Math 138 5/5/09.

Last time

Thm: $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

$\Rightarrow$ $u * \varphi$ defined by
$$(u*\varphi)(\psi) = u(\tilde{\varphi} * \psi) \quad \text{for } \psi \in \mathcal{S}.$$

is the function $f$ s.t. $f(x) = u(\tau_x \tilde{\varphi})$.

$f \in C^\infty$, and it and all its derivatives are slowly increasing.

Differentiation

If $u, \varphi \in \mathcal{S}$ then

$$\int_{\mathbb{R}^n} D^\beta u(x)\, \varphi(x)\, dx = (-1)^{|\beta|} \int_{\mathbb{R}^n} u(x)\, D^\beta \varphi(x)\, dx$$

$D^\beta u$: $\varphi \to \int_{\mathbb{R}^n} D^\beta u(x)\, \varphi(x)\, dx$

are els. of $\mathcal{S}'$

$u$: $\psi \to \int_{\mathbb{R}^n} u(x)\, \psi(x)\, dx$

$$D^\beta u(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi) \quad \forall\, u, \varphi \in \mathcal{S}.$$

and this allows us to extend the definition to $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

(since $(-1)^{|\beta|} u(D^\beta \varphi)$ is the composition of two continuous functions).

<u>Translation</u> if $u, \varphi \in \mathcal{S}$.

$$(\tau_h u)(\varphi) = \int_{\mathbb{R}^n} u(x-h)\, \varphi(x)\, dx.$$

$$= \int_{\mathbb{R}^n} u(y)\, (\varphi(y+h))\, dx.$$

$$= u(\tau_{-h}\varphi)$$ <u>extend as before</u>

<u>reflection</u> if $u, \varphi \in \mathcal{S}$.

$$\tilde{u}(\varphi) = \int_{\mathbb{R}^n} u(-x)\, \varphi(x)\, dx.$$

$$= \int_{\mathbb{R}^n} u(y)\, \varphi(-y)\, dy.$$

$$= u(\tilde{\varphi})$$ <u>extend as before</u>

## Fourier Transform

if $u, \varphi \in \mathcal{S}$. we have

(multiplication formula).

$$\int_{\mathbb{R}^n} u(x)\,\hat{\varphi}(x)\,dx = \int_{\mathbb{R}^n} \hat{u}(x)\,\varphi(x)\,dx.$$

or $u(\hat{\varphi}) = \hat{u}(\varphi)$. , we extend the definition of $\hat{u}$ to $\mathcal{S}'$ as before.

**Note:** if $f \in L^p$ $1 \le p \le 2$, $\varphi \in \mathcal{S}$

$$\int_{\mathbb{R}^n} f\,\hat{\varphi}\,dx = \int_{\mathbb{R}^n} \hat{f}\,\varphi\,dx.$$

where $\hat{f}$ ~~is the limit in $L^p$ of functions $\hat{f_k}$~~, ~~[illegible]~~ ~~$f_n$~~

is defined as the transform of $f \in L^1 + L^2$.

## Operators which commute with translations

$$\tau_h B = B \tau_h \qquad B : \underbrace{V \to W}_{\text{certain fcn. spaces.}}$$

e.g. $f \in L^p(\mathbb{R}^n)$. fixed

$$Bg = f * g \qquad g \in L^1$$

$B : L^1 \to L^p$ is bounded

since $\|f * g\|_p \le \|g\|_1 \|f\|_p$.

$$( \|B\| \le \|f\|_p ).$$

$$[\tau_h (Bg)](x) \quad \tau_h \int_{\mathbb{R}^n} f(u)\, g(\cancel{x-u})\, du.$$

$$= \int_{\mathbb{R}^n} f(u)\, g(x-h-u)\, du$$

$$B(\tau_h g)(x) = \int_{\mathbb{R}} f(u)\, (\tau_h g)(x-u)\, du$$

$$= \int_{\mathbb{R}^n} f(u)\, g(x-h-u)\, du$$

**Thm:** $B : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ $\quad 1 \le p, q \le \infty.$

linear, bdd. commutes w/ translations.

$\exists !\ u \in \mathcal{S}'$ s.t. $B\varphi = u * \varphi \quad \forall \varphi \in \mathcal{S}$

**Recall** from last time.

$f$ has $L^p$ derivatives, all orders $\le n+1$.

