# Math 138 4/30/09.

## A technical lemma

Recall that $f \in L^p$ has an $L^p$ derivative w.r.t $x_k$

if: (take $h = (0, \dots, 0, h_k, 0 \dots 0)$.) $\exists\, g \in L^p$

s.t. $$\left( \int_{\mathbb{R}^n} \left| \frac{f(x+h) - f(x)}{h_k} - g(x) \right|^p dx \right)^{1/p} \to 0$$

as $h_n \to 0$.

higher order $L^p$-derivatives are then similarly defined.

Lemma: if $f \in L^p(\mathbb{R}^n)$ has $L^p$ derivatives of all orders $\leq n+1$, then $f$ is a.e. equal to a continuous fcn. $g$ s.t.

$$|g(0)| \leq C \sum_{|\alpha| \leq n+1} \|D^\alpha f\|_p$$

with $C = C(n, p)$..

pf: take $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\exists\ C' = C'(n)$ s.t.

$$(1+|x|^2)^{\frac{n+1}{2}} \le (1+|x_1|+\dots+|x_n|)^{n+1}$$

$$\le C' \sum_{|\alpha| \le n+1} |x^\alpha|.$$

~~$C'$ is a max of multinomial coefficients.~~

Suppose $p = 1$.

$$|\hat{f}(x)| \le \frac{C' \sum_{|\alpha|\le n+1} |x^\alpha|}{(1+|x|^2)^{\frac{n+1}{2}}} |\hat{f}(x)|.$$

$$= C' (1+|x|^2)^{-\left(\frac{n+1}{2}\right)} \sum_{|\alpha|\le n+1} \left|\widehat{(2\pi^{-|\alpha|} D^\alpha f)}(x)\right|.$$

$$\le C'' (1+|x|^2)^{-\left(\frac{n+1}{2}\right)} \sum_{|\alpha|\le n+1} \|D^\alpha f\|_1.$$

$\therefore\ \hat{f} \in L^1(\mathbb{R}^n)$.

and $\|\hat{f}\|_{L^1} \le C''' \sum_{|\alpha|\le n+1} \|D^\alpha f\|_1$.

with $C''' = C'' \cdot \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}} dx$.

$\therefore$ $f$ is a.e. equal to a continuous function $g$ and.

$$|g(0)| \le \|f\|_\infty \le \|\hat{f}\|_1 \le C \sum_{|\alpha| \le n+1} \|D^\alpha f\|_2 .$$

Now suppose $p > 1$.

Take $\varphi \in C^\infty$ with $\varphi(x) = 1$ if $|x| \le 1$.
$= 0$ if $|x| > 2$.

Then $\varphi f \in L^1$ and has $L^1$ derivatives of all orders $\le n+1$.

$\therefore$ $\varphi f$ ~~is~~ ~~at~~ a.e equal to a continuous $h$ s.t.

$$|h(0)| \le C \sum_{|\alpha| \le n+1} \| D^\alpha (\varphi f) \|_1 .$$

Now $D^\alpha(\varphi f) = \sum_{\mu + \nu = \alpha} C^{\mu\nu} (D^\mu f)(D^\nu \varphi)$

so

$$\| D^\alpha(\varphi f)\|_1 \le \int_{|x| \le 2} \sum_{\mu+\nu=\alpha} C^{\mu\nu} |D^\mu f| \, |D^\nu \varphi| \, dx .$$

$$\leq \sum_{\mu+\nu=\alpha} C\left(\sup_{|x|\leq 2}|D^\nu \varphi|\right)\int_{|x|\leq 2}|D^\mu f(x)|\,dx$$

$$\leq A \cdot \sum_{|\nu|\leq|\alpha|}\int_{|x|\leq 2}|D^\nu f(x)|\,dx.$$

$$\leq AB \sum_{|\nu|\leq|\alpha|}\|D^\nu f\|_p.$$

where $\begin{cases} B = \left(\int_{|x|\leq 2} dx\right)^{1/q} & \text{with } \frac{1}{p}+\frac{1}{q}=1. \\ A \geq \|D^\nu \varphi\|_\infty & \forall\ |\nu|\leq|\alpha|. \end{cases}$

$\therefore\ \exists\ K>0$ s.t.

