1^{-1}\eta_1\delta^{1-\tau}}) \subset \zeta_1 A_2'$, where we define

$$A_2' = A_1' + B_{C\delta^{1-3\tau}} \subset A + B_{C\delta^{1-3\tau}}\,. \tag{2.135}$$

From (2.130), (2.134)

$$\mathcal{N}_\delta(A_2' + A_2') < C(\alpha\delta^{d\tau})^{-C} N \tag{2.136}$$

and

$$\mathcal{N}_\delta\big(\mathcal{G} \cap (A_2' \times B_2')\big) > c(\alpha\delta^{d\tau})^C N^2\,. \tag{2.137}$$

Iterating ℓ_0 times clearly produces the required sets $A' = A_{\ell_0}'$, $B' = B_{\ell_0}'$.

Remark. Taking the previous remark on multiple sums and differences (2.80)–(2.82) into account, one may again in Lemma 2.96, replace (2.100) by

$$\mathcal{N}_\delta(\underbrace{A' \pm A' \pm \cdots \pm A'}_{h \text{ terms}}) < C\alpha^{-C_h} N^{1+} \tag{2.138}$$

for any fixed h. Similarly in Lemma 2.113, (2.118) has the following strengthening

$$\mathcal{N}_\delta(A' \pm A' \pm \cdots \pm A') < C(\alpha\delta^{d\tau})^{-C} N\,. \tag{2.139}$$

3 Proof of Propositions 1.5 and 1.7

Consider first Proposition 1.7. By Baire's theorem there is a nonempty open subset $O \subset S_{d-1}$ and $\gamma = 0$ such that for $\xi \in O$, there are points a_ξ, b_ξ in E satisfying

$$|a_\xi - b_\xi| > \gamma \tag{3.1}$$

$$a_\xi - b_\xi /\!\!/ \xi \tag{3.2}$$

$$\tfrac{1}{2}(a_\xi + b_\xi) \in E\,. \tag{3.3}$$

Apply Lemma 2.83 with $A = B = E$ and

$$\mathcal{G} = \big\{(a_\xi, b_\xi) \in E \times E \;\big|\; \xi \in O\big\}\,. \tag{3.4}$$

Choose $\delta > 0$ small. In (2.84), (2.85), we may take $N = \mathcal{N}_\delta(2E) < C\mathcal{N}_\delta(E)$. On the other hand, by (3.1), (3.2)

$$\mathcal{N}_\delta\big(\{a_\xi - b_\xi \mid \xi \in O\}\big) > c(O,\gamma)\delta^{-(d-1)} \tag{3.5}$$

and (2.86) implies thus that for $\delta \to 0$

$$c(O,\gamma)\delta^{-(d-1)} < \mathcal{N}_\delta(E)^{\frac{25}{13}+} \tag{3.6}$$

$$\mathcal{N}_\delta(E) > c\left(\tfrac{1}{\delta}\right)^{\frac{13}{25}(d-1)-}\,. \tag{3.7}$$

This proves (1.10).

The main new difficulty to derive Proposition 1.5 is that Minkowski-dimension is replaced by Hausdorff-dimension. Fix a small number $\kappa > 0$. One may then write

$$E = \bigcup_{\delta < \delta_0} E_\delta \text{ where we take } \delta \text{ of the form } \delta = 2^{-(1+\kappa)^j} \quad (j \in \mathbb{Z}_+) \tag{3.8}$$

with

$$\mathcal{N}_\delta(E_\delta) > \left(\tfrac{1}{\delta}\right)^{\nu+\kappa d}, \quad \nu = H-\dim E\,. \tag{3.9}$$

For each $\xi \in S_{d-1}$, denote $I_\xi /\!\!/ \xi$ a line segment of unit length contained in E.

Clearly, there is $j > j_0 \sim \log\log 1/\delta_0$ such that for $\delta = 2^{-(1+\kappa)^j}$

$$\operatorname{mes}(I_\xi \cap E_\delta) > \tfrac{1}{j^2} \tag{3.10}$$

for ξ in a subset $\mathcal{D}$ of S_{d-1} of measure

$$\operatorname{mes}\mathcal{D} > \tfrac{1}{j^2}\,. \tag{3.11}$$

We will moreover assume that

$$\left|P_{e_d^\perp}(\xi)\right| < \tfrac{1}{10} \text{ for } \xi \in \mathcal{D} \tag{3.12}$$

where e_d is the d unit vector.

Define $q = [\delta^{-1+\kappa}]$ and for $r = 0, 1, \dots, q-1$

$$\mathcal{E}_r = \left\{x \in \mathbb{R}^d \;\middle|\; [\delta^{-1}x_d] \equiv r (\operatorname{mod} q)\right\}. \tag{3.13}$$

Let

$$E_\delta^r = E_\delta \cap \mathcal{E}_r\,, \quad I_\xi^r = I_\xi \cap \mathcal{E}_r \tag{3.14}$$

$$E_\delta = \bigcup_{r=0}^{q-1} E_\delta^r\,, \quad I_\xi = \bigcup_{r=0}^{q-1} I_\xi^r\,. \tag{3.15}$$

Hence, from (3.10), (3.11)

$$\operatorname{mes} I_\xi^r \lesssim q^{-1}\,; \quad \int_{\mathcal{D}} \sum_{r=0}^{q-1} \operatorname{mes}(I_\xi^r \cap E_\delta)\sigma(d\xi) > \tfrac{1}{j^4}\,. \tag{3.16}$$

Thus there is a set $\mathcal{R} \subset \{0, 1, \dots, q-1\}$ satisfying

$$|\mathcal{R}| > \tfrac{q}{2j^4} \tag{3.17}$$

and for $r \in \mathcal{R}$

$$\int_{\mathcal{D}} \operatorname{mes}(I_\xi^r \cap E_\delta)\sigma(d\xi) > \tfrac{1}{2qj^4}\,. \tag{3.18}$$

For $r \in \mathcal{R}$, there is a subset $\mathcal{D}_r \subset \mathcal{D}$ such that

$$\operatorname{mes} D_r > \tfrac{1}{4j^4} \tag{3.19}$$

$$\operatorname{mes}(I_\xi^r \cap E_\delta) > \tfrac{1}{4j^4 q} \text{ for } \xi \in \mathcal{D}_r\,. \tag{3.20}$$