$\Rightarrow$ $f =$ a.e a cont. $g$. s.t.

$$|g(0)| \le C \sum_{|\alpha| \le n+1} \|D^\alpha f\|_p.$$

$$C = C(n, p).$$

**Pf:** (Thm)

**Claim:** $\varphi \in \mathcal{S} \Rightarrow B\varphi$ has $L^q$ derivatives of all orders.

let $h = (0, \ldots 0, h_j, 0, \ldots 0)$.

then

$$(\tau_h(B\varphi) - B\varphi)/h_j$$

$$= \frac{B(\tau_h\varphi) - B\varphi}{h_j}$$

$$= B((\tau_h\varphi - \varphi)/h_j)$$

$$\frac{\tau_h\varphi - \varphi}{h_j} \to -\frac{\partial\varphi}{\partial x_j} \quad \text{in } \mathcal{S}. \quad \text{as } |h| \to 0$$

$\therefore$ in $L^p$ norm.

$$\therefore \quad B\left(\frac{\tau_h\varphi - \varphi}{h_j}\right) \to -B\left(\frac{\partial\varphi}{\partial x_j}\right) \text{ in } L^q.$$

$$\| $$

$$\frac{\tau_h(B\varphi) - B\varphi}{h_j} \to -\frac{\partial(B\varphi)}{\partial x_j} \quad \left(\begin{array}{l}\text{also} \\ \text{in } L^q\end{array}\right.$$

We can repeat, starting with $-\frac{\partial\varphi}{\partial x_j}$ instead of $\varphi$ to see that $B\varphi$ has an $L^q$ derivative of each order.

and $\quad B(D^\alpha \varphi) = D^\alpha(B\varphi).$

$$\forall\ \alpha = (\alpha_1, \dots, \alpha_n)$$

By our lemma from last time then

$B\varphi =$ a.e. a cont $g$ s.t

$(g = g_\varphi)$.

$$|g_\varphi(0)| \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha(B\varphi)\|_q.$$

$$= C \sum_{|\alpha| \leq n+1} \|B(D^\alpha \varphi)\|_q$$

$$\leq \|B\|\, C \sum_{|\alpha| \leq n+1} \|D^\alpha \varphi\|_p.$$

Since each $L^p$ norm is controlled by a linear comb. of the $\rho_{\alpha,\beta}$ in the sum $\varphi \to g_\varphi(0)$ is an el of $\mathcal{S}'$.

which we denote by $u_1$

$u_1(\varphi) = g_\varphi(0)$.

We claim that $B\varphi = \tilde{\hat{u}}_1 * \varphi \quad \forall \varphi \in \mathcal{S}$.

$$\tilde{\hat{u}}_1 * \varphi(x) = \tilde{\hat{u}}_1(\tau_x \tilde{\varphi})$$ (by our characterization of convolution of $u \in \mathcal{S}'$ w/ $\varphi \in \mathcal{S}$)

$$\cancel{\neq \psi_1((\tau_x \tilde{\varphi})^{\sim}))}$$

$$\cancel{\neq \psi_1}$$

$$= \tilde{\hat{u}}_1((\tau_{-x}\varphi)^{\sim})$$

$$= u_1(\tau_{-x}\varphi)$$

$$= B(\tau_{-x}\varphi)(0)$$

$\boxed{\text{where } B(\tau_{-x}\varphi) \text{ is identified with the cont function that it equals a.e.}}$

$$= \tau_{-x} B(\varphi)(0).$$

$$= B(\varphi)(x).$$

Since these steps are reversible $u = \hat{u}_1$ is unique

For $\varphi \in \mathcal{S}$.

<u>Note</u>: $B\varphi$ is a.e. equal to a $C^\infty$ function which (with all derivatives) is slowly increasing.

<u>Def</u>: $(L^p, L^q) \subset \mathcal{S}'$

$= \{u \in \mathcal{S}' : \exists A > 0$ s.t. $\|u * \varphi\|_q \leq A \|\varphi\|_p \ \forall \varphi \in \mathcal{S}\}$

When $p < \infty$ this is exactly (in 1-1 corresp) with set of bdd linear operators from $L^p$ to $L^q$ which commute with translations.

( $\varphi \to u * \varphi$ is given by $u(\tau_x \tilde{\varphi})$ ).

Thm: $u \in \mathcal{S}'$ is in $(L^1, L^1)$ iff

$u$ is a finite Borel measure.

In this case, the norm of

$$B : L^1 \cap \mathcal{S} \to L^1$$

$$B\varphi = u * \varphi \quad \forall \varphi \in \mathcal{S}.$$

is the total variation of the measure $u$.

