Annals of Mathematics, **102** (1975), 159-182

Inequalities in Fourier analysis

By WILLIAM BECKNER

1. Introduction

Inequalities are a basic tool in the study of Fourier analysis. The classical result relating L^p estimates for a function and its Fourier transform is the Hausdorff-Young theorem. For an integrable function on $\mathbf{R}^n$ the Fourier transform is given by

$$(\mathcal{F}f)(x) = \int e^{2\pi i x y} f(y)dy \ .$$

The Fourier transform $\mathcal{F}$ then extends to a bounded linear operator on $L^p(\mathbf{R}^n)$, $1 \leqq p \leqq 2$, and

$$(1) \qquad \|\mathcal{F}f\|_{p'} \leqq \|f\|_p \ .$$

Here p' denotes the dual exponent, $1/p + 1/p' = 1$. This result can be obtained by using the M. Riesz convexity theorem to interpolate between the end-point estimates for $L^1(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$.

This theorem holds generally for analysis on any locally compact abelian group. The first proofs were given for the case of the circle group $\mathbf{T} \approx \mathbf{R}/\mathbf{Z}$ and developed from the efforts of W. H. Young to extend the Parseval theorem for Fourier series to other L^p classes. In terms of the basic relationship between the Fourier transform and convolution, Young observed that an inequality for the Fourier transform could be obtained from a convolution inequality. For integrable functions on $\mathbf{R}^n$ the convolution of two functions is given by

$$(f * g)(x) = \int f(x - y)g(y)dy$$

and under the action of the Fourier transform, convolution goes over to pointwise multiplication.

$$\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g) \ .$$

By the careful application of Hölder's inequality, one can obtain Young's inequality for convolution:

$$(2) \qquad \|f * g\|_r \leqq \|f\|_p \|g\|_q$$

where $1 \leqq p, q, r \leqq \infty$ and $1/r = 1/p + 1/q - 1$. In some sense there is a duality between inequalities (1) and (2) reflecting the basic identification by the Fourier transform of the L^1 convolution algebra with a pointwise multiplication algebra of bounded functions, and the essential object of study here is the group action of translation on $\mathbf{R}^n$. For p' an even integer, that is, $p' = 2, 4, 6, \cdots$, the Fourier transform inequality (1) can be obtained from the convolution inequality (2) (this was Young's observation); and in turn for $1 \leqq p, q, r' \leqq 2$, the convolution inequality can be obtained from the Fourier transform inequality. However, we should remark that convolution is essentially a positive operation, and so convolution arguments are likely to be conceptually easier than arguments for the Fourier transform.

These inequalities are sharp on the circle group $\mathbf{T}$; that is, there exist extremal functions for which equality between norms is attained. In fact, there is a theorem of Hardy and Littlewood that equality in (1) will be attained only for exponential functions

$$f(x) = Ae^{2\pi i m x} \qquad m \in \mathbf{Z}$$

(see Zygmund, Vol. II, page 105). However, on the real line a much sharper inequality for special values of p was obtained by Babenko in 1961 ([1]). For p' an even integer, i.e., $p' = 2, 4, 6, \cdots$, Babenko proved, using methods of entire functions, that the best norm for the Fourier transform inequality would be attained for gaussian functions,

$$f(x) = e^{-\alpha x^2}\,, \qquad \alpha > 0\,.$$

In Babenko's proof the Fourier transform is regularized by composition with the classical Mehler kernel for Hermite functions. The resulting operator is compact, and therefore by a weak compactness argument a solution will exist to the corresponding extremal problem for the maximum norm of this operator over a bounded set. This extremal solution will satisfy two non-linear integral equations, and then using Phragmen-Lindelöf methods and rearranging contour integrals in the complex plane, one can calculate the value of the maximum norm for this extremal problem. Then a limiting argument gives a sharp L^p inequality for the Fourier transform. The importance of the even integer values of p' is that all functions appearing in the integral equations will have entire extensions.

Though Babenko's proof held only for the special values $p' = 2, 4, 6, \cdots$, it was clear by a convexity argument that inequalities (1) and (2) would have new sharp forms on $\mathbf{R}^n$ for general p. In the work described in this paper we have obtained sharp L^p inequalities for both the Fourier transform and convolution on $\mathbf{R}^n$. That is, we give precise values for the norms

$$\mathcal{A} = \sup \frac{\|\mathcal{F}f\|_{p'}}{\|f\|_p}$$

$1 \leqq p \leqq 2$, $1/p + 1/p' = 1$; and

$$\mathcal{C} = \sup \frac{\| f * g \|_r}{\| f \|_p \, \| g \|_q}$$

with $1/r = 1/p + 1/q - 1$. These values will be attained for gaussian functions.

On reflection, the role of the gaussian function should not be considered too surprising. First, the function $\exp(-\pi x^2)$ is invariant under the action of the Fourier transform, and the Hermite functions are eigenfunctions of the Fourier transform. But consider the problem in n dimensions. Here one might expect that if an extremal function exists, it should be rotationally invariant. Also, we might suppose this extremal function to be essentially unique, that is, up to the basic symmetry operations of Fourier analysis. The norm in n dimensions will be a power of the one-dimensional norm so we would want an extremal function for which a product of functions radial in separate variables is also radial in the variables jointly. This is possible only for a gaussian function. This simple heuristic idea suggested the basic structure of our proof for sharp convolution inequalities.

Suppose we assume that Babenko's inequality held on the line for general p and write out the resulting inequality in terms of Hermite expansions. Let

$$f(x) = \sum a_n H_n(x) e^{-\pi x^2} ,$$
$$(\mathcal{F} f)(x) = \sum a_n i^n H_n(x) e^{-\pi x^2} .$$

Here the $\{H_n(x)\}$ are the Hermite polynomials corresponding to the gaussian measure $dw(x) = \sqrt{2} \cdot \exp(-2\pi x^2) dx$.

$$\| \mathcal{F} f \|_{p'} \leqq [p^{1/p}/p'^{1/p'}]^{1/2} \| f \|_p ,$$

$$\left\{ \sqrt{p'} \int | \sum a_n i^n H_n(x) |^{p'} e^{-p' \pi x^2} dx \right\}^{1/p'} \leqq \left\{ \sqrt{p} \int | \sum a_n H_n(x) |^p e^{-\pi p x^2} dx \right\}^{1/p} .$$

This format is very suggestive that, notwithstanding some interplay between the dilation group and the Hermite semigroup, Babenko's inequality is related to multiplier problems on the Hermite semigroup.

The gaussian function has an intrinsic character both as an entire function and as a probability distribution. Analysis of the Hermite semigroup and gaussian measures has played an important role in the study of quantization in quantum field theory, particularly in the work of Segal, Nelson, Glimm, and Gross. The application of probabilistic methods to the study of the Hermite semigroup in recent work of Nelson ([6]) and Gross ([2]) suggested a different approach to obtaining sharp inequalities for the Fourier transform on $\mathbf{R}^n$. Nelson used stochastic integrals and gaussian processes to obtain a basic multiplier inequality on the Hermite semigroup (see Section IV). Gross then gave a proof of Nelson's result directly on the line by using the central limit theorem to obtain the gaussian measure from a sequence of Bernoulli trials. Both proofs emphasized the product character of the multiplier operator. The structure of our proof follows the general methods of Nelson and Gross in using the product character of multiplier operators to obtain sharp L^p estimates.

In Section II we prove a sharp Hausdorff-Young result, that is, Babenko's inequality, for the Fourier transform on $\mathbf{R}^n$ for the full range of values of p, $1 \leqq p \leqq 2$;

$$\|\mathcal{F}f\|_{p'} \leqq (A_p)^n \|f\|_p ,$$
$$A_p = [p^{1/p}/p'^{1/p'}]^{1/2} .$$

In Section III we obtain a sharp Young's inequality for convolution on $\mathbf{R}^n$,

$$\|f * g\|_r \leqq (A_p A_q A_{r'})^n \|f\|_p \|g\|_q .$$

In Section IV we make some brief remarks about the relation of these results to other problems in harmonic analysis.

I would especially like to express my appreciation to Elias M. Stein for having given advice, encouragement and many helpful suggestions during the course of this work.