Fix $\xi \in \mathcal{D}_r$. Consider the set

$$Q = \left\{\tfrac{1}{q}([\delta^{-1}x_d] - r) \;\middle|\; x \in I_\xi^r \cap E_\delta\right\} \tag{3.21}$$

contained in an arithmetic progression of $M \sim \delta^{-\kappa}$ consecutive integers. By (3.20)

$$|Q| > c\mathcal{N}_\delta(I_\xi^r \cap E_\delta) > \tfrac{c}{j^4 q\delta} = \tfrac{c}{j^4} M\,. \tag{3.22}$$

At this point, we invoke results of Heath-Brown and Szemerédi (cf. [He], [S]) on existence of triples in arithmetic progression in subsets $Q_0 \subset \{1, \dots, M\}$ satisfying a density condition $|Q_0| > M/(\log M)^c$ (for some explicit constant $c > 0$). The original result of Roth [R] required $|Q_0| > M/\log\log M$ and is a bit too weak for our purpose. Observe indeed that by (3.8)

$$j \sim \kappa^{-1}\left(\log\log \tfrac{1}{\delta}\right) \sim \kappa^{-1} \log\log M < (\log\log M)^2 \tag{3.23}$$

for $\delta < \delta_0$ small enough. Thus (3.22) largely suffices to ensure that Q contains a triple in progression. This means that for each $\xi \in \mathcal{D}_r$, there exist $a^r_\xi \neq b^r_\xi$ in $I^r_\xi \cap E_\delta$ satisfying

$$\tfrac{1}{2}(a^r_\xi + b^r_\xi) \in E^r_\delta + B_{2\delta}\,. \tag{3.24}$$

Clearly

$$a^r_\xi - b^r_\xi /\!/ \xi \tag{3.25}$$

$$|a^r_\xi - b^r_\xi| > \delta q \sim \delta^\kappa\,. \tag{3.26}$$

Next we apply again Lemma 2.83. Let

$$A = B = E^r_\delta + B_{2\delta} \tag{3.27}$$

$$\mathcal{G} = \big\{(a^r_\xi, b^r_\xi) \;\big|\; \xi \in \mathcal{D}_r\big\} \tag{3.28}$$

$$N = C\mathcal{N}_\delta(E^r_\delta)\,. \tag{3.29}$$

From (2.86), we conclude that

$$\mathcal{N}_\delta\big(\{a_\xi - b_\xi \mid \xi \in \mathcal{D}_r\}\big) < c\mathcal{N}_\delta(E^r_\delta)^{\frac{25}{13}+}\,. \tag{3.30}$$

Recalling (3.19), (3.26), the left of (3.30) is at least

$$\mathcal{N}_\delta\big(\{a_\xi - b_\xi \mid \xi \in \mathcal{D}_r\}\big) > c\left(\tfrac{1}{\delta}\right)^{d-1}(\delta^\kappa)^{d-1}(\operatorname{mes}\mathcal{D}_r) > cj^{-4}\left(\tfrac{1}{\delta}\right)^{(1-\kappa)(d-1)} \tag{3.31}$$

and (3.30) implies for $r \in \mathcal{R}$

$$\mathcal{N}_\delta(E^r_\delta) > cj^{-3}\left(\tfrac{1}{\delta}\right)^{\frac{13}{25}(1-\kappa)(d-1)-}\,. \tag{3.32}$$

Thus, from (3.9), (3.14), (3.15), (3.17), (3.32), (3.23)

$$\left(\tfrac{1}{\delta}\right)^{\nu+\kappa d+} > \mathcal{N}_\delta(E_\delta) > c|\mathcal{R}|j^{-3}\left(\tfrac{1}{\delta}\right)^{\frac{13}{25}(1-\kappa)(d-1)-} > j^{-7}\left(\tfrac{1}{\delta}\right)^{(1-\kappa)(\frac{13}{25}d+\frac{12}{25})-}$$
$$= \left(\tfrac{1}{\delta}\right)^{(1-\kappa)(\frac{13}{25}d+\frac{12}{25})-}\,. \tag{3.27}$$

Consequently

$$\nu + \kappa d > (1-\kappa)\left(\tfrac{13}{25}d + \tfrac{12}{25}\right) - \tag{3.34}$$

and (2.6) follows by letting $\kappa \to 0$.

This proves Proposition 1.5.

4 Reciprocals of Arithmetic Progressions in Sumsets

Another ingredient in the proof of Proposition 1.20 is the following.

LEMMA 4.1. *Let $f \geq 0$ be a function on $\mathbb{R}$ such that*

$$\operatorname{supp} f \subset [-1, 1] \tag{4.2}$$

$$\int f dx > \varepsilon \tag{4.3}$$

$$\widehat{f} \geq 0 \tag{4.4}$$

$$\int \widehat{f} \leq 1\,. \tag{4.5}$$

Then, for some absolute constant C

$$I = \iint f\left(1-\frac{1}{\lambda}\right) f\left(1-\frac{1}{\lambda+\mu}\right) f\left(1-\frac{1}{\lambda+2\mu}\right) \varphi_1(\lambda)\varphi_2(\mu)d\lambda\, d\mu > \varepsilon^C \tag{4.6}$$

where we assume $0 \leq \varphi_\alpha \leq 1$ *localizing functions s.t.*

$$\begin{cases} \varphi_1 = 1 \text{ on a neighborhood of } 1 \\ \varphi_2 = 1 \text{ on a neighborhood of } 0\,. \end{cases}$$

Proof. Write

$$\lambda = 1 + x\,, \quad \mu = y \tag{4.7}$$

where $x, y = o(1)$. Thus

$$I = \iint f\left(1-\frac{1}{1+x}\right) f\left(1-\frac{1}{1+x+y}\right) f\left(1-\frac{1}{1+x+2y}\right) \varphi_2(x)\varphi_2(y)dx\, dy\,. \tag{4.8}$$

