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## A FIXED POINT FREE NONEXPANSIVE MAP

DALE E. ALSPACH<sup>1</sup>

**ABSTRACT.** In this note we give an example of a weakly compact convex subset of  $L_1[0, 1]$  that fails to have the fixed point property for nonexpansive maps. This answers a long-standing question which was recently raised again by S. Reich [7].

**1. Introduction.** A (usually nonlinear) map  $T$  on a subset  $K$  of a Banach space  $X$  is said to be nonexpansive if for every  $k_1, k_2$  in  $K$ ,  $\|Tk_1 - Tk_2\| \leq \|k_1 - k_2\|$ . Many authors have given conditions on the set  $K$  that guarantee that a nonexpansive map  $T$  on  $K$  has a fixed point, e.g., [1], [2], [5], [6]. Usually  $K$  is assumed to be weakly compact and convex. Of course, if  $T$  is weakly continuous, then  $T$  has a fixed point by the Schauder-Tychonoff fixed point theorem. For  $T$  nonexpansive, (and not weakly continuous) positive results have been obtained only by placing additional requirements on  $K$ ; however, it was unknown whether any of these additional requirements on  $K$  were necessary. Our example shows that in fact some additional assumptions on  $K$  are necessary.

**2. The example.** Let  $X = L_1[0, 1]$  and let

$$K = \left\{ f \in L_1[0, 1] : \int_0^1 f = 1, 0 \leq f \leq 2, \text{ a.e.} \right\}.$$

It is easy to see that  $K$  is a weakly closed, convex subset of the order interval  $\{f: 0 \leq f \leq 2\}$ , and thus  $K$  is weakly compact, because order intervals in  $L_1[0, 1]$  are weakly compact. (This is a direct consequence of uniform integrability, [3, p. 292].) Define the map  $T$  from  $K$  to  $K$  by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 2, & 0 \leq t \leq \frac{1}{2}, \\ [2f(2t - 1) - 2] \vee 0, & \frac{1}{2} < t \leq 1. \end{cases}$$

(We will use equality throughout with the understanding that there may be an exceptional set of measure zero.) We leave it to the reader to check that  $T$  is an *isometry* on  $K$ .

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Suppose that  $T$  has a fixed point  $g$ . We note first that  $g = 21_A$  for some set  $A$  of measure one-half. Indeed,

$$\begin{aligned} \{t: g(t) = 2\} &= \{t: Tg(t) = 2\} \\ &= \{t/2: g(t) = 2\} + \left\{ \frac{1+t}{2} : g(t) = 2 \right\} \\ &\quad + \{t/2: 1 \leq g(t) < 2\}. \end{aligned}$$

(We are using  $+$  to denote disjoint union.) Because the measure of  $\{t/2: g(t) = 2\} + \{(1+t)/2: g(t) = 2\}$  is equal to the measure of  $\{t: g(t) = 2\}$ , it follows that  $\{t: 1 \leq g(t) < 2\}$  is of measure zero. Iteration of this argument shows that

$$\{t: 0 < g(t) < 2\} = \bigcup_{n=0}^{\infty} \{t: 2^{-n} \leq g(t) < 2^{-n+1}\}$$

is of measure zero, as well.

Next observe that for  $g = 21_A$

$$\{t: T^n g(t) = 2\} = \sum_{\epsilon_i \in \{0,1\}} \left\{ \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_n}{2^n} + \frac{t}{2^n} : t \in A \right\}$$

for all  $n$ . We have this for  $n = 1$  above, and induction establishes it in general. Because  $g$  is fixed,  $A = \{t: T^n g(t) = 2\}$  for all natural numbers  $n$  and thus, the intersection of  $A$  with any interval with dyadic end points has measure exactly half the measure of the interval. Obviously no such measurable set exists. This contradiction shows that  $T$  has no fixed point.

REMARK 1. The set  $K$  has diameter two, but  $\|f - 1\| \leq 1$  for all  $f \in K$  and thus,  $K$  cannot be the minimal weakly compact convex subset invariant under  $T$ . In particular, the set

$$\bigcap_{i=1}^{\infty} \{f: \|f - (1 + r_i)\| \leq 1\} \cap \{f: \|f - 1\| \leq 1\} \cap K,$$

where  $r_i = \text{sgn}[\sin 2\pi i t]$ , the  $i$ th Rademacher function, is invariant.

REMARK 2. It remains open whether there is a closed, bounded, convex subset of a reflexive space (hence, weakly compact) without the fixed point property for nonexpansive maps.

REMARK 3. When viewed as a transformation acting on the sets  $\{(x, y): 0 < y \leq f(x)\}$ . This example is essentially the baker's transformation from ergodic theory [4]. The various properties of our example can be derived from the well-known properties of that transformation.

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