II. Babenko's inequality

We obtain the following sharp inequality for the Fourier transform on $\mathbf{R}^n$:

THEOREM 1. *$\mathcal{F}\colon L^p(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$, $1 < p \leqq 2$ with*

$$\|\mathcal{F}f\|_{p'} \leqq (A_p)^n \|f\|_p , \qquad (3)$$
$$A_p = [p^{1/p}/p'^{1/p'}]^{1/2} .$$

Consider the case $n = 1$; the Fourier transform on $\mathbf{R}^n$ splits naturally into a product of one-dimensional operators and the result for n dimensions will follow from an application of Lemma 2 (on products of operators) below. As suggested in the preceding section, inequalities for the Fourier transform are equivalent to multiplier inequalities on the Hermite semigroup. Let $\{H_m(x)\}$ denote the Hermite polynomials corresponding to the gaussian measure[1]

[1] Note that we have changed normalization for the Hermite polynomials from our remark in Section I. The usual properties for the Hermite polynomials are easily obtained by using the generating function below. Also, see Vilenkin, *Special Functions and the Theory of Group Representations*; and Szegö, *Orthogonal Polynomials*.

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \;;$$

that is,

$$H_m(x) = \int (x + iy)^m d\mu(y)$$

and we have the generating function

$$\exp\left(-\frac{t^2}{2} + xt\right) = \sum_{m=0}^{\infty} \frac{1}{m!} t^m H_m(x) \,.$$

For $|\omega| < 1$ consider the multiplier operator defined on basis elements by

$$T_\omega \colon H_m \longrightarrow \omega^m H_m \,.$$

Observe that $T_{\omega_1} T_{\omega_2} = T_{\omega_1 \omega_2}$. This operator can be expressed as an integral operator on $L^2(d\mu)$ defined by the Mehler kernel

$$T_\omega(x, y) = (1 - \omega^2)^{-1/2} \exp\left\{-\frac{\omega^2(x^2 + y^2)}{2(1 - \omega^2)} + \frac{\omega x y}{1 - \omega^2}\right\},$$

$$(T_\omega g)(x) = \int T_\omega(x, y) g(y) d\mu(y) \,.$$

Then Theorem 1 for $n=1$ is equivalent to the following multiplier inequality.

THEOREM 2. *For $\omega = i\sqrt{p-1}$ and $1 < p \leqq 2$,*

$$\begin{aligned} &T_\omega \colon L^p(d\mu) \longrightarrow L^{p'}(d\mu)\,, \\ &\| T_\omega g \|_{L^{p'}(d\mu)} \leqq \| g \|_{L^p(d\mu)} \,. \end{aligned} \tag{4}$$

Suppose we consider equation (4) for polynomials $g(x)$. Then by a simple change of variables this equation is equivalent to equation (3), holding for $n = 1$ and functions $f(x) = g(\sqrt{2\pi p}x) \exp(-\pi x^2)$. This is sufficient to establish the equivalence of the two theorems since each will hold for a dense set of functions. Explicitly, equation (4) can be written

$$\left\{\int \left| \int T_\omega(x, y) g(y) d\mu(y) \right|^{p'} d\mu(x)\right\}^{1/p'} \leqq \left\{\int | g(x) |^p \, d\mu(x)\right\}^{1/p} .$$

On the left-hand side of the equation make the change of variables $x = \sqrt{2\pi p'}\,u$ and $y = \sqrt{2\pi p}\,v$, and substitute the value $\omega = i\sqrt{p-1}$; on the right-hand side of the equation make the change of variables $x = \sqrt{2\pi p}\,u$. Then we obtain

$$\left\{\int \left| \int e^{2\pi i u v} g(\sqrt{2\pi p}\,v) e^{-\pi v^2} dv \right|^{p'} du\right\}^{1/p'} \leqq A_p \left\{\int \left| g(\sqrt{2\pi p}\,u) e^{-\pi u^2} \right|^p du\right\}^{1/p} .$$

We prove this multiplier inequality for the Hermite semigroup by using the classical central limit theorem to obtain the gaussian measure $d\mu$ as a limiting probability measure. Suppose we consider a sequence of Bernoulli trials; that is, let $d\nu(x)$ be the discrete probability measure with positive weight 1/2 at the points $x = \pm 1$, and let $d\nu_n(x)$ be the n-fold convolution of the measure $d\nu(\sqrt{n}x)$ with itself. Then $d\nu_n$ converges to $d\mu$ in $C_0(\mathbf{R})^*$, and moreover the moments of $d\nu_n$ will converge to the moments of $d\mu$. For $h \in C_0(\mathbf{R})$

$$\int h(x)d\nu_n(x) = \int h(x_1 + \cdots + x_n)d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n) ,$$
$$\lim \int h(x)d\nu_n(x) = \int h(x)d\mu(x) .$$

At each stage in the convergence process we prove an analogue of the basic multiplier inequality of Theorem 2 with respect to the product measure $d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n)$; then we obtain the final multiplier inequality for the Hermite semigroup as a limit of inequalities with respect to these product measures. The product measures $d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n)$ are discrete measures; that is, the x_i can assume only two values, $\pm 1/\sqrt{n}$. So all functions over these measure spaces will be polynomials of degree at most one in each of the n variables. By imitating the action of the multiplier operator T_ω with respect to the initial Hermite polynomials (that is, $H_0(x) = 1$ and $H_1(x) = x$), we define an analogue C of the multiplier T_ω on the measure space over $d\nu$:

$$T_\omega\colon aH_0 + bH_1 \longrightarrow aH_0 + \omega bH_1$$
$$C\colon a + bx \longrightarrow a + \omega bx .$$

The initial problem is to show that C is a bounded linear operator on $L^p(d\nu)$ to $L^{p'}(d\nu)$ with norm one; this result is contained in Lemma 1 below. In general, we define operators

$$C_{n,k}\colon a + bx_k \longrightarrow a + \omega bx_k$$

where a and b are functions of the remaining $n - 1$ variables; and

$$K_n = C_{n,1} \cdots C_{n,n} .$$

By a lemma on products of operators (Lemma 2 below), the "two-point inequality" for C, that is the operator C having norm one, will imply that K_n is a bounded linear operator with norm one on $L^p[d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n)]$ to $L^{p'}[d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n)]$. The restriction $\bar{K}_n$ of the operator K_n to the subspace of functions symmetric in the n variables will also be a linear operator of norm one. We denote this function space of symmetric functions over the product measure $d\nu(\sqrt{n}x_1) \cdots d\nu(\sqrt{n}x_n)$ by X_n.

The functions

$$\varphi_{n,l}(x_1, \cdots, x_n) = l!\, \sigma_l(x_1, \cdots, x_n)$$

form an orthogonal basis in $L^2[X_n]$. The σ's are the elementary symmetric functions in n variables;

$$\sigma_l(x_1, \cdots, x_n) = \sum_{m_1 < \cdots < m_l} x_{m_1} \cdots x_{m_l}$$

and $1 \leqq m_i \leqq n$. The generating function for the symmetric functions is given by

$$\begin{aligned}\mathcal{T}(x_1, \cdots, x_n; t) &= \prod_{k=1}^n (1 + x_k t) = \sum_{l=0}^n t^l \sigma_l(x_1, \cdots, x_n) \\ &= \sum_{l=0}^n \frac{1}{l!} t^l \varphi_{n,l}(x_1, \cdots, x_n) .\end{aligned}$$

For the Hermite polynomials the generating function is given by

$$T(x; t) = e^{-t^2/2 + tx} = \sum_{m=0}^\infty \frac{1}{m!} t^m H_m(x) .$$

Observe that if $x = x_1 + \cdots + x_n$ with $(x_i)^2 = 1/n$, then we can obtain the following relations between the two generating functions

$$\begin{aligned}T(x_1 + \cdots + x_n; t) &= e^{-t^2/2}[\cosh (t/\sqrt{n})]^n \mathcal{T}[x_1, \cdots, x_n; \sqrt{n} \tanh (t/\sqrt{n})] , \\ \mathcal{T}(x_1, \cdots, x_n; t) &= \exp\left\{\frac{1}{2}[\sqrt{n} \tanh^{-1} (t/\sqrt{n})]^2\right\}\left(1 - \frac{t^2}{n}\right)^{n/2} \\ &\quad \times T[x_1 + \cdots + x_n; \sqrt{n} \tanh^{-1} (t/\sqrt{n})] .\end{aligned}$$

The first relation follows from

$$\begin{aligned}T(x_1 + \cdots + x_n; t) &= e^{-t^2/2} \prod_{k=1}^n e^{tx_k} \\ &= e^{-t^2/2} \prod_{k=1}^n [\cosh (t/\sqrt{n}) + x_k \sqrt{n} \sinh (t/\sqrt{n})]\end{aligned}$$

and the second relation is obtained by inversion. Then by differentiation one can express $\varphi_{n,l}(x_1, \cdots, x_n)$ as a linear combination of the Hermite polynomials $H_k(x_1 + \cdots + x_n)$, $0 \leqq k \leqq l$. In fact, we will have

$$(5) \quad \varphi_{n,l}(x_1, \cdots, x_n) = H_l(x_1 + \cdots + x_n) + \frac{1}{n} \sum_{r=1}^{[l/2]} a_{l,r} H_{l-2r}(x_1 + \cdots + x_n)$$

where the coefficients $a_{l,r}$ are bounded with respect to n for fixed l (see Appendix, Part 1). The functions $\varphi_{n,l}$ are eigenfunctions of the operators $\bar{K}_n$:

$$\begin{aligned}\bar{K}_n\colon x_{m_1} \cdots x_{m_l} &\longrightarrow \omega^l x_{m_1} \cdots x_{m_l} \\ \bar{K}_n \varphi_{n,l} &= \omega^l \varphi_{n,l} ,\end{aligned}$$

and they are the natural analogues of the Hermite polynomials over the function spaces $L^2[X_n]$.