For $0 < \gamma < 1/10$, define

$$I_\gamma = \iint f\left(1-\frac{1}{1+\gamma x}\right) f\left(1-\frac{1}{1+\gamma x+\gamma y}\right) f\left(1-\frac{1}{1+\gamma x+2\gamma y}\right) \cdot \varphi_2(x)\varphi_2(y)dx\, dy\,. \tag{4.9}$$

Thus

$$I \geq \gamma^2 I_\gamma. \tag{4.10}$$

Write I_δ in Fourier. Thus

$$I_\gamma = \iiint dk_1 dk_2 dk_3\, \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3)\; I_\gamma(k_1,k_2,k_3) \tag{4.11}$$

where $I_\gamma(\hbar)$ denotes the integral

$$J_\gamma(\hbar) = \iint e^{i[k_1(1-\frac{1}{1+\gamma x})+k_2(1-\frac{1}{1+\gamma x+\gamma y})+k_3(1-\frac{1}{1+\gamma x+2\gamma y})]}\varphi(x)\varphi(y)dx\, dy\,. \tag{4.12}$$

Write

$$\begin{aligned}\theta(x,y) &= k_1\left(1-\frac{1}{1+\gamma x}\right)+k_2\left(1-\frac{1}{1+\gamma x+\gamma y}\right)+k_3\left(1-\frac{1}{1+\gamma x+2\gamma y}\right) \\ &= \gamma(k_1+k_2+k_3)x+\gamma(k_2+2k_3)y-\tfrac{1}{2}\gamma^2\left[k_1x^2+k_2(x+y)^2+k_3(x+2y)^2\right] \\ &\quad + 0(\gamma^3 K)\end{aligned} \tag{4.13}$$

where

$$K = |k_1| + |k_2| + |k_3|\,. \tag{4.14}$$

It follows from (4.13) that

$$\Big|J_\gamma(\bar{k})-\Big(\int\varphi\Big)^2\Big|\leq\big[\gamma(|k_1+k_2+k_3|+|k_2+2k_3|)+\gamma^2K\big]\Big(\int\varphi\Big)^2. \tag{4.15}$$

Further

$$|\partial_x\theta|>\gamma|k_1+k_2+k_3|-\gamma^2K \tag{4.16}$$

$$|\partial_y\theta|>\gamma|k_2+k_3|-\gamma^2K\,. \tag{4.17}$$

Hence, integrating in x (resp. in y), we get

$$\big|J_\gamma(\bar{k})\big|<C\big(1+\gamma|k_1+k_2+k_3|\big)^{-1}\text{ if }|k_1+k_2+k_3|>10\gamma K \tag{4.18}$$

resp.

$$\big|J_\gamma(\bar{k})\big|<C\big(1+\gamma|k_2+2k_3|\big)^{-1}\text{ if }|k_2+k_3|>10\gamma K\,. \tag{4.19}$$

Assume next

$$|k_1+k_2+k_3|\leq 10\gamma K\,,\quad |k_2+2k_3|\leq 10\gamma K\,. \tag{4.20}$$

Hence

$$k_1=k_3+0(\gamma K)\,,\quad k_2=-2k_3+0(\gamma K) \tag{4.21}$$

and

$$k_1x^2+k_2(x+y)^2+k_3(x+2y)^2=2k_3y^2+0(\gamma K)(x^2+y^2)\,. \tag{4.22}$$

Thus

$$|\partial^2_{yy}\theta|\sim K\gamma^2 \tag{4.23}$$

and stationary phase implies the bound

$$\begin{aligned}\big|J_\gamma(\bar{k})\big|&<C(\gamma^2K)^{-1/2}\\ &\overset{\text{by (4.20)}}{=}C\big[\gamma(|k_1+k_2+k_3|+|k_2+2k_3|)+\gamma^2K\big]^{-1/2}.\end{aligned} \tag{4.24}$$

Hence, from (4.18), (4.19), estimate (4.24) is clearly always valid.

Define

$$\nu(\gamma,\bar{k})=\gamma\big(|k_1+k_2+k_3|+|k_2+2k_3|\big)+\gamma^2K \tag{4.25}$$

and introduce the following subsets of $\mathbb{R}^3$

$$\mathfrak{S}_\gamma=\big\{(k_1,k_2,k_3)\in\mathbb{R}^3\ \big|\ \nu(\gamma,\bar{k})<\tfrac{1}{10}\big\} \tag{4.26}$$

$$\mathfrak{S}_{\gamma,r}=\big\{(k_1,k_2,k_3)\in\mathbb{R}^3\ \big|\ \nu(\gamma,\bar{k})\sim\tfrac{1}{10}2^r\big\}\,. \tag{4.27}$$

By (4.15)

$$\Big|J_\gamma(\bar{k})-\Big(\int\varphi\Big)^2\Big|<\tfrac{1}{10}\Big(\int\varphi\Big)^2\text{ for }\bar{k}\in\mathfrak{S}_\gamma\,. \tag{4.28}$$

By(4.24)

$$\big|J_\gamma(\bar{k})\big|\leq C2^{-r/2}\text{ for }\bar{k}\in\mathfrak{S}_{\gamma,r}\,. \tag{4.29}$$

Rewrite (4.11) as

$$I_\gamma = \int_{\mathfrak{S}_\gamma} d\bar{k}\, \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3)J_\gamma(\bar{k}) + \sum_{r\geq 0}\int_{\mathfrak{S}_{\gamma,r}} d\bar{k}\widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3)J_\gamma(\bar{k})\,. \quad (4.30)$$

By (4.28) and (4.4)

$$(4.30) \sim \int_{\mathfrak{S}_\gamma} d\bar{k}\ \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3) = \rho_\gamma\,. \quad (4.32)$$

where by (4.2), (4.3), (4.4)

$$\rho_\gamma \geq \varepsilon^3\,. \quad (4.33)$$

By (4.29)

$$(4.31) \leq C\sum_{r\geq 0} 2^{-r/2}\int_{\mathfrak{S}_{\gamma,r}} d\bar{k}\, \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3)\,. \quad (4.34)$$