Pf: If $u$ is a finite Borel measure. $u = d\mu$

$$(u * \varphi)(x) = \int \varphi(x-y)\, d\mu$$

and $$\|u * \varphi(x)\|_1 = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi(x-y)\, d\mu \right| dx.$$

$$\leq \|\mu\| \, \|\varphi\|_1 .$$

If $u \in (L^1, L^1)$

let $u_\epsilon = u * W_\epsilon \qquad \epsilon > 0$

where $W_\epsilon(x) = \frac{1}{(4\pi\epsilon)^{n/2}} e^{-|x|^2/4\epsilon}$

is the Gauss-Weierstrass kernel.

$(\in \mathcal{S} \;\; \forall \epsilon > 0)$

Since $u \in (L^1, L^1)$ and $\int W_\epsilon \, dx = 1$.

$$\|u_\epsilon\|_1 \leq A \, \|W(\cdot,\epsilon)\| = A \quad \forall \epsilon > 0.$$

i.e $\|u_\epsilon\|_1$ is uniformly bounded.

We can consider $L^1(\mathbb{R}^n)$ as a subspace of $M(\mathbb{R}^n)$: the finite Borel measures on $\mathbb{R}^n$

$$f \in L^1(\mathbb{R}^n) \to \mu_f$$

$$\mu_f(E) = \int_E f(x)\, dx.$$

Since $M(\mathbb{R}^n)$ is the dual space of $C_0(\mathbb{R}^n)$ and the unit ball of $M(\mathbb{R}^n)$ is compact in the $w^*$ topology, $\exists\ \epsilon_k \searrow 0$ s.t $u_{\epsilon_k}$ converges $w^*$ to a measure $d\mu \in M(\mathbb{R}^n)$.

i.e. for each $\varphi \in C_0(\mathbb{R}^n)$

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x) u_{\epsilon_k}(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)$$

<u>Claim</u>: $\mu = u$ as a distribution.

i.e. $\forall\ \psi \in \mathcal{S}$.

$$u(\psi) = \int_{\mathbb{R}^n} \psi(x)\,d\mu(x).$$

Let $\psi_\epsilon(x) = \int_{\mathbb{R}^n} \psi(x-t)\, w(t,\epsilon)\, dt.$

then

$$D^\alpha \psi_\epsilon(x) = \int_{\mathbb{R}^n} D^\alpha \psi(x-t)\, w(t,\epsilon)\, dt.$$

so

$$D^\alpha \psi_\epsilon(x) \longrightarrow D^\alpha \psi(x) \quad \text{uniformly.}$$

( our ~~theorem~~ stated that for $f \in C^0 \subset L^\infty$.
$\|f * \varphi_\epsilon - f\|_\infty \to 0$ as $\epsilon \to 0$ ).
where $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$, $\varphi \in L^1$
$\int \varphi\, dx = 1$. )

It follows that $\psi_\epsilon \to \psi$ in $\mathcal{S}$ as $\epsilon \to 0$

so that

$$u(\psi_\epsilon) \to u(\psi)$$

but,

$$u(\psi_\epsilon) = u(\psi * W_\epsilon) = u(\psi * \tilde{W}_\epsilon) = (u * W_\epsilon)(\psi)$$
$$= \int_{\mathbb{R}^n} \psi(x)\, u_\epsilon(x)\, dx$$

$\therefore$

$$u(\psi_{\epsilon_k}) = \int_{\mathbb{R}^n} \psi(x)\, u_{\epsilon_k}(x)\, dx$$

$$\to \int_{\mathbb{R}^n} \psi(x)\, d\mu \qquad \text{as } \epsilon_k \to 0.$$

The bound on the norm of $B$ by the total variation is clear.

<u>Thm:</u> $u \in \mathcal{S}'$ is in $(L^2, L^2)$ iff $\exists\ b \in L^\infty(\mathbb{R}^n)$ s.t. $\hat{u} = b$.