We have argued that $\bar{K}_n$ maps $L^p[X_n]$ into $L^{p'}[X_n]$ with norm one; it remains to show that these inequalities imply the multiplier inequality of

Theorem 2. It suffices to prove this result for a dense set of functions, namely polynomials. Any polynomial g can be expressed as a finite linear combination of Hermite polynomials

$$g(x) = \sum_{l=1}^{M} b_l H_l(x) .$$

Define corresponding polynomials in the discrete variables $x_1, \cdots, x_n$ by

$$g_n(x_1, \cdots, x_n) = \sum_{l=1}^{M} b_l \varphi_{n,l}(x_1, \cdots, x_n) .$$

Then

$$T_\omega \colon \sum b_l H_l(x) \longrightarrow \sum \omega^l b_l H_l(x)$$
$$\bar{K}_n \colon \sum b_l \varphi_{n,l}(x_1, \cdots, x_n) \longrightarrow \sum \omega^l b_l \varphi_{n,l}(x_1, \cdots, x_n) .$$

In the limit $n \longrightarrow \infty$

$$\int |(T_\omega g)(x_1 + \cdots + x_n) - (\bar{K}_n g_n)(x_1, \cdots, x_n)|^{p'} d\nu(\sqrt{n}\, x_1) \cdots d\nu(\sqrt{n}\, x_n) \longrightarrow 0$$

because of the relation expressed in equation (5) between the Hermite polynomials and the basis functions $\varphi_{n,l}$. By the triangle inequality

$$\big|\, \| T_\omega g \|_{L^{p'}(d\nu_n)} - \| \bar{K}_n g_n \|_{L^{p'}[X_n]} \big|$$
$$\leqq \left\{ \int |(T_\omega g)(x_1 + \cdots + x_n) - (\bar{K}_n g_n)(x_1, \cdots, x_n)|^{p'} d\nu(\sqrt{n}\, x_1) \cdots d\nu(\sqrt{n}\, x_n) \right\}^{1/p'} .$$

As noted above in our remarks on Bernoulli trials, the moments of $d\nu_n$ will converge to moments of $d\mu$, so in fact $d\nu_n$ converges to $d\mu$ weakly with respect to functions of polynomial growth, and since g is a polynomial we have

$$\lim \| \bar{K}_n g_n \|_{L^{p'}[X_n]} = \lim \| T_\omega g \|_{L^{p'}(d\nu_n)} = \| T_\omega g \|_{L^{p'}(d\mu)} .$$

By a similar argument

$$\lim \| g_n \|_{L^p[X_n]} = \| g \|_{L^p(d\mu)} .$$

Thus

$$\| \bar{K}_n g_n \|_{L^p[X_n]} \leqq \| g_n \|_{L^p[X_n]}$$

implies

$$\| T_\omega g \|_{L^{p'}(d\mu)} \leqq \| g \|_{L^p(d\mu)} .$$

We now prove the basic "two-point inequality" and a lemma on products of operators.

LEMMA 1. *$C \colon a + bx \longrightarrow a + \omega bx$ is a bounded linear operator with norm one on $L^p(d\nu)$ to $L^{p'}(d\nu)$ with $\omega = i\sqrt{p-1}$, $1 < p \leqq 2$ and $1/p + 1/p' = 1$; that is, for all complex numbers a and b*

$$\left\{\frac{|a+\omega b|^{p'}+|a-\omega b|^{p'}}{2}\right\}^{1/p'} \leqq \left\{\frac{|a+b|^{p}+|a-b|^{p}}{2}\right\}^{1/p}. \tag{6}$$

It suffices to show this result for $a = 1$ and b any complex number; by an appropriate scaling this is equivalent to showing $G(\xi, \eta) \leqq 1$ for ξ and η real where

$$G(\xi,\eta) = \frac{\left\{\dfrac{[(1+\eta)^2+(p-1)\xi^2]^{p'/2}+[(1-\eta)^2+(p-1)\xi^2]^{p'/2}}{2}\right\}^{1/p'}}{\left\{\dfrac{[(1+\xi)^2+(p'-1)\eta^2]^{p/2}+[(1-\xi)^2+(p'-1)\eta^2]^{p/2}}{2}\right\}^{1/p}}.$$

Note that $p' - 1 = 1/(p - 1)$. Using Minkowski's inequality we obtain

$$\left\{\frac{[(1+\eta)^2+(p-1)\xi^2]^{p'/2}+[(1-\eta)^2+(p-1)\xi^2]^{p'/2}}{2}\right\}^{2/p'}$$
$$\leqq \left[\frac{|1+\eta|^{p'}+|1-\eta|^{p'}}{2}\right]^{2/p'} + (p-1)\xi^2$$

and

$$\left\{\frac{[(1+\xi)^2+(p'-1)\eta^2]^{p/2}+[(1-\xi)^2+(p'-1)\eta^2]^{p/2}}{2}\right\}^{2/p}$$
$$\geqq \left[\frac{|1+\xi|^{p}+|1-\xi|^{p}}{2}\right]^{2/p} + (p'-1)\eta^2 .$$

In the second equation the sign of the inequality reverses because $(p/2) \leqq 1$. Thus

$$G(\xi,\eta) \leqq \left[\frac{\left[\dfrac{|1+\eta|^{p'}+|1-\eta|^{p'}}{2}\right]^{2/p'} + (p-1)\xi^2}{\left[\dfrac{|1+\xi|^{p}+|1-\xi|^{p}}{2}\right]^{2/p} + (p'-1)\eta^2}\right]^{1/2}.$$

But

$$\left\{\frac{|1+\eta|^{p'}+|1-\eta|^{p'}}{2}\right\}^{1/p'} \leqq [1+(p'-1)\eta^2]^{1/2}$$

and

$$[1+(p-1)\xi^2]^{1/2} \leqq \left\{\frac{|1+\xi|^{p}+|1-\xi|^{p}}{2}\right\}^{1/p}$$

then imply

$$\left[\frac{\left[\dfrac{|1+\eta|^{p'}+|1-\eta|^{p'}}{2}\right]^{2/p'} + (p-1)\xi^2}{\left[\dfrac{|1+\xi|^{p}+|1-\xi|^{p}}{2}\right]^{2/p} + (p'-1)\eta^2}\right]^{1/2} \leqq 1 .$$

The two inequalities above are obtained by an elementary minimization argument. First, observe that the inequality

$$\left[\frac{|1+y|^m+|1-y|^m}{2}\right]^{1/m} \leqq \sqrt{1+(m-1)y^2}$$

for $0 \leqq y < 1$ and $m > 2$ will also imply the case of $y > 1$, if we divide through by a factor of y. We want to show this inequality for fixed $m > 2$ and y varying between 0 and 1. It suffices to consider the function

$$\varphi(y) = \frac{1}{m}\ln\left\{\frac{(1+y)^m+(1-y)^m}{2}\right\} - \frac{1}{2}\ln\left[1+(m-1)y^2\right]$$

for $0 < y < 1$ and $2 \leqq m < \infty$.

$$\begin{aligned}
\varphi'(y) &= [(1+y)^m+(1-y)^m]^{-1}[1+(m-1)y^2]^{-1}\Delta(y)\\
\Delta(y) &= (1+y)^{m-1}[1-(m-1)y] - (1-y)^{m-1}[1+(m-1)y]\\
\Delta'(y) &= -m(m-1)y\{[1+y]^{m-2}-[1-y]^{m-2}\}\,.
\end{aligned}$$

For $0 < y < 1$ and $m \geqq 2$ we have $\Delta'(y) \leqq 0$ which implies $\varphi'(y) < 0$ which implies $\varphi(y) < 0$. For $1 < m < 2$ all inequalities reverse sign. Observe that these two basic inequalities above are special cases of the result that for fixed $y > 0$, the function

$$\left\{\frac{\left|1+\frac{y}{\sqrt{p-1}}\right|^p + \left|1-\frac{y}{\sqrt{p-1}}\right|^p}{2}\right\}^{1/p} \tag{7}$$

is monotone decreasing as a function of p, $1<p<\infty$ (see Appendix, Part 2). Thus, we have shown that $G(\xi, \eta) \leqq 1$ for $1<p\leqq 2$, and hence the "two-point inequality" in Lemma 1 holds.