Assume first that for all r

$$\int_{\mathfrak{S}_{\gamma,r}} d\bar{k}\, \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3) < c2^{r/10}\rho_\gamma \quad (4.35)$$

for an appropriate constant $c > 0$. Then

$$(4.31) \leq cC\sum_{r\geq 0} 2^{-r/2}2^{r/10}\rho_\gamma < \tfrac{1}{2}\ (4.30) \quad (4.36)$$

and hence by (4.32), (4.33), (4.36)

$$I_\gamma > \tfrac{1}{2}(4.30) \gtrsim \rho_\gamma \gtrsim \varepsilon^3\,. \quad (4.37)$$

Otherwise, denote

$$r_0 = \max\left\{ r \;\middle|\; \int_{\mathfrak{S}_{\gamma,r}} \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3)d\bar{k} > c2^{r/10}\rho_\gamma \right\} \quad (4.38)$$

and let

$$\gamma_1 = 2^{-r_0}\gamma\,. \quad (4.39)$$

By (4.25)

$$\nu(\gamma_1;\bar{k}) \leq 2^{-r_0}\nu(\gamma;\bar{k}) < \tfrac{1}{10} \text{ for } \bar{k} \in \mathfrak{S}_\gamma \cup \bigcup_{r\leq r_0} \mathfrak{S}_{\gamma,r}\,. \quad (4.40)$$

It follows that

$$\mathfrak{S}_\gamma \cup \bigcup_{r\leq r_0} \mathfrak{S}_{\gamma,r} \subset \mathfrak{S}_{\gamma_1} \quad (4.41)$$

and thus, by (4.4)

$$\rho_{\gamma_1} \geq \rho_\gamma + \int_{\mathfrak{S}_{\gamma,r_0}} \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_3) > (1 + c2^{r_0/10})\rho_\gamma > (1+c)\rho_\gamma\,. \quad (4.42)$$

Consequently, by (4.39), (4.42)

$$\frac{\gamma}{\gamma_1} \le C\big(\tfrac{\rho_{\gamma_1}}{\rho_\gamma}\big)^{10} < \big(\tfrac{\rho_{\gamma_1}}{\rho_\gamma}\big)^{C} . \tag{4.43}$$

Starting from $\gamma_1 = 1/10$, $\rho_{\gamma_1} \gtrsim \varepsilon^3$, we construct a decreasing sequence $\{\gamma_s\}$ as follows.

By (4.37), if

$$I_{\gamma_s} \ll \varepsilon^3 \tag{4.44}$$

we get r_0 and let

$$\gamma_{s+1} = 2^{-r_0}\gamma_s \tag{4.45}$$

for which, by (4.42), (4.43)

$$\rho_{\gamma_{s+1}} > (1+c)\rho_{\gamma_s} \tag{4.46}$$

$$\frac{\gamma_s}{\gamma_{s+1}} > \left(\frac{\rho_{\gamma_{s+1}}}{\rho_{\gamma_s}}\right)^C . \tag{4.47}$$

Since $\rho_\gamma \le 1$ by (4.5) it follows from (4.46) that

$$I_{\gamma_{s_*}} \gtrsim \varepsilon^3 \tag{4.48}$$

has to hold for some $s = s_*$. By (4.47), (4.33)

$$\frac{1}{\gamma_{s_*}} = \frac{\gamma_{s_*-1}}{\gamma_{s_*}}\ \frac{\gamma_{s_*-2}}{\gamma_{s_*-1}} \cdots \frac{\gamma_1}{\gamma_2}\ \frac{1}{\gamma_1} \le 10\left(\frac{\rho_{\gamma_{s_*}}}{\rho_{\gamma_{s_*-1}}}\right)^C \left(\frac{\rho_{\gamma_{s_*-1}}}{\rho_{\gamma_{s_*-2}}}\right)^C \cdots \left(\frac{\rho_{\gamma_2}}{\rho_{\gamma_1}}\right)^C < \varepsilon^{-C} . \tag{4.49}$$

Recalling (4.10), (4.48), (4.49) imply that

$$I \ge \gamma_{s_*}^2 I_{\gamma_{s_*}} \gtrsim \varepsilon^{2C+3} \tag{4.50}$$

and thus (4.6).

This proves Lemma 4.1.

Lemma 4.1 is the result that will be applied in the next section. We want however to point out the following consequence.

LEMMA 4.51. *Let A be a measurable subset of $[0,1]$ of measure*

$$\operatorname{mes} A > \delta \tag{4.52}$$

(i) *Given $\rho \in \mathbb{R}$, $|\rho| < 1$, there are $\lambda, \mu \in \mathbb{R}$ such that*

$$\frac{1}{\lambda}, \frac{1}{\lambda+\mu}, \frac{1}{\lambda+2\mu} \in A - A + \rho \tag{4.53}$$

$$|\mu| > \delta^C . \tag{4.54}$$

(ii) *In particular, there are $\lambda, \mu \in \mathbb{R}$ satisfying*

$$\frac{1}{\lambda}, \frac{1}{\lambda+\mu}, \frac{1}{\lambda+2\mu} \in A - A + A \tag{4.55}$$

$$|\mu| > \delta^C. \tag{4.56}$$

Proof. Statement (ii) obviously follows from (i). Rescaling arguments permit us to take $\rho = 1$ in (i). Apply then Lemma 4.1 with $f = \chi_A * \chi_{-A}$ satisfying (4.2), (4.5) with $\varepsilon = \delta^2$ (in fact, we consider $\rho^{-1}A \cap I$ with I an interval of unit length, such that $|\rho^{-1}A \cap I| > \delta$). Hence, by (4.6), there are $\lambda, \mu \in \mathbb{R}$, $|\mu| > \delta^C$ s.t.

$$\frac{1}{\lambda}, \frac{1}{\lambda+\mu}, \frac{1}{\lambda+2\mu} \in 1 + \operatorname{supp} f \subset 1 + A - A\,. \tag{4.57}$$

REMARKS. (i) By similar stationary phase arguments, the preceding may be extended to arithmetic progressions of arbitrary length.