In this case, $\|b\|_\infty$ is the norm of

$$B : L^2 \cap \mathcal{S} \to L^2$$
$$B\varphi = u * \varphi$$

and also, $\widehat{(u * \varphi)} = \hat{u}\hat{\varphi}$

<u>Pf</u>:

Let $\varphi_0 = e^{-\pi|x|^2}$ $(\in \mathcal{S})$

(and recall that $\hat{\varphi_0} = \varphi_0$ )

If $u \in (L^2, L^2)$ then

$$\Phi_0 \equiv \hat{u}\,\hat{\varphi_0} = \widehat{(u * \varphi_0)} \in L^2(\mathbb{R}^n).$$

Let $b(x) = e^{\pi|x|^2}\Phi_0(x) = \dfrac{\Phi_0(x)}{\hat{\varphi_0}(x)}.$

<u>Claim:</u> $\widehat{(u * \varphi)} = b\,\hat{\varphi}.$ $\forall \varphi \in \mathcal{S}.$

Since $\widehat{u * \varphi} = \hat{u}\,\hat{\varphi}$ (as a dist.).

its enough to show

$\hat{u}\,\hat{\varphi}(\psi) = b\,\hat{\varphi}(\psi)$ $\forall \psi \in \mathcal{D}.$
($\mathcal{D} \subseteq \mathcal{S}$ is dense

<u>multiplication:</u>

For $v \in \mathcal{A}'$ $\psi \in \mathcal{A}$.

let $v\psi \in \mathcal{A}'$ be defined by

$$(v\psi)(\varphi) = v(\psi\varphi) \quad \forall \varphi \in \mathcal{A}.$$

With this definition we claim that for $u \in \mathcal{A}'$ and $\varphi \in \mathcal{A}$

$$\widehat{(u * \varphi)} = \hat{u}\hat{\varphi}$$

i.e. that $\forall \psi \in \mathcal{A}$

$$\widehat{u * \varphi}(\psi) = \hat{u} \cdot \hat{\varphi}(\psi).$$

We have

$$\hat{\varphi}\psi = \mathcal{F}^{-1}(\tilde{\varphi} * \hat{\psi}),$$

so

$$(\widehat{u * \varphi})(\psi) = (u * \varphi)(\hat{\psi}) = u(\tilde{\varphi} * \hat{\psi})$$

$$= \hat{u}(\hat{\varphi}\psi) = \hat{u}\hat{\varphi}(\psi)$$

If $\psi \in \mathcal{D}$ then

$$\frac{\psi}{\hat{\varphi}_0}(x) = \psi(x)\, e^{\pi|x|^2} \in \mathcal{D}$$

$$\therefore \ \hat{u}\hat{\varphi}(\psi) = \hat{u}(\hat{\varphi}\psi) = \hat{u}(\hat{\varphi}\hat{\varphi}_0 \psi/\hat{\varphi}_0)$$

$$= (\hat{u}\hat{\varphi}_0)(\hat{\varphi}\psi/\hat{\varphi}_0)$$

$$= \Phi(\hat{\varphi}\psi/\hat{\varphi}_0)$$

$$= \int_{\mathbb{R}^n} \Phi_0(x)\,\hat{\varphi}(x)\, e^{\pi|x|^2}\psi(x)\,dx$$

$$= \int_{\mathbb{R}^n} b(x)\,\hat{\varphi}(x)\,\psi(x)\,dx$$

$$= (b\hat{\varphi})(\psi).$$

$$\therefore \ \hat{u}(\hat{\varphi}\psi) = b\hat{\varphi}(\psi) = b(\hat{\varphi}\psi) \quad \forall \varphi \in \mathcal{S}, \ \psi \in \mathcal{D}.$$

Choose $\varphi$ s.t $\hat{\varphi} = 1$ in $\operatorname{supp}(\psi)$

to see that

$$\hat{u}(\varphi) = b(\varphi). \qquad \forall \varphi \in \mathcal{D}.$$

$\therefore \hat{u} = b.$

Since $u \in (L^2, L^2)$ $\exists$ $A > 0$ s.t.

$$\|b\hat{\varphi}\|_2 \underset{\text{just showed}}{=} \|(\widehat{u * \varphi})\|_2 \underset{\text{Plancherel}}{=} \|u * \varphi\|_2 \underset{(L_2, L_2)}{\leq} A \|\varphi\|_2.$$

$\forall \varphi \in \mathcal{S}.$

$\Rightarrow$ $b \in L^\infty$

(Choose $\hat{\varphi}$ to be a $C_c^\infty$ bump function approximation to the identity).

Conversely, if

$\hat{u} = b \in L^\infty(\mathbb{R}^n)$

$(\widehat{u * \varphi}) = \hat{u}\hat{\varphi}$ so $\|\widehat{u * \varphi}\|_2 = \|b\hat{\varphi}\|_2$

$\therefore \|u * \varphi\|_2 \leq \|b\|_\infty \|\varphi\|_2$ (Plancherel)