$$|h(0)| \leq K \sum_{|\alpha|\leq n+1}\|D^\alpha f\|_p.$$

Since $\varphi = 1$ in $|x|\leq 1$,

f equals a.e. in $|x|\leq 1$, a cont fcn. $g$ s.t

$$|g(0)| = |h(0)| \leq K \sum_{|\alpha|\leq n+1}\|D^\alpha f\|_p.$$

But by changing $\varphi$, the argument shows that f = a.e. a continuous fcn on any ball centered at 0. //

Last time: We defined $\mathcal{S}$ and its topology (induced by the norms $\rho_{\alpha\beta}$) and the space of continuous linear functionals on $\mathcal{S}$ ($\mathcal{S}'$).

We saw that ~~this~~ $\mathcal{S}'$ includes $L^p$, tempered functions and tempered measures

Many important operations in analysis can be extended to $\mathcal{S}'$. $\left[\begin{array}{l}\text{and Fourier} \\ \text{Differentiation} \\ \text{operators}\end{array}\right.$

<u>e.g.</u> <u>convolution of $u \in \mathcal{S}'$ with $g \in \mathcal{S}$.</u>

If $g$ is a function on $\mathbb{R}^n$

let $\tilde{g}(x) = g(-x)$.

Then,

If $u, \varphi, \psi \in \mathcal{S}$

$$\int_{\mathbb{R}^n} (u * \varphi)(x)\, \psi(x)\, dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y)\varphi(x-y)\,dy \;\; \psi(x)\,dx$$

$$= \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \psi(x)\,\tilde{\varphi}(y-x)\,dx \;\; dy.$$

$$= \int_{\mathbb{R}^n} u(y)\,(\psi * \tilde{\varphi})(y)\,dy.$$

$$= \int_{\mathbb{R}^n} u(y)\,(\tilde{\varphi} * \psi)(y)\,dy.$$

Denote the linear functionals

$$\psi \to \int_{\mathbb{R}^n} (u * \varphi)(x)\, \psi(x)\, dx$$

$$\theta \to \int_{\mathbb{R}^n} u(x)\, \theta(x)\, dx$$

by $u * \varphi$ and $u$ respectively

then we have.

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi).$$

and note that the right hand side is well defined for any $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$.

and $\psi \to u(\tilde{\varphi} * \psi)$

is continuous.

We <u>define</u> the convolution of $u \in \mathcal{S}'$ with $\varphi \in \mathcal{S}$ by. $(u * \varphi \in \mathcal{S}')$.

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi)$$

fact $(u * \varphi) * \psi = u * (\varphi * \psi)$.

$\forall u \in \mathcal{S}'$ $\varphi, \psi \in \mathcal{S}$.

Thm (3.13). If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$.

then $u * \varphi$ is the function $f$

$$f(x) = u(\tau_x \tilde{\varphi})$$

where $(\tau_x g)(y) = g(y-x)$.

(so $(\tau_x \tilde{\varphi})(y) = \tilde{\varphi}(y-x) = \varphi(x-y)$.).

We have $f \in C^\infty$ and $f$ and all its derivatives are slowly increasing.

Pf: 1st. $f \in C^\infty$ and is slowly increasing.

Let $h = (0 \ldots 0, h_j, 0 \ldots 0)$.

then

$$\frac{\tau_{x+h} \tilde{\varphi} - \tau_x \tilde{\varphi}}{h_j} \longrightarrow - \tau_x \frac{\partial \tilde{\varphi}}{\partial x_j} \quad \text{in } \mathcal{S}.$$

as $|h| \to 0$

Since $u \in \mathcal{S}'$ is continuous

$$\frac{f(x+h)-f(x)}{h_j} = u\left(\frac{\tau_{x+h}\tilde{\varphi} - \tau_x\tilde{\varphi}}{h_j}\right) \to u\left(-\tau_x\left(\frac{\partial\tilde{\varphi}}{\partial x_j}\right)\right)$$

as $h_j \to 0$.

Since translations are continuous in $\mathcal{S}$.