(ii) It is remarkable to notice that if we consider instead the problem of separated progressions

$$\lambda, \lambda+\mu,\ \lambda+2\mu \in A + A - A \tag{4.58}$$

or, more generally

$$\lambda, \lambda+\mu,\ \lambda+2\mu \in \underbrace{A \pm A \pm \cdots \pm A}_{h \text{ terms}} \tag{4.59}$$

with A satisfying (4.52), the known results only permit us to get (4.58), (4.59) with μ satisfying a much weaker property when $\delta \to 0$

$$|\mu| > \exp\left(-\left(\tfrac{1}{\delta}\right)^c\right) \tag{4.60}$$

for some $c > 0$. In this context, we mention for instance the result of [FHRu] (see also [Bo2]): Let $Q \subset [1, N]$ be a set of integers and write $\delta = |Q|/N$. Then $3Q = Q + Q + Q$ contains an arithmetic progression of length at least

$$[c\delta N^{c\delta^3}] \tag{4.61}$$

for some $c > 0$. It is reasonable to conjecture that the true dependence on δ in (4.61) only should involve $\log 1/\delta$ in the exponent.

5 Proof of Proposition 1.20 (I)

Consider next the validity of a maximal inequality

$$\|f_\delta^*\|_{L^p(S_{d-1})} \ll \left(\tfrac{1}{\delta}\right)^{\frac{d}{p}-1+\varepsilon} \|f\|_p\,. \tag{5.0}$$

Hence, using for simplicity the notation $|\cdot|$ to denote the measure in the appropriate spaces,

$$\lambda^p |\mathcal{D}| \ll \left(\tfrac{1}{\delta}\right)^{d-p+\varepsilon} |\mathcal{A}| \tag{5.1}$$

where

$$\mathcal{D} = \left\{\xi \in S_{d-1} \;\middle|\; \chi_\delta^*(\xi) > \lambda\right\} \tag{5.2}$$

$$\chi = \chi_{\mathcal{A}} = \text{ indicator function of } \mathcal{A} \subset \mathbb{R}^d . \tag{5.3}$$

We claim that (5.0), (5.1) may be fulfilled for some

$$p = \left(\tfrac{1}{2} + c\right) d \tag{5.4}$$

for some constant $c > 0$.

Since f_δ^* is obtained by averaging on some tube of width δ, we may clearly in our problem replace f by $f * (\delta^{-d}\chi_{B_\delta})$ and thus assume the set $\mathcal{A}$ considered above is a union of δ-balls.

If then for a δ-tube τ in direction ξ we have

$$\chi_\delta^*(\xi) = |\tau|^{-1} \int_\tau \chi > \lambda \tag{5.5}$$

it follows from Hölder's inequality that

$$|\tau|^{-1} \big(\lambda|\tau|\big)^{1-\frac{1}{p}} \, |\mathcal{A}|^{1/p} > \lambda$$

thus

$$|\mathcal{A}| > \lambda|\tau| \sim \lambda\delta^{d-1} . \tag{5.6}$$

Hence, if $\lambda < \delta$, (5.6) implies for $p \geq 1$

$$|\mathcal{A}| > \delta^{d-p}\lambda^p \tag{5.7}$$

and we may thus assume

$$1 > \lambda > \delta . \tag{5.8}$$

Assume first

$$\lambda > \delta^\tau \tag{5.9}$$

where $\tau > 0$ is a small number (independent of the dimension and to be specified in the next section).

Case (5.9) is the most difficult one. In the next section, we will prove

Lemma 5.10. *With previous setup, assume*

$$|\mathcal{A}| = M\delta^{d/2} \text{ and } |\mathcal{D}| = \kappa . \tag{5.11}$$

Then there exist positive numbers $\tau > 0$, $c > 0$ (independent of d) such that if

$$\chi_\delta^*(\xi) > \delta^\tau \text{ for } \xi \in \mathcal{D} \tag{5.12}$$

one has

$$M > \kappa\delta^{-cd} . \tag{5.13}$$

Consequently, one may take $p = \left(\frac{1}{2} + c\right)d$ in case (5.9).

Next, consider the general case (5.8).

For $\xi \in \mathcal{D} = [\chi_\delta^* > \lambda]$, let $L_\xi /\!/ \xi$ satisfy

$$|L_\xi \cap \mathcal{A}| > \lambda . \tag{5.14}$$

Write

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \tag{5.15}$$

where we define

$$\xi \in \mathcal{D}_1 \text{ if } \operatorname{diam}(L_\xi \cap \mathcal{A}) > \left(\tfrac{\lambda}{\delta}\right)^\tau \lambda \tag{5.16}$$

$$\xi \in \mathcal{D}_2 \text{ if } \operatorname{diam}(L_\xi \cap \mathcal{A}) \leq \left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\,. \tag{5.17}$$

Distinguish the following cases.

Case (i):

$$|\mathcal{D}_1| > \tfrac{1}{2}|\mathcal{D}|\,. \tag{5.18}$$

Consider the map

$$\Phi : \left\{(a, a') \in \mathcal{A} \times \mathcal{A} \;\middle|\; |a - a'| > \left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\right\} \to \left(\tfrac{\lambda}{\delta}\right)^\tau \lambda S_{d-1}$$

given by

$$\Phi(a, a') = \left(\frac{\lambda}{\delta}\right)^\tau \lambda \frac{a - a'}{|a - a'|}\,. \tag{5.19}$$

Since Φ is Lipschitz and $\operatorname{Im}\Phi \supset (\lambda/\delta)^\tau \lambda \mathcal{D}_1$, we have for the δ-entropy-numbers

$$\mathcal{N}_\delta(\mathcal{A})^2 \geq \mathcal{N}_\delta\big(\left(\tfrac{\lambda}{\delta}\right)^\tau \lambda \mathcal{D}_1\big) > c\big(\left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\big)^{d-1} \operatorname{mes}(\mathcal{D}_1)\,.\delta^{1-d} \tag{5.20}$$

hence, by (5.18) and the assumption on $\mathcal{A}$

$$\mathcal{N}_\delta(\mathcal{A}) > c\big(\left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\big)^{\frac{d-1}{2}} |\mathcal{D}|^{1/2} \delta^{\frac{1-d}{2}} \tag{5.21}$$

$$|\mathcal{A}| > c\big(\left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\big)^{\frac{d-1}{2}} |\mathcal{D}|^{1/2} \delta^{\frac{1+d}{2}}\,. \tag{5.22}$$

Thus, from (5.22), in order to fulfill (5.1), it suffices that

$$\lambda^p \ll \left(\tfrac{1}{\delta}\right)^{d-p+\varepsilon} \big(\left(\tfrac{\lambda}{\delta}\right)^\tau \lambda\big)^{\frac{d-1}{2}} \delta^{\frac{1+d}{2}} \tag{5.23}$$

$$\lambda^{p - \frac{d-1}{2}(1+\tau)} \ll \delta^{p - \frac{d-1}{2}(1+\tau) - \varepsilon} \tag{5.24}$$

hence, by (5.8)

$$p \leq \frac{d-1}{2}(1 + \tau)\,. \tag{5.25}$$

For d sufficiently large, this gives again (5.4).