The following lemma is used to show inequalities for products of operators. It generalizes an important lemma used in the work of Segal and Nelson.

LEMMA 2. *Consider two linear operators T_1 and T_2 which are integral operators defined by kernels; suppose*

$$\begin{aligned}
&T_1\colon L^p(d\rho_1) \longrightarrow L^q(d\lambda_1)\,, && \|T_1\| \leqq 1\,,\\
&T_2\colon L^p(d\rho_2) \longrightarrow L^q(d\lambda_2)\,, && \|T_2\| \leqq 1\,,
\end{aligned}$$

where $d\rho_i$ and $d\lambda_i$ are σ-finite measures; then if $p \leqq q$, we have for the product of the two operators

$$T_1T_2\colon L^p[d\rho_1 \times d\rho_2] \longrightarrow L^q[d\lambda_1 \times d\lambda_2]\,, \qquad \|T_1T_2\| \leqq 1\,.$$

The proof of the lemma is contained in the following steps.

$$\left\{\iint |(T_1T_2f)(x_1, x_2)|^q\, d\lambda_1(x_1)d\lambda_2(x_2)\right\}^{1/q}$$
$$\leqq \left\{\int d\lambda_2(x_2)\left[\int |(T_2f)(y_1, x_2)|^p\, d\rho_1(y_1)\right]^{q/p}\right\}^{1/q}$$
$$\leqq \left\{\int d\rho_1(y_1)\left[\int |(T_2f)(y_1, x_2)|^q\, d\lambda_2(x_2)\right]^{p/q}\right\}^{1/p}$$
$$\leqq \left\{\iint |f(y_1, y_2)|^p\, d\rho_1(y_1)d\rho_2(y_2)\right\}^{1/p}.$$

Here we have used the fact that both T_1 and T_2 are operators of norm one. We have interchanged orders of integration using Minkowski's inequality for integrals;[2] that is for $r \geqq 1$

$$\left\{\int dx\left[\int |F(x, y)|\, dy\right]^r\right\}^{1/r} \leqq \int dy\left\{\int |F(x, y)|^r\, dx\right\}^{1/r}.$$

In the computation above, we take $r = q/p \geqq 1$.

III. Young's inequality

The relation between the Fourier transform and convolution is basic to the study of harmonic analysis. We have obtained the following sharp form of Young's inequality for convolutions on $\mathbf{R}^n$.

THEOREM 3. *For* $f \in L^p(\mathbf{R}^n)$, $g \in L^q(\mathbf{R}^n)$, $1 \leqq p, q, r \leqq \infty$ *and* $1/r = 1/p + 1/q - 1$,

$$\begin{aligned} &\|f*g\|_r \leqq (A_pA_qA_{r'})^n \|f\|_p \|g\|_q, \\ &A_m = [m^{1/m}/m'^{1/m'}]^{1/2}, \qquad 1/m + 1/m' = 1. \end{aligned} \tag{8}$$

But as a consequence of the sharp Hausdorff-Young inequality of Theorem 1, we can obtain immediately the following partial result.

THEOREM 3′. *For* $1 \leqq p, q, r' \leqq 2$ *and* $1/r = 1/p + 1/q - 1$,

$$\|f*g\|_r \leqq (A_pA_qA_{r'})^n \|f\|_p \|g\|_q.$$

Note that at least two of these exponents will always be less than or equal to two. Consider $n = 1$ and observe that

$$\begin{aligned} \|f*g\|_r &\leqq A_{r'} \|(\mathcal{F}f)(\mathcal{F}g)\|_{r'} \leqq A_{r'} \|\mathcal{F}f\|_{p'} \|\mathcal{F}g\|_{q'} \\ &\leqq A_{r'}(A_p\|f\|_p)(A_q\|g\|_q), \end{aligned}$$

for $1/r' = 1/p' + 1/q'$. In addition, the sharp Hausdorff-Young inequality of Theorem 1 for the special case where p' is an even integer can be obtained directly from the sharp Young's inequality for convolution in Theorem 3. Let $p' = 2k$ and so $p = 2k/(2k-1)$; then

---

[2] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, 1970), page 271.

$$[\|\mathcal{F}f\|_{2k}]^k = \|\overbrace{f*\cdots*f}^{k}\|_2 \leqq [A_p\|f\|_p]^k .^3$$

To obtain the general result of sharp convolution inequalities in Theorem 3, the basic problem is to calculate the one-dimensional convolution norm

$$\mathcal{C} = \sup \frac{\|f*g\|_r}{\|f\|_p\|g\|_q} = \sup \frac{\|f*g*h\|_\infty}{\|f\|_p\|g\|_q\|h\|_{r'}} \tag{9}$$

with $1 \leqq p, q, r \leqq \infty$ and $1/r = 1/p + 1/q - 1$. The equality of these norms is easily seen by noting the relations

$$\sup_h \frac{\|f*g*h\|_\infty}{\|f\|_p\|g\|_q\|h\|_{r'}} \leqq \frac{\|f*g\|_r}{\|f\|_p\|g\|_q}$$
$$= \sup_h \frac{\int \tilde{h}(f*g)dx}{\|h\|_{r'}\|f\|_p\|g\|_q} \leqq \sup_h \frac{\|f*g*h\|_\infty}{\|f\|_p\|g\|_q\|h\|_{r'}}$$

with $\tilde{h}(x) = h(-x)$. Observe that on $\mathbf{R}^n$ the convolution operation has a product structure, in that it acts on the variables separately with respect to dimension. Also, it takes positive functions to positive functions.

LEMMA 3. *The convolution norm for n dimensions will be $\mathcal{C}^n$ where $\mathcal{C}$ is the one-dimensional norm.*

Consider $n = 2$, and observe that for positive functions on $\mathbf{R}^2$

$$\begin{aligned}(f*g*h)(x) &= \int f(x-y-z)g(y)h(z)dydz \\ &\leqq \mathcal{C}\int\left\{\left[\int |f(x_1-y_1-z_1,t)|^p\,dt\right]^{1/p}\left[\int |g(y_1,t)|^q\,dt\right]^{1/q}\right. \\ &\qquad \times \left.\left[\int |h(z_1,t)|^{r'}\,dt\right]^{1/r'}\right\}dy_1dz_1 \\ &\leqq \mathcal{C}^2\|f\|_p\|g\|_q\|h\|_{r'} .\end{aligned}$$

But the two-dimensional norm is seen to be at least $\mathcal{C}^2$ by considering products of functions in one variable. Note that the content of this lemma extends to the case where we consider the convolution of an arbitrary number of functions; in fact, this argument applies to any positive operation with a product structure.

In the general consideration of convolution inequalities the following lemma allows a restriction to radial functions that are decreasing ([7] and [8]).

LEMMA (Hardy-Littlewood, F. Riesz, Sobolev).

[3] Theorems 1, 2, and 3′ were obtained about a year ago. Influenced by these results, H. Brascamp and E. Lieb have recently obtained an independent proof of Theorem 3.

$$\int_{\mathbf{R}^n} h(x)(f_1 * \cdots * f_m)(x)dx \leqq \int_{\mathbf{R}^n} \bar{h}(x)(\bar{f}_1 * \cdots * \bar{f}_m)(x)dx \; ; \tag{10}$$

$\bar{f}$ denotes the equimeasurable symmetric decreasing rearrangement of f.