~~$f$ has continuous 1st partial derivatives.~~

~~Since~~ $\frac{\partial\tilde{\varphi}}{\partial x_j} \in \mathcal{S}$ also, we can repeat the argument to show that $D^\beta f$ exist and is continuous for all $\beta = (\beta_1, \ldots, \beta_n)$

and

$$D^\beta f(x) = (-1)^{|\beta|} u(\tau_x D^\beta \tilde{\varphi}).$$

So, by induction, (since $D^\beta\tilde{\varphi} \in \mathcal{S}$) if $f$ is slowly increasing then so are all the $(D^\beta f)$.

To see that $f$ is slowly increasing

Since $u$ is continuous on $\mathcal{S}$, $\exists$ integers. $l, m$ and a const. $C > 0$ s.t.

$$|f(x)| = |u(\tau_x \tilde{\varphi})| \leq C \sum_{|\alpha| \leq l,\, |\beta| \leq m} \rho_{\alpha\beta}(\tau_x \tilde{\varphi})$$

but

$$\rho_{\alpha\beta}(\tau_x \tilde{\varphi}) = \sup_{w \in \mathbb{R}^n} |w^\alpha (D^\beta \tilde{\varphi})(w - x).|$$

$$= \sup_{w \in \mathbb{R}^n} |(w + x)^\alpha D^\beta \tilde{\varphi}(w)|.$$

which is bounded by a polynomial in $x$.

We now show that $u * \varphi = f$ in $\mathcal{S}'$.

i.e. that $(u * \varphi)(\psi) = \int_{\mathbb{R}^n} \psi(t) f(t)\, dt.$

$\forall \psi \in \mathcal{S}.$

We have.

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi)$$

$$= u\left(\int_{\mathbb{R}^n} \tilde{\varphi}(x-t)\psi(t)\,dt\right)$$

$$(*) \quad = u\left(\int_{\mathbb{R}^n} \tau_t \tilde{\varphi}(x)\psi(t)\,dt\right)$$

On $\mathcal{S}$, the finite Riemann sum approximations

$$\sum_{i=1}^{N} \tau_{t_i^*} \tilde{\varphi}(x)\,\psi(t_i^*)\,|(\Delta t)_i|$$

converge to $\int_{\mathbb{R}^n} \tau_t \tilde{\varphi}(x)\psi(t)\,dt$.

(its enough to check that they do so in any of the norms $\rho_{\alpha\beta}$ ).

So

$$(*) = \int_{\mathbb{R}^n} u(\tau_t \tilde{\varphi})\psi(t)\,dt.$$

$$= \int_{\mathbb{R}^n} \psi(t) f(t)\,dt.$$

since $u$ is continuous and linear.

Fourier Transform:

$\forall\ \varphi \in \mathcal{S}$ and $u \in \mathcal{S}$.

$$u(\hat{\varphi}) = \int_{\mathbb{R}^n} u(x)\,\hat{\varphi}(x)\,dx$$

$$= \int \hat{u}(x)\,\varphi(x)\,dx \qquad \text{(multiplication formula).}$$

$$= \hat{u}(\varphi)$$

So we define. (for $u \in \mathcal{S}'$).

$$\hat{u}(\varphi) = u(\hat{\varphi}).$$

If $f \in L^p$ $\quad 1 \le p \le 2$

$\hat{f}$ is the same as in the previous definition (since $\mathcal{S}$ is dense in every $L^p$ $\ 1 \le p \le \infty$).

(but see 4.13 for the case $p > 2$).

We can also extend the operation of differentiation to elements of $\mathcal{S}'$:

- If $u, \varphi \in \mathcal{S}$ then we have

$$\int_{\mathbb{R}^n} D^\beta u(x)\,\varphi(x) = (-1)^{|\beta|}\int_{\mathbb{R}^n} u(x)(D^\beta\varphi)(x)\,dx$$

by integration by parts.

$$\varphi \longrightarrow \int_{\mathbb{R}^n} D^\beta u(x)\,\varphi(x)\,dx$$

$$\psi \rightarrow \int_{\mathbb{R}^n} u(x)\,\psi(x)\,dx$$

are continuous linear functionals on $\mathcal{S}$ which we may denote by $D^\beta u$ and $u$ respectively we then have.

$$(D^\beta u)(\varphi) = (-1)^{|\beta|}\,u(D^\beta\varphi).$$

but the right hand side is well defined $\forall\, u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$.

and

$\varphi \to u(D^\beta \varphi)$ is continuous on $\mathcal{S}$. (composition of two cont. maps).