Case (ii): $|\mathcal{D}_2| > \frac{1}{2}|\mathcal{D}|$.

Rescale the problem multiplying by $(\lambda/\delta)^{-\tau}\lambda^{-1}$. Thus

$$\delta \to \delta' = \left(\tfrac{\delta}{\lambda}\right)^{1+\tau} \tag{5.26}$$

$$\lambda \to \lambda' = \left(\tfrac{\delta}{\lambda}\right)^\tau \tag{5.27}$$

$$\mathcal{A} \to \mathcal{A}' = \lambda^{-1} \left(\tfrac{\delta}{\lambda}\right)^\tau \mathcal{A}\,. \tag{5.28}$$

For $\xi \in \mathcal{D}_2$, (5.14), (5.17) imply

$$(\chi_{\mathcal{A}'})^*_{\delta'}(\xi) > \lambda' > (\delta')^\tau . \tag{5.29}$$

If τ is choosen as in (5.10), it follows from (5.26)–(5.28) that for $p = \big(\frac{1}{2}+c\big)d$

$$(\lambda')^p|\mathcal{D}_2| < \big(\tfrac{1}{\delta'}\big)^{d-p}\,|\mathcal{A}'|$$

$$\big(\tfrac{\delta}{\lambda}\big)^{\tau p}\,|\mathcal{D}| \lesssim \big(\tfrac{\lambda}{\delta}\big)^{(1+\tau)(d-p)}\,\big(\lambda^{-1}\,\big(\tfrac{\delta}{\lambda}\big)^\tau\,\big)^d|\mathcal{A}|$$

hence

$$\lambda^p|\mathcal{D}| \lesssim \big(\tfrac{1}{\delta}\big)^{d-p}\,|\mathcal{A}| \tag{5.30}$$

which is (5.1). This proves Proposition 1.20.

6 Proof of Proposition 1.20 (II)

It remains to prove Lemma 5.10.

Assume thus $\mathcal{A} \subset B(0,1) \subset \mathbb{R}^d$ a union of b-balls such that, letting $\chi = \chi_{\mathcal{A}}$

$$\chi^*_\delta(\xi) > \delta^\tau \text{ for } \xi \in \mathcal{D} \subset S^{d-1} \tag{6.1}$$

with $\tau > 0$ small (independently of d).

Thus for $\xi \in \mathcal{D}$, there are $a_\xi, b_\xi \in \mathcal{A}$ such that

$$a_\xi - b_\xi /\!/ \xi \tag{6.2}$$

$$\|a_\xi - b_\xi\| > \delta^\tau \tag{6.3}$$

$$\text{mes}\big([a_\xi, b_\xi] \cap \mathcal{A}\big) > \delta^\tau . \tag{6.4}$$

Define a function $F = F(\lambda, \xi)$ as follows

$$F(\lambda,\xi) = 1 \text{ if } \lambda a_\xi + (1-\lambda)b_\xi \in \mathcal{A} \tag{6.5}$$

$$= 0 \text{ otherwise}. \tag{6.6}$$

We may by (6.4) clearly assume that

$$\int_{1/2}^1 F(\lambda,\xi)d\lambda > \delta^\tau \text{ for } \xi \in \mathcal{D} \tag{6.7}$$

and restrict F to $[1/2, 1]$.

Define

$$f(x,\xi) = \int_1^2 F\left(\frac{1}{x+x'},\xi\right) F\left(\frac{1}{x'},\xi\right) dx' . \tag{6.8}$$

Hence

$$\text{supp}\, f_\xi \subset [-1,1] \tag{5.9}$$

$$\int f_\xi(x)dx = \left(\int_1^2 F\left(\tfrac{1}{y},\xi\right) dy\right)^2 \gtrsim \delta^{2\tau} \tag{6.10}$$

and also

$$\widehat{f_\xi}(k) = \widehat{F_\xi\left(\tfrac{1}{\cdot}\right)}(k)\widehat{F_\xi\left(\tfrac{1}{\cdot}\right)}(-k) = \left|\widehat{F_\xi\left(\tfrac{1}{\cdot}\right)}(k)\right|^2 \geq 0 \tag{6.11}$$

$$\int \widehat{f_\xi}(k)dk = \int_1^2 F\left(\tfrac{1}{y}, \xi\right) dy \leq 1\,. \tag{6.12}$$

Thus $f = f_\xi$ satisfies (4.2)–(4.5) with $\varepsilon \sim \delta^{2\tau}$.

Applying then Lemma 4.1 gives for $\xi \in \mathcal{D}$ fixed

$$\iint f\left(1-\frac{1}{\lambda}, \xi\right) f\left(1-\frac{1}{\lambda+\mu}, \xi\right) f\left(1-\frac{1}{\lambda+2\mu}, \xi\right) \varphi_1(\lambda)\varphi_2(\mu)d\lambda d\mu > \delta^{C\tau}\,. \tag{6.13}$$

Next, substituting (6.8) and integrating (6.13) in $\xi \in \mathcal{D}$, gives

$$\int \varphi_1(\lambda)\varphi_2(\mu)d\lambda\, d\mu \cdot \int_1^2 dx_0dx_1dx_2 \int_{\mathcal{D}} d\xi\, \Pi_{\ell=0,1,2} F\left(\frac{1}{1-\frac{1}{\lambda+\ell\mu}+x_\ell}, \xi\right) F\left(\frac{1}{x_\ell}, \xi\right) > \delta^{C\tau}|\mathcal{D}|\,. \tag{6.14}$$