By equimeasurable we mean that f and $\bar{f}$ have the same distribution function

$$m\{x: |f(x)| > \alpha\} = m\{x: \bar{f}(x) > \alpha\} \, ,$$

where m denotes Lebesgue measure on $\mathbf{R}^n$. The original lemma was proved in n dimensions by Sobolev for rearrangements of three functions, but it is not difficult to extend this result to an arbitrary number of functions. In his proof of the one-dimensional result, Riesz remarks that this extension is an immediate consequence of the method used by Hardy and Littlewood for the rearrangements of series. First, one observes that convolution preserves the class of radial decreasing functions. Convolution takes radial functions to radial functions since this is true for the Fourier transform. Suppose that $\bar{f}$ and $\bar{g}$ are radial decreasing functions; to see that $\bar{f} * \bar{g}$ is also a decreasing function, consider

$$\begin{aligned} \|\bar{f} * \bar{g}\|_r &= \sup \int h(x)(\bar{f} * \bar{g})(x)dx \qquad \|h\|_{r'} = 1, \; h \geqq 0 \\ &\leqq \sup \int \bar{h}(x)(\bar{f} * \bar{g})(x)dx \leqq \|\bar{f} * \bar{g}\|_r \, . \end{aligned}$$

But, by the conditions for equality in Hölder's inequality, we must have

$$(\bar{h})^{r'/r} = \bar{f} * \bar{g} \quad \text{a.e.}$$

so $\bar{f} * \bar{g}$ is a radial decreasing function. Then

$$\begin{aligned} \int h(x)(f_1 * f_2 * f_3)(x)dx &\leqq \int \bar{h}(x)[\bar{f}_1 * (\overline{f_2 * f_3})](x)dx \\ &= \int (\bar{h} * \bar{f}_1)(x)(\overline{f_2 * f_3})(x)dx \\ &= \int g_\sharp(x)(f_2 * f_3)(x)dx \\ &\leqq \int \bar{g}_\sharp(x)(\bar{f}_2 * \bar{f}_3)(x)dx \\ &= \int (\bar{h} * \bar{f}_1)(x)(\bar{f}_2 * \bar{f}_3)(x)dx \\ &= \int \bar{h}(x)(\bar{f}_1 * \bar{f}_2 * \bar{f}_3)(x)dx \end{aligned}$$

where $g_\sharp$ is some rearrangement of the function $\bar{h} * \bar{f}_1$, which is itself a radial decreasing function. This argument is clearly independent of dimension.[4]

In a rough sense the interplay between these two lemmas would force an essentially unique extremal solution for which the maximum norm is attained to consist of gaussian functions. That is, for measurable functions on $\mathbf{R}^n$ the only way for a product of functions, each radial in separate variables, also to be radial in the variables jointly is for the functions to be gaussian. This remark provides the underlying basis for the fact that on euclidean spaces extremal solutions to these inequalities in Fourier analysis are given by gaussian functions.

To solve the basic problem defined by equation (9) we modify this convolution problem in a natural way so that a smooth extremal solution will exist in two dimensions. In making this modification, or regularization, we retain the product structure of the convolution operation so that Lemma 3 extends to the modified problem, and then use this fact to show that a smooth extremal solution must consist of gaussian functions. We then calculate the norm for the modified problem, and through a limiting argument obtain the norm for the original convolution inequality.

By using the rearrangement lemma, we restrict our attention to radial decreasing functions in two dimensions. Let

$$k(x_1, x_2) = A \exp[-\alpha(x_1^2 + x_2^2)] , \qquad \alpha > 0, \ \|k\|_{p_4} = 1$$

be a fixed gaussian function. Consider the two-dimensional norm for the convolution of four functions with one being a fixed gaussian. That is,

$$\mathfrak{D}^2 = \sup \frac{\|k * f * g * h\|_\infty}{\|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}} \tag{11}$$

where $1 < p_1, p_2, p_3, p_4 < \infty$ and $1/p_1' + 1/p_2' + 1/p_3' + 1/p_4' = 1$. Because the functions are radial and decreasing,

$$\begin{aligned} \|k * f * g * h\|_\infty &= (k * f * g * h)(0) \\ &= \int k(x + y + z) f(x) g(y) h(z)\, dx\, dy\, dz , \qquad x, y, z \in \mathbf{R}^2 . \end{aligned}$$

Using a weak compactness argument there will exist an overall sequence $\{f_n, g_n, h_n\}$ such that

$$(k * f_n * g_n * h_n)(0) \longrightarrow \mathfrak{D}^2$$

[4] For a different argument (also independent of dimension) see G. Sampson, "Sharp estimates of convolution transforms in terms of decreasing functions," Pac. J. Math. **38** (1971), 213–231. Recently a generalization of the Sobolev lemma was given by H. Brascamp, E. Lieb and J. Luttinger, "A general rearrangement inequality for multiple integrals," J. Functional Analysis **17** (1974), 227–237, using arguments similar to those of Sobolev.

with $\|f_n\|_{p_1} = 1$, $\|g_n\|_{p_2} = 1$, $\|h_n\|_{p_3} = 1$ and $f_n \to f$ weakly in L^{p_1}, $g_n \to g$ weakly in L^{p_2}, and $h_n \to h$ weakly in L^{p_3}. The fixed gaussian k is a smooth and rapidly decreasing function. The radial decreasing functions f_n, g_n, h_n have uniform majorizations on bounded sets; i.e., if $\|f_n\|_{p_1} = 1$, then

$$f_n(r) \leqq [1/(\pi r^2)]^{1/p_1} = \psi(r) .$$

First,

$$k * f_n \longrightarrow k * f$$

as a pointwise limit because of the weak convergence of f_n. But $k * f_n$ is majorized by $k * \psi$, and this latter function is a good function in L^r, $1/r = 1/p_1 + 1/p_4 - 1$. So by the Lebesgue dominated convergence theorem $k * f_n$ converges to $k * f$ in L^r. Then

$$\begin{aligned} |(k * f_n * g_n * h_n)(0) - (k * f * g_n * h_n)(0)| &= |([k * f_n - k * f] * g_n * h_n)(0)| \\ &\leqq \|[k * f_n - k * f] * g_n * h_n\|_\infty \\ &\leqq \|g_n\|_{p_2} \|h_n\|_{p_3} \|(k * f_n) - (k * f)\|_r \\ &= \|(k * f_n) - (k * f)\|_r \end{aligned}$$

and thus

$$\lim (k * f_n * g_n * h_n)(0) = \lim (k * f * g_n * h_n)(0) .$$

By repeating this argument for the functions g_n and h_n we obtain

$$\mathfrak{D}^2 = (k * f * g * h)(0) .$$

But by definition

$$(k * f * g * h)(0) \leqq \mathfrak{D}^2 \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}$$

and since these functions are weak limits, we must have $\|f\|_{p_1} = 1$, $\|g\|_{p_2} = 1$, and $\|h\|_{p_3} = 1$. Thus, we have used a weak compactness argument to show the existence of an extremal solution for which the maximum norm is attained in equation (11). Since the supremum is attained, the conditions for equality in the case of Hölder's inequality require that the following integral equations be satisfied:

$$\begin{aligned} \mathfrak{D}^2 f^{p_1 - 1} &= k * g * h \quad \text{a.e.} , \\ \mathfrak{D}^2 g^{p_2 - 1} &= k * f * h \quad \text{a.e.} , \\ \mathfrak{D}^2 h^{p_3 - 1} &= k * f * g \quad \text{a.e.} \end{aligned}$$

For example,

$$\begin{aligned} \mathfrak{D}^2 = (k * f * g * h)(0) &= \int f(x)(k * g * h)(x)dx \\ &= \sup_{\|f'\|_{p_1} = 1} \int f'(x)(k * g * h)(x)dx . \end{aligned}$$

So we can in fact choose representative functions for this extremal solution which are smooth (even real analytic) by specifying that the above integral equations are satisfied identically (at most they can differ at the origin).