We define (for $u \in \mathcal{S}'$)

$D^\beta u \in \mathcal{S}'$ by

$$(D^\beta u)(\psi) = (-1)^{|\beta|} u(D^\beta \psi)$$

$\forall\, \psi \in \mathcal{S}$.

<u>Similarly</u>: (translation and reflection)

$$(\tau_h u)(\psi) = u(\tau_{-h} \varphi).$$

$u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

$$\tilde{u}(\varphi) = u(\tilde{\varphi})$$

$u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

If $f \in L^p$ is fixed and

$$Bg = f * g \qquad \text{for } g \in L^1(\mathbb{R}^n)$$

then

$$\|Bg\|_p \le \|f\|_p \, \|g\|_1 .$$ (An earlier thm which followed from Minkowski's integral inequality).

and

$$\tau_h (Bg) = B(\tau_h g)$$

(by change of variables.

We say that $B$ ~~is translation~~ commutes with translations.

$$( \tau_h B = B \tau_h ).$$

"All" bounded operators which commute with translation are of this type.

**Thm:** Suppose $B: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$

$1 \leq p, q \leq \infty$

is linear, bounded and commutes with translations; then $\exists !$

$u \in \mathcal{S}'$ s.t. $B\varphi = u * \varphi \quad \forall \varphi \in \mathcal{S}$.

**Pf:**

**Claim:** $\varphi \in \mathcal{S} \Rightarrow B\varphi$ has $L^q$ derivatives of all orders:

If $h = (0, \ldots, h_j, \ldots, 0)$ then

$$\left(\frac{\tau_h(B\varphi) - B\varphi}{h_j}\right) = \left(\frac{B(\tau_h\varphi) - B\varphi}{h_j}\right)$$

$$= B\left(\frac{\tau_h\varphi - \varphi}{h_j}\right)$$

$$\frac{\tau_h\varphi - \varphi}{h_j} \to -\varphi_j \equiv -\frac{\partial\varphi}{\partial x_j} \quad \text{in } \mathcal{S}.$$

$\therefore$ in $L^p$ norm.

($B: L^p \to L^q$ is bdd). $\therefore$ $$\frac{\tau_h(B\varphi) - B\varphi}{h_j} \to -\frac{\partial(B\varphi)}{\partial x_j} = -B\varphi_j \quad \text{in } L^q.$$

Iterating the argument shows that $B\varphi$ has $L^q$ derivatives of all orders.

and $\quad B(D^\alpha \varphi) = D^\alpha (B\varphi) \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n)$.

From our earlier lemma, we have that $B\varphi = $ a.e. a cont. fcn $g_\varphi$ s.t.

$$|g_\varphi(0)| \leq C \sum_{|\alpha| \leq n+1} \| D^\alpha (B\varphi) \|_q$$

$$= C \sum_{|\alpha| \leq n+1} \| B(D^\alpha \varphi) \|_q$$

$$\leq \|B\| \cdot C \sum_{|\alpha| \leq n+1} \| D^\alpha \varphi \|_p.$$

Since the $\|D^\alpha \varphi\|_p$ are controlled by a pair of ~~semi~~ norms $\rho_{\alpha', \beta'}$ we have that

$\varphi \to g_\varphi(0)$ is a cont. linear functional on $\mathcal{S}$.

let $u_1(\varphi) = g_\varphi(0)$ as defined above.

<u>Claim</u> $u = \tilde{u}_1$ has $B\varphi(x) = u * \varphi$ $\forall \varphi \in \mathcal{S}$.

If $\varphi \in \mathcal{S}$ then

$$u * \varphi(x) = u(\tau_x \tilde{\varphi}) = u([\tau_{-x}\varphi]^\sim)$$

$$= \tilde{u}(\tau_{-x}\varphi)$$

$$= u_1(\tau_{-x}\varphi)$$

$$= B(\tau_{-x}\varphi))(0)$$

$$= \tau_{-x}(B\varphi)(0)$$

$$= (B\varphi)(x).$$

($u$ is clearly unique) ✓.