Thus, by Fubini, there are $\lambda, \mu, x_0, x_1, x_2$ and $\mathcal{D}_1 \subset \mathcal{D}$ such that

$$|\mu| > \delta^{C\tau} \tag{6.15}$$

$$|\mathcal{D}_1| > \delta^{C\tau}|\mathcal{D}| \tag{6.16}$$

$$F\left(\frac{1}{x_\ell}, \xi\right) = 1 = F\left(\frac{1}{1-\frac{1}{\lambda+\ell\mu}+x_\ell}, \xi\right),\ \ \ell = 0,1,2 \text{ and } \xi \in \mathcal{D}_1\,. \tag{6.17}$$

Denote then

$$\lambda_\ell = \tfrac{1}{x_\ell} \quad (\ell = 0,1,2) \tag{6.18}$$

$$\frac{1}{\lambda+\ell\mu} = 1 + \frac{1}{\lambda_\ell} - \frac{1}{\wedge_\ell} \quad (\ell = 0,1,2)\,. \tag{6.19}$$

Thus $1/2 \leq \lambda_\ell, \wedge_\ell$ and we clearly assume also (from the Fubini argument)

$$\tfrac{1}{2} \leq \lambda_\ell\,, \quad \wedge_\ell < 1 - \delta^{C\tau}\,. \tag{6.20}$$

(Recall that in exponent notation C refers to a constant independent of d).

It follows from (6.17) that for $\xi \in \mathcal{D}_1$

$$F(\lambda_\ell, \xi) = 1 = F(\wedge_\ell, \xi) \quad (\ell = 0,1,2)$$

which means that, by (6.5)–(6.6), for $\xi + \mathcal{D}_1$

$$\lambda_\ell a_\xi + (1-\lambda_\ell)b_\xi \in \mathcal{A} \tag{6.21}$$

$$\wedge_\ell a_\xi + (1-\wedge_\ell)b_\xi \in \mathcal{A}\,. \tag{6.22}$$

Recall (5.11)

$$|\mathcal{A}| = M\delta^{d/2} \tag{6.23}$$

$$|\mathcal{D}| = \kappa \tag{6.24}$$

hence, by (6.16)

$$|\mathcal{D}_1| > \delta^{C\tau}\kappa\,. \tag{6.25}$$

At this stage, apply Lemma 2.113 with $A = B = \mathcal{A}$, $\mathcal{G} \subset \{(a_\xi, b_\xi) \mid \xi \in \mathcal{D}_1\}$ and considering the positive numbers

$$\xi_\ell = \lambda_\ell \qquad \eta_\ell = 1 - \lambda_\ell \qquad (\ell = 0,1,2) \tag{6.26}$$

$$\xi_\ell = \wedge_{\ell-3} \qquad \eta_\ell = 1 - \wedge_{\ell-3} \qquad (\ell = 3,4,5) \tag{6.27}$$

(thus $\ell_0 = 6$). Since $\mathcal{A}$ was assumed a collection of δ-balls, (6.23) implies

$$\mathcal{N}_\delta(\mathcal{A}) < CM\delta^{-d/2} \equiv N\,. \tag{6.28}$$

By (6.20), condition (2.116) holds with τ replaced by $C\tau$. Condition (2.117) is implied by (6.21), (6.22). By (6.25), there is a finite subset $\mathcal{D}_2 \subset \mathcal{D}_1 \subset S_{d-1}$ such that

$$\|\xi - \xi'\| > \delta \text{ for } \xi \neq \xi' \text{ in } \mathcal{D}_2 \tag{6.29}$$

and

$$|\mathcal{D}_2| > c\left(\tfrac{1}{\delta}\right)^{d-1}\delta^{C\tau}\kappa\,. \tag{6.30}$$

Define

$$\mathcal{G} = \Big\{(a_\xi, b_\xi) \;\Big|\; \xi \in \mathcal{D}_2\Big\}\,. \tag{6.31}$$

Again considering the Lipschitz map $\{(a,b)\in\mathcal{A}\times\mathcal{A} \mid \|a-b\|>\delta^\tau\} \xrightarrow{\Phi} \delta^\tau S_{d-1}$.

$$\Phi(a,b) = \delta^\tau \frac{a-b}{\|a-b\|} \tag{6.32}$$

and recalling (6.2), (6.3), it follows thus from (6.29), (6.30) that

$$\mathcal{N}_\delta(\mathcal{G}) \geq \mathcal{N}_\delta(\delta^\tau \mathcal{D}_2) > c\delta^{\tau(d-1)}|\mathcal{D}_2| > c\delta^{\tau(d+C)+1}\kappa\delta^{-d}\,. \tag{6.33}$$

Therefore, from (6.28), (6.33), condition (2.115) holds with

$$\alpha = \kappa M^{-2}\delta^{\tau(d+C)+1}\,. \tag{6.34}$$

Lemma 2.113 gives then

$$\begin{aligned} A' &\subset \mathcal{A} + B_{C\delta^{1-C\tau}} \\ B' &\subset \mathcal{A} + B_{C\delta^{1-C\tau}} \end{aligned} \tag{6.35}$$

such that the following properties hold (cf. also (2.139)).

$$\mathcal{N}_\delta(A' + A' - A') < (\alpha\delta^{d\tau})^{-C} N \tag{6.36}$$

$(1-\lambda_\ell)B'$ is contained in a translate of $\lambda_\ell A'$ (ℓ=0, 1, 2) (6.37)

$(1-\wedge_\ell)B'$ is contained in a translate of $\wedge_\ell A'$ (ℓ=0, 1, 2) (6.38)

$$\big|\{\xi \in \mathcal{D}_2 \mid a_\xi \in A', b_\xi \in B'\}\big| \geq \mathcal{N}_\delta\big(\mathcal{G} \cap (A' \times B')\big) > (\alpha\delta^{d\tau})^C N^2\,. \tag{6.39}$$

Returning then to (6.19), it follows that for $\ell = 0, 1, 2$

$$A' + \left(\frac{1}{\lambda + \ell\mu} - 1\right)B' \subset A' + \left(\frac{1}{\lambda_\ell} - 1\right)B' - \left(\frac{1}{\wedge_\ell} - 1\right)B' \quad (6.40)$$

is contained in a translate of $A' + A' - A'$. Hence, from (6.36)

$$\mathcal{N}_\delta\left(A' + \left(\frac{1}{\lambda + \ell\mu} - 1\right)B'\right) < (\alpha\delta^{d\tau})^{-C} N \quad (6.41)$$

$$\mathcal{N}_\delta\big((\lambda + \ell\mu)A' + (1 - \lambda - \ell\mu)B'\big) < C(\alpha\delta^{d\tau})^{-C} N \quad (6.42)$$

for $\ell = 0, 1, 2$.