The fixed function k is gaussian so it splits into a product of gaussian functions

$$k(x) = k_0(x_1)k_0(x_2)$$

with one-dimensional norm $\| k_0 \|_{p_4} = 1$. So for this smooth extremal solution we have

$$\begin{aligned}\mathfrak{D}^2 &= \int k_0(x_1 + y_1 + z_1)k_0(x_2 + y_2 + z_2)f(\sqrt{x_1^2 + x_2^2})g(\sqrt{y_1^2 + y_2^2}) \\ &\quad \times h(\sqrt{z_1^2 + z_2^2})dx_1dx_2dy_1dy_2dz_1dz_2 ; \\ \mathfrak{D}^2 &= \int k_0(x_1 + y_1 + z_1)\Phi(x_1, y_1, z_1)\left[\int |f(\sqrt{x_1^2 + t^2})|^{p_1} dt\right]^{1/p_1} \\ &\quad \times \left[\int |g(\sqrt{y_1^2 + t^2})|^{p_2} dt\right]^{1/p_2}\left[\int |h(\sqrt{z_1^2 + t^2})|^{p_3} dt\right]^{1/p_3} dx_1dy_1dz_1 ; \\ \Phi(x_1, y_1, z_1) &= \int k_0(x_2 + y_2 + z_2)F_{x_1}(x_2)G_{y_1}(y_2)H_{z_1}(z_2)dx_2dy_2dz_2 ; \\ F_{x_1}(x_2) &= \frac{f(\sqrt{x_1^2 + x_2^2})}{\left[\int |f(\sqrt{x_1^2 + t^2})|^{p_1} dt\right]^{1/p_1}} ; \\ G_{y_1}(y_2) &= \frac{g(\sqrt{y_1^2 + y_2^2})}{\left[\int |g(\sqrt{y_1^2 + t^2})|^{p_2} dt\right]^{1/p_2}} ; \\ H_{z_1}(z_2) &= \frac{h(\sqrt{z_1^2 + z_2^2})}{\left[\int |h(\sqrt{z_1^2 + t^2})|^{p_3} dt\right]^{1/p_3}} .\end{aligned}$$

Smoothness and monotonicity will insure that the function Φ is continuous, and $\Phi(x_1, y_1, z_1) \leqq \mathfrak{D}$ as it corresponds to a one-dimensional convolution problem. Hence

$$\begin{aligned}\mathfrak{D} &= \int k_0(x_1 + y_1 + z_1)\left[\int |f(\sqrt{x_1^2 + t^2})|^{p_1} dt\right]^{1/p_1} \\ &\quad \times \left[\int |g(\sqrt{y_1^2 + t^2})|^{p_2} dt\right]^{1/p_2}\left[\int |h(\sqrt{z_1^2 + t^2})|^{p_3} dt\right]^{1/p_3} dx_1dy_1dz_1\end{aligned}$$

and so we must have $\Phi(x_1, y_1, z_1) = \mathfrak{D}$ almost everywhere, and by continuity $\Phi(x_1, y_1, z_1) = \mathfrak{D}$ for all values of the variables x_1, y_1, z_1. Observe that the functions $F_{x_1}(x_2)$, $G_{y_1}(y_2)$, $H_{z_1}(z_2)$ give a smooth one-dimensional extremal solution as functions of x_2, y_2, z_2 for all values x_1, y_1, z_1 (which can then be varied independently as "parameters"). That is,

$$\mathfrak{D} = \int k_0(x_2 + y_2 + z_2)F_{x_1}(x_2)G_{y_1}(y_2)H_{z_1}(z_2)dx_2dy_2dz_2 \ .$$

Suppose we fix x_1, y_1 and allow z_1 to vary. Then the function

$$H_{z_1}(z_2) = \frac{h(\sqrt{z_1^2 + z_2^2})}{\left[\int |h(\sqrt{z_1^2 + t^2})|^{p_3} dt\right]^{1/p_3}}$$

in terms of the variable z_2 will be determined by an integral equation in terms of the functions f, g and the parameters x_1, y_1 because of the conditions for equality in Hölder's inequality

$$\mathfrak{D}(H_{z_1})^{p_3-1} = F_{x_1} * G_{y_1} * k_0 \ .$$

Thus, this continuous function $H_{z_1}(z_2)$ must be independent of the "parameter" z_1. Setting $z_1 = 0$ and defining

$$u(z_1^2 + z_2^2) = \frac{h(\sqrt{z_1^2 + z_2^2})}{h(0)} \ ,$$

we obtain

$$\frac{u(z_1^2 + z_2^2)}{\left[\int |u(z_1^2 + t^2)|^{p_3} dt\right]^{1/p_3}} = \frac{u(z_2^2)}{\left[\int |u(t^2)|^{p_3} dt\right]^{1/p_3}} \ .$$

Now set $z_2 = 0$ and we find

$$u(z_1^2 + z_2^2) = u(z_1^2)u(z_2^2) \ . \tag{12}$$

This relation holds for all z_1, z_2 so u must be exponential and h is a gaussian function. The same argument will show that f and g must also be gaussian functions.

Now if we consider the one-dimensional convolution problem

$$\mathfrak{D} = \frac{\| f * g * h * k \|_\infty}{\| f \|_{p_1} \| g \|_{p_2} \| h \|_{p_3} \| k \|_{p_4}}$$

where f, g, h, k are one-dimensional gaussian functions, a variational argument will show that

$$\mathfrak{D} = A_{p_1}A_{p_2}A_{p_3}A_{p_4} \tag{13}$$

where $A_m = [m^{1/m}/m'^{1/m'}]$. Now let $1/p_1 + 1/p_2 + 1/p_3 = 2$, $1/s' = \varepsilon^2$ and $1/q_i = 1/p_i + 1/(3s')$. Let f, g, h be one-dimensional step functions with $k(x) = \exp(-\pi s' x^2)$. Then

$$\frac{|(f * g * h * k)(x)|}{\| f \|_{q_1} \| g \|_{q_2} \| h \|_{q_3} \| k \|_s} \leqq A_{q_1}A_{q_2}A_{q_3}A_s$$

and in the limit $\varepsilon \to 0$ (i.e., $s \to 1$) we obtain

$$\frac{|(f*g*h)(x)|}{\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3}} \leqq A_{p_1}A_{p_2}A_{p_3}\,,$$

$$\mathcal{C} = A_{p_1}A_{p_2}A_{p_3}\,. \tag{14}$$

More generally, we obtain the following sharp form of Young's inequality for convolutions.

THEOREM 4. *For* $f_i \in L^{p_i}(\mathbf{R}^n)$ *with* $1 < p_i < \infty$ *and* $1/p'_1 + \cdots + 1/p'_m = 1/r'$, $1 \leqq r \leqq \infty$, *then*

$$\|f_1 * \cdots * f_m\|_r \leqq [A_{p_1} \cdots A_{p_m} A_{r'}]^n \|f_1\|_{p_1} \cdots \|f_m\|_{p_m}\,, \tag{15}$$

$A_p = [p^{1/p}/p'^{1/p'}]^{1/2}$ *and in the limit* $p \to 1$ *or* $p \to \infty$, $\lim A_p = 1$.

As illustrated in our argument to obtain the sharp convolution inequalities, gaussian functions will be extremal functions on which the maximum norms are attained for Theorems 1, 3 and 4.

IV. Further remarks

1. *Nelson's inequality*. In the study of mathematical problems in quantum field theory, analysis of quartic interactions and other problems (particularly in the work of Nelson and Glimm) has used basic L^p estimates for the quantization operator. This problem can be formulated on Fock space, or in terms of stochastic integrals and gaussian processes, or as a Hermite multiplier inequality on the line. The best possible estimate has been called Nelson's "hypercontractive inequality" (see [6]). This estimate viewed as a multiplier inequality is in fact a special case of the sharp form of Young's inequality contained in Theorem 3 for one dimension.

THEOREM 5. *For real* ω, $0 \leqq \omega \leqq [(p-1)/(r-1)]^{1/2}$ *and* $p \leqq r$ *the multiplier inequality on the Hermite semigroup*

$$\|T_\omega g\|_{L^r(d\mu)} \leqq \|g\|_{L^p(d\mu)} \tag{16}$$

is equivalent to the convolution inequality on the line

$$\|k*f\|_r \leqq A_p A_q A_{r'} \|k\|_q \|f\|_p \tag{17}$$

where k *is a gaussian function,* $f \in L^p(\mathbf{R})$ *and* $1/q = 1/r + 1 - 1/p$, $1 < p, q, r' < \infty$.