Next, we will apply Lemma 2.83. Take

$$A = \lambda A' + (1 - \lambda)B' \quad (6.43)$$

$$B = (\lambda + 2\mu)A' + (1 - \lambda - 2\mu)B' \quad (6.44)$$

$$\mathcal{G} = \big\{(\lambda a' + (1-\lambda)b', (\lambda+2\mu)a' + (1-\lambda-2\mu)b') \bigm| a' \in A', b' \in B'\big\} \subset A \times B \quad (6.45)$$

satisfying (2.84), 2.85) with N replaced by $(\alpha\delta^{d\tau})^{-C}N$, from (6.42).

Consequently, (2.86) implies that

$$\mathcal{N}_\delta\big(\{a - b \mid (a, b) \in \mathcal{G}\}\big) < CN^{\frac{25}{13}+} \quad (6.46)$$

hence, from (6.43)–(6.45) and (6.15)

$$\mathcal{N}_\delta(A' - B') < C\mu^{-d} N^{\frac{25}{13}+} < C\delta^{-Cd\tau} N^{\frac{25}{13}}\,. \quad (6.47)$$

Define

$$\mathcal{D}_3 = \{\xi \in \mathcal{D}_2 \mid a_\xi \in A',\, b_\xi \in B'\} \quad (6.48)$$

which satisfies thus, by (6.29), (6.39)

$$|\mathcal{D}_3| > (\alpha\delta^{d\tau})^C N^2 \quad (6.49)$$

$$\mathcal{N}_\delta(\mathcal{D}_3) > c(\alpha\delta^{d\tau})^C N^2\,. \quad (6.50)$$

Hence, by (6.47), (6.50) and (6.2), (6.3)

$$\begin{aligned} C\delta^{-Cd\tau} N^{25/13} &> \mathcal{N}_\delta\big(\{a_\xi - b_\xi \mid \xi \in \mathcal{D}_3\}\big) > c\mathcal{N}_\delta(\delta^\tau \mathcal{D}_3) \\ &> c\delta^{d\tau}\mathcal{N}_\delta(\mathcal{D}_3) > c(\alpha\delta^{d\tau})^C N^2\,. \end{aligned} \quad (6.51)$$

Substituting (6.28), (6.34) in (6.51) implies

$$(CM\delta^{-\frac{d}{2}})^{\frac{1}{13}} = N^{\frac{1}{13}} < C(\alpha\delta^{d\tau})^{-C} < C(\kappa M^{-2}\delta^{1+Cd\tau})^{-C} \quad (6.52)$$

$$\left(\tfrac{1}{\delta}\right)^{d(\frac{1}{26} - C\tau) - C} < \left(\tfrac{M}{\kappa}\right)^C\,. \quad (6.53)$$

Assuming τ small enough and d large enough, (5.13) clearly follows. This completes the proof of Lemma 5.10 and hence of Proposition 1.20.

REMARK. The improvement $p \leq \left(\frac{1}{2}+c\right)d$ obtained in Proposition 1.20 over the known result $p \leq \frac{d}{2}+1$ [W1] requires d to be sufficiently large and in particular does not improve on he $p = 5/2$ exponent for $d = 3$. It is based on some different combinatorial aspects however and establishes a connection between the Besicovitch problem and a collection of classical results in additive number theory, cf. [N].

References

[BS] A. BALOG, E. SZEMERÉDI, A statistical theorem of set addition, Combinatorica 14 (1994), 263–268.

[Bo1] B1 J. BOURGAIN, Besicovitch type maximal operators and applications to Fourier Analysis, GAFA 1 (1991), 147–187.

[Bo2] J. BOURGAIN, On arithmetic progressions in sums of sets of integers, in “A Tribute to L. Erdös” (Baker, Bollabás, Hajnal, eds.), Cambridge UP, (1990), 105–110.

[FHRu] G. FREIMAN, H. HALBERSTAM, I. RUZSA, Integer sum sets containing long arithmetic progressions, J. London Math. Soc. (2) 46 (1992), 193–201.

[G] T. GOWERS, A new proof of Szemerédi’s theorem for arithmetic progressions of length four, GAFA 8 (1998), 529–551.

[D] R.O. DAVIES, Some remarks on the Kakeya problem, Proc. Cambridge Phil. Soc. 69 (1971), 417–421.

[He] D. HEATH-BROWN, Integer sets containing no arithmetic progressions, J. London Math. Soc. (2) 35 (1987), 385–394.

[N] M. NATHANSON, Additive Number Theory: Inverse Problems and the Geometry of Sumsets, Springer Ergibnisse der Math, Springer Verlag, 1996.

[R] K. ROTH, On certain sets of integers, J. London Math. Soc. 28 (1953), 104–109.

[Ru] I. RUZSA, Sums of finite sets, in “Number Theory: New York Seminar” (D.V. Chudnovsky, G.V. Chudnovsky, M.B. Nathanson, eds.), Springer-Verlag, 1996.

[S] E. SZEMERÉDI, Integer sets containing no arithmetic progressions, Acta Math. Hungar. 56 (1990), 155–158.

[W1] T. WOLFF, An improved bound for Kakeya type maximal functions, Revista Math Iberoamericana 11 (1995), 651–674.

[W2] T. WOLFF, Recent work connected with the Kakeya problem, in “Prospects in Mathematics” (Princeton, NJ, 1996), AMS (1999), 129–162.

Jean Bourgain, Inst. of Advanced Study, Olden Lane, Princeton, NJ 08540, USA
bourgain@math.ias.edu
Submitted: September 1998