The notation here is the same as in Theorem 2. It suffices to consider the maximum value of ω, and by dilation we normalize the functions in the second inequality so that $k(x) = \exp(-\pi q' x^2)$. The equivalence is evident by considering polynomials $g(x)$ in equation (16), and noting that by a simple change of variables equation (17) will hold for functions

$$f(x) = g(\sqrt{2\pi pp'}x)\exp(-\pi p'x^2)\ .$$

Both of these sets of functions are dense in their respective spaces. Explicitly, equation (16) can be written

$$\left\{\int\left|\int T_\omega(x, y)g(y)d\mu(y)\right|^r d\mu(x)\right\}^{1/r} \leqq \left\{\int |g(x)|^p du(x)\right\}^{1/p}\ .$$

On the left-hand side of the equation make the change of variables $x = \sqrt{2\pi rr'}u$ and $y = \sqrt{2\pi pp'}v$, and substitute the value $\omega = [(p-1)/(r-1)]^{1/2}$; on the right-hand side make the change of variables $x = \sqrt{2\pi pp'}u$. Then we obtain

$$\left\{\int\left|\int e^{-\pi q'(u-v)^2}g(\sqrt{2\pi pp'}v)e^{-\pi p'v^2}dv\right|^r du\right\}^{1/r}$$
$$\leqq A_pA_qA_{r'}\left(\frac{1}{qq'}\right)^{1/2q}\left\{\int |g(\sqrt{2\pi pp'}u)e^{-\pi p'u^2}|^p du\right\}^{1/p}\ .$$

2. *Convolution extremal functions.* In our proof to obtain sharp convolution inequalities in Section III, we used the product structure of convolution to show that gaussian functions provide an extremal solution. An extension of that argument will in fact show that up to the basic symmetry operations of translation, character multiplication and dilation, any extremal function will almost everywhere be gaussian. Details of this remark will appear in a later paper.

3. *Hirschman's inequality.* Prior to the idea of Babenko there were no definitive results to indicate that the classical Hausdorff-Young inequality could be improved on the line. But Hirschman had earlier suggested a relation between the Hausdorff-Young inequality and the Weyl-Heisenberg uncertainty inequality ([5]). For $f \in \mathcal{S}(\mathbf{R})$ and

$$\lambda = \int x\,|f(x)|^2\,dx\ , \qquad \hat{\lambda} = \int x\,|(\mathcal{F}f)(x)|^2\,dx$$

then the Weyl-Heisenberg inequality is given by

$$\frac{\left[\int (x-\lambda)^2\,|f(x)|^2\,dx\right]^{1/2}}{\|f\|_2}\,\frac{\left[\int (x-\hat{\lambda})^2\,|(\mathcal{F}f)(x)|^2\,dx\right]^{1/2}}{\|\mathcal{F}f\|_2} \geqq \frac{1}{4\pi}\ . \tag{18}$$

Hirschman argued that by differentiating the Hausdorff-Young inequality as a function of the exponent p, then the equality of L^2 norms, i.e., $\|\mathcal{F}f\|_2 = \|f\|_2$, would imply an inequality of the form

$$\int |f|^2 \ln |f|^2\,dx + \int |\mathcal{F}f|^2 \ln |\mathcal{F}f|^2\,dx \leqq E_H \tag{19}$$

where we have made the normalization $\|f\|_2 = \|\mathcal{F}f\|_2 = 1$.[5]

Hirschman conjectured this inequality would hold for the constant $E_H = \ln 2 - 1$ (which is the value attained on the left-hand side for a gaussian function). He demonstrated that this result would then imply the Weyl-Heisenberg inequality by using the classical entropy inequality for a probability distribution; i.e.,

$$\int \varphi(x) \ln \varphi(x) dx \geqq -1/2 - (1/2) \ln \left\{ 2\pi \int (x-\mu)^2 \varphi(x) dx \right\} \tag{20}$$

with

$$\int \varphi(x) dx = 1 , \qquad \int x\varphi(x) dx = \mu .$$

If one uses the classical Hausdorff-Young inequality in Hirschman's argument, then one can only obtain the value $E_H = 0$ in equation (19), but using the sharp Hausdorff-Young result one obtains the best constant, $E_H = \ln 2 - 1$, which is attained for gaussian functions.

4. *Inequalities on the torus.* It is easy to give a limiting argument to show that the sharp inequalities on the line imply the classical inequalities on the circle group. For example, let $p(x)$ be a finite trigonometric sum defined on $\mathbf{T} \approx \mathbf{R}/\mathbf{Z}$; then consider functions on $\mathbf{R}$ of the form

$$h(x) = \lambda^{1/2p} p(x) e^{-\pi\lambda x^2} , \qquad \lambda > 0 ;$$

then in the limit $\lambda \to 0$, the sharp Hausdorff-Young inequality on $\mathbf{R}$

$$\|\mathcal{F}h\|_{L^{p'}(\mathbf{R})} \leqq A_p \|h\|_{L^p(\mathbf{R})}$$

will imply the classical Hausdorff-Young inequality on $\mathbf{T}$;

$$\|\mathcal{F}p\|_{L^{p'}(\mathbf{Z})} \leqq \|p\|_{L^p(\mathbf{T})} .$$

5. *Locally compact abelian groups.* For the problem of sharp L^p inequalities in analysis on a locally compact abelian group, we can use the structure theorem (van Kampen) which states that any locally compact abelian group is topologically isomorphic to a product $\mathbf{R}^n \times G_0$, where G_0 is a locally compact abelian group which contains an open compact subgroup H_0 and the dimension n is an invariant of the group ([4, Theorem 24.30]). For groups G_0 the classical inequalities are sharp, and this was demonstrated in work by Hewitt and Hirschman ([4, Theorem 43.13]). This product structure for the group together with Lemma 2 above on products of operators

[5] This argument is similar to the one used by Gross ([3]) to obtain a logarithmic inequality for his proof of Nelson's inequality.

then implies that the sharp norms for these inequalities on locally compact abelian groups will be given as a product of the sharp $\mathbf{R}^n$ norms with the G_0 norm which is by the Hewitt-Hirschman theorem equal to one. This last remark confirms the conjecture of Hewitt and Ross (see [4, Vol. II, page 630]).

Appendix

1. We give here a short proof of equation (5),

$$\begin{aligned}\varphi_{n,l}(x_1, \cdots, x_n) &= H_l(x_1 + \cdots + x_n)\\ &\quad + \frac{1}{n}\sum_{r=1}^{[l/2]} a_{l,r}H_{l-2r}(x_1 + \cdots + x_n)\,,\end{aligned}$$

with $(x_i)^2 = 1/n$ and the coefficients $a_{l,r}$ bounded with respect to n for fixed l. By explicit computation

$$\begin{aligned}\varphi_{n,0}(x_1, \cdots, x_n) &= H_0(x_1 + \cdots + x_n)\,,\\ \varphi_{n,1}(x_1, \cdots, x_n) &= H_1(x_1 + \cdots + x_n)\,,\\ \varphi_{n,2}(x_1, \cdots, x_n) &= H_2(x_1 + \cdots + x_n)\,,\\ \varphi_{n,3}(x_1, \cdots, x_n) &= H_3(x_1 + \cdots + x_n) + \frac{2}{n}H_1(x_1 + \cdots + x_n)\,.\end{aligned}$$

The general result in equation (5) can be proved by a recursion argument.

Using the definition of the Hermite polynomials as

$$H_l(x) = \int (x + iy)^l d\mu(y)\,,$$

we obtain through an integration by parts the recursion relation

$$\begin{aligned}H_l(x) &= xH_{l-1}(x) - (l-1)H_{l-2}(x)\\ &= H_1(x)H_{l-1}(x) - (l-1)H_{l-2}(x)\,.\end{aligned}$$

For the basis functions $\varphi_{n,l}(x_1, \cdots, x_n)$ we have

$$\begin{aligned}\varphi_{n,l}(x_1, \cdots, x_n) &= \varphi_{n,1}(x_1, \cdots, x_n)\varphi_{n,l-1}(x_1, \cdots, x_n)\\ &\quad - \frac{(l-1)}{n}[n - (l-2)]\varphi_{n,l-2}(x_1, \cdots, x_n)\,.\end{aligned}$$

In terms of the elementary symmetric functions this is simply

$$\begin{aligned}l\sigma_l(x_1, \cdots, x_n) &= \sigma_1(x_1, \cdots, x_n)\sigma_{l-1}(x_1, \cdots, x_n)\\ &\quad - \frac{1}{n}[n - (l-2)]\sigma_{l-2}(x_1, \cdots, x_n)\,.\end{aligned}$$

To obtain this recursion relation use the generating function

$$\mathcal{T}(x_1, \cdots, x_n; t) = \prod_{k=1}^n (1 + x_k t) = \sum_{l=0}^n t^l \sigma_l(x_1, \cdots, x_n) ,$$

$$\frac{\partial}{\partial t}\mathcal{T}(x_1, \cdots, x_n; t) = \sum_{k=1}^n x_k \mathcal{T}(x_1, \cdots, \hat{x}_k, \cdots, x_n; t)$$

$$= \sum_{l=0}^n t^{l-1} l \sigma_l(x_1, \cdots, x_n) ,$$

$$\sigma_1(x_1, \cdots, x_n)\mathcal{T}(x_1, \cdots, x_n; t) = \sum_{k=1}^n x_k(1 + x_k t)\mathcal{T}(x_1, \cdots, \hat{x}_k, \cdots, x_n; t) ,$$

$$\sigma_1(x_1, \cdots, x_n)\mathcal{T}(x_1, \cdots, x_n; t) - \frac{\partial}{\partial t}\mathcal{T}(x_1, \cdots, x_n; t)$$

$$= \frac{1}{n} t \sum_{k=1}^n \mathcal{T}(x_1, \cdots, \hat{x}_k, \cdots, x_n; t) .$$

Observe that

$$\sum_{k=1}^n \sigma_l(x_1, \cdots, \hat{x}_k, \cdots, x_n) = (n - l)\sigma_l(x_1, \cdots, x_n) ;$$

that is, by symmetry the left-hand side must be a constant multiple of $\sigma_l(x_1, \cdots, x_n)$. We note that for $n \geqq l$ there are $\binom{n}{l}$ terms in the expression for $\sigma_l(x_1, \cdots, x_n)$, and find the constant by setting all the x_i equal to the same value. Thus we have

$$\sigma_1(x_1, \cdots, x_n)\mathcal{T}(x_1, \cdots, x_n; t) - \frac{\partial}{\partial t}\mathcal{T}(x_1, \cdots, x_n; t)$$

$$= \frac{1}{n} t \sum_{m=0}^{n-1} (n - m)\sigma_m(x_1, \cdots, x_n)t^m$$

and by comparing powers of t this gives the desired recursion relation for the elementary symmetric functions in the case $(x_i)^2 = 1/n$.

It is seen explicitly that equation (5) holds for the initial values of l. Then the two recursion relations for the Hermite polynomials $H_l(x_1 + \cdots + x_n)$ and the symmetric basis functions $\varphi_{n,l}(x_1, \cdots, x_n)$ together with an induction argument on l show that equation (5) holds in general.

2. We give a proof for the remark contained in equation (7); that is, the function

$$\left\{\frac{\left|1 + \frac{y}{\sqrt{p-1}}\right|^p + \left|1 - \frac{y}{\sqrt{p-1}}\right|^p}{2}\right\}^{1/p}$$

is monotone decreasing as a function of p, for $p > 1$ and fixed $y > 0$. The monotonicity of this function is equivalent to the inequality

$$\left\{\frac{\left|1 + \sqrt{\frac{p-1}{q-1}}\,y\right|^q + \left|1 - \sqrt{\frac{p-1}{q-1}}\,y\right|^q}{2}\right\}^{1/q} \leqq \left\{\frac{|1 + y|^p + |1 - y|^p}{2}\right\}^{1/p}$$

for $1 < p \leqq q < \infty$. Suppose we have proved this inequality for $0 < y \leqq 1$.

Then we can obtain the case $y > 1$ from this result. For observe that if $0 < w \leqq 1$,

$$\left| w \pm \sqrt{\frac{p-1}{q-1}} \right|^2 \leqq \left| 1 \pm \sqrt{\frac{p-1}{q-1}}\, w \right|^2 ;$$

so we have

$$\begin{aligned} &\left\{ \frac{\left| w + \sqrt{\frac{p-1}{q-1}} \right|^q + \left| w - \sqrt{\frac{p-1}{q-1}} \right|^q}{2} \right\}^{1/q} \\ &\quad \leqq \left\{ \frac{\left| 1 + \sqrt{\frac{p-1}{q-1}}\, w \right|^q + \left| 1 - \sqrt{\frac{p-1}{q-1}}\, w \right|^q}{2} \right\}^{1/q} \\ &\quad \leqq \left\{ \frac{|1+w|^p + |1-w|^p}{2} \right\}^{1/p} . \end{aligned}$$

Now dividing through both sides of this equation by a factor w and setting $y = 1/w$, we obtain the above inequality for the case $y > 1$.

This inequality is equivalent to a multiplier inequality on "two-point spaces" (with real-valued functions) corresponding to the multiplier factor $\gamma = \sqrt{(p-1)/(q-1)}$; that is,

$$T\colon a + bx \longrightarrow a + \gamma bx$$

for a and b real, $\gamma = \sqrt{(p-1)/(q-1)}$, and x with the values ± 1 with equal weight. If we can show that T has norm one as a linear mapping from $L^p(d\nu)$ to $L^q(d\nu)$ (here we want real-valued function spaces) for the restricted case $1 \leqq p \leqq q \leqq 2$, then this result will imply by duality the inequality for $2 \leqq q' \leqq p' < \infty$ since $\gamma = \sqrt{(p-1)/q-1)} = \sqrt{(q'-1)/(p'-1)}$; that is,

$$T^*\colon a + bx \longrightarrow a + \gamma bx$$

is a linear mapping of norm one on $L^{q'}(d\nu)$ to $L^{p'}(d\nu)$ (again real-valued function spaces); and these two results together will imply the general inequality.

Thus we need only to show the inequality

$$\left\{ \frac{\left| 1 + \sqrt{\frac{p-1}{q-1}}\, y \right|^q + \left| 1 - \sqrt{\frac{p-1}{q-1}}\, y \right|^q}{2} \right\}^{1/q} \leqq \left\{ \frac{|1+y|^p + |1-y|^p}{2} \right\}^{1/p}$$

for the restricted case $0 < y \leqq 1$ and $1 < p \leqq q \leqq 2$. With the use of the binomial expansion, this inequality is equivalent to

$$\left[\sum_{k=0}^{\infty} \binom{q}{2k} \left(\frac{p-1}{q-1} \right)^k y^{2k} \right]^{p/q} \leqq \sum_{k=0}^{\infty} \binom{p}{2k} y^{2k} ,$$

and for $1 < p \leqq q \leqq 2$ the binomial coefficients $\binom{p}{2k}$ and $\binom{q}{2k}$ are both positive, and in addition

$$\frac{p}{q}\binom{q}{2k}\left(\frac{p-1}{q-1}\right)^k \leqq \binom{p}{2k} .$$

Using the elementary result that for $0 < \lambda \leqq 1$ and $x > 0$

$$(1 + x)^\lambda \leqq 1 + \lambda x ,$$

we have

$$\begin{aligned}\left[1 + \sum_{k=1}^\infty \binom{q}{2k}\left(\frac{p-1}{q-1}\right)^k y^{2k}\right]^{p/q} &\leqq 1 + \frac{p}{q}\sum_{k=1}^\infty \binom{q}{2k}\left(\frac{p-1}{q-1}\right)^k y^{2k} \\ &\leqq 1 + \sum_{k=1}^\infty \binom{p}{2k} y^{2k} .\end{aligned}$$

We remark that the basic inequality of this section is the natural analogue *of Nelson's inequality for* the "two-point space."

PRINCETON UNIVERSITY

REFERENCES

[1] K. I. BABENKO, An inequality in the theory of Fourier integrals, Izv. Akad. Nauk SSSR, Ser. Mat. **25** (1961), 531-542; English transl., Amer. Math. Soc. Transl. (2) **44**, 115-128.

[2] L. GROSS, Logarithmic Sobolev inequalities, Amer. J. Math., in press.

[3] G. H. HARDY, J. E. LITTLEWOOD, and G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, 1952.

[4] E. HEWITT and K. A. ROSS, *Abstract Harmonic Analysis*, 2 vols., Springer-Verlag, 1963, 1970.

[5] I. I. HIRSCHMAN, Jr., A note on entropy, Amer. J. Math. **79** (1957), 152-156.

[6] E. NELSON, The free Markoff field, J. Func. Anal. **12** (1973), 211-227.

[7] F. RIESZ, Sur une inégalité intégrale, J. London Math. Soc. **5** (1930), 162-168.

[8] S. L. SOBOLEV, On a theorem of functional analysis, Mat. Sb. (N.S.) **4** (46) (1938), 471-497; English transl., Amer. Math. Soc. Transl. (2) **34**, 39-68.

[9] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[10] E. M. STEIN, and G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[11] N. VILENKIN, *Special Functions and the Theory of Group Representations* Vol. 22, Translations of Mathematical Monographs, Amer. Math. Soc., 1968.

[12] A. ZYGMUND, *Trigonometric Series*, 2nd edition, 2 vols., Cambridge University Press, 1959.

(Received May 19, 1975)