Math 138 4/28/09.

Final assignment.

Folland: 8.18 (Heisenberg).

Folland: 9.15.
9.29 (you may use Stein).

details of 4.13 from Stein.

- Explain how to use conditional expectations to prove the Lebesgue-Radon-Nikodym theorem (see Neveu).

Tempered Distributions

$\mathcal{S}$: the Schwartz space

$= C^\infty$ functions φ on $\mathbb{R}^n$ s.t.

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty.$$

$\forall$ $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$

non-negative integers

e.g. $\varphi(x) = e^{-\delta |x|^2} \in \mathcal{S}$ $\quad \forall\ \delta > 0$

This is a minimal requirement to be closed under differentiation and $\mathcal{F}$.

$\mathcal{D} =$ C^∞ functions with compact support.

e.g.

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

$f \in C^\infty$

and $\varphi(t) = f(1+t) f(1-t)$

$$= \begin{cases} e^{-2/(1-t^2)} & |t| < 1 \\ 0 & \text{else.} \end{cases}$$

$\in \mathcal{D}(\mathbb{R})$.

a) $\Psi(\vec{x}) = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_n) \in \mathcal{D}(\mathbb{R}^n)$.

b) $\psi(\vec{x}) = \begin{cases} e^{-2/(1-|x|^2)} & |x| < 1. \\ 0 & \text{else} \end{cases} \in \mathcal{D}(\mathbb{R}^n)$.

c) if $\eta \in C^\infty$ and ψ as in b) then $\psi(\epsilon x)\eta(x) \in \mathcal{D}(\mathbb{R}^n)$

and $\quad \epsilon^2 \psi(\epsilon x)\, \eta(x) \to \eta(x) \quad$ as $\epsilon \to 0$

(in the topology of $\mathcal{S}$ which we are about to define).

Note that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < +\infty \quad \forall \alpha, \beta$$

iff $\displaystyle \sup_{x \in \mathbb{R}^n} |D^\beta (x^\alpha \varphi(x))| < +\infty \quad \forall \alpha, \beta.$

$\therefore$ if $\varphi \in \mathcal{S}$ then

$$P(x)\, \varphi(x) \quad \text{and} \quad P(D)\, \varphi(x)$$

are in $\mathcal{S}$

for any polynomial P. (in n variables).

$\mathcal{S}$ is a dense subset of C_0 and of L^p $\quad 1 \leq p < \infty$.

In fact $\mathcal{D}$ is

We have (for the L^p norm of $\varphi \in \mathcal{S}$)

$$\left(\int_{\mathbb{R}^n} |\varphi(x)|^p\,dx\right)^{1/p} \leq \left(\int_{|x|<1} |\varphi(x)|^p\,dx\right)^{1/p} + \left(\int_{|x|\geq 1} |\varphi(x)|^p\,dx\right)^{1/p}$$

$$\leq \|\varphi\|_\infty \left(\int_{|x|<1} dx\right)^{1/p}$$

$$+ \left(\int_{|x|\geq 1} \left(|x|^{2n}|\varphi(x)|\right)^p \cdot \frac{1}{|x|^{2np}}\,dx\right)^{1/p}$$

$$\leq \|\varphi\|_\infty \left(\mathrm{Vol}_{(n)}(B_1)\right)^{1/p}$$

$$+ \sup_{x\in\mathbb{R}^n}\left(|x|^{2n}|\varphi(x)|\right)\left(\int_{|x|\geq 1} \frac{1}{|x|^{2np}}\,dx\right)^{1/p}$$

$$\int_{|x|\geq 1} \frac{1}{|x|^{2np}}\,dx = \left(\left(\int_1^\infty \frac{1}{r^{2np}}\, r^{n-1}\,dr\right)\left(\mathrm{Area}_{n-1}(S_{(1)})\right)\right)$$

$$= \left(\int_1^\infty \frac{1}{r^{2np-n+1}}\,dr\right)\left(\mathrm{Area}_{(n-1)}(S_{(1)})\right)$$

$r^{-2np+n-1}$

$$\frac{r^{-2np+n}}{-2np+n}\Bigg|_1^\infty = \frac{1}{n(2p-1)}$$

Since $P(D)\hat{f}(x) = (P(-2\pi i t)f(t))^\wedge(x)$,

~~each $f \in \mathcal{S}$ has~~

and $P(2\pi i x)\hat{f}(x) = (P(D)f)^\wedge(x)$

each $f \in \mathcal{S}$ has $\hat{f} \in \mathcal{S}$.

$$x^\alpha \cdot D^\beta \hat{f}(x) = x^\alpha \cdot ((-2\pi i t)^\beta f(t))^\wedge(x)$$

$$= \Big(\underbrace{D^\alpha\big((-2\pi i t)^\beta\big) f(t)}_{\in \mathcal{S}}\Big)^\wedge(x)$$

$\therefore$ $|x^\alpha D^\beta \hat{f}|$ is bounded.

In fact

Since $(\mathcal{F}^{-1}g)(x) = \mathcal{F}g(-x)$ for $g \in \mathcal{S}$.

$\mathcal{F}$ is a 1-1 map of $\mathcal{S}$ onto $\mathcal{S}$.

if $\varphi, \psi \in \mathcal{S}$. then

$\hat{\varphi}, \hat{\psi}$ are in $\mathcal{S}$

$\therefore \hat{\varphi}\hat{\psi} \in \mathcal{S}$

$\therefore \varphi * \psi \in \mathcal{S}$.

The countable family of (~~semi~~)norms.

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)|$$

induces a topology on $\mathcal{S}$ which is metrizable.

Our earlier results show that a linear functional L on $\mathcal{S}$ is continuous iff $\exists\ C > 0$ ~~s.t.~~ and integers m, ℓ s.t.

$$|L(\varphi)| \le C \sum_{|\alpha| \le \ell,\ |\beta| \le m} \rho_{\alpha\beta}(\varphi) \quad \forall \varphi \in \mathcal{S}.$$

Some topological facts:

- $\varphi(x) \Rightarrow x^{\alpha} D^{\beta} \varphi(x)$ is continuous
- $\varphi \in \mathcal{S} \Rightarrow \lim\limits_{h \to 0} \tau_h \varphi = \varphi$
- for $\varphi \in \mathcal{S}$ and $h = (0, \ldots, h_i, \ldots 0)$

$$[\varphi - \tau_h \varphi]/h_i \rightarrow \frac{\partial \varphi}{\partial x_i} \quad \text{as } |h| \to 0.$$

- $\mathcal{S}$ is complete in the metric.
- $\mathcal{F}$ is a homeomorphism of $\mathcal{S} \leftrightarrow \mathcal{S}$.
- $\mathcal{D}$ is a dense subset of $\mathcal{S}$.
- $\mathcal{S}$ is separable.

The collection of all continuous linear functionals on $\mathcal{S}$ is called the space of Tempered Distributions

e.g.:

1) $f \in L^p(\mathbb{R}^n) \quad 1 \le p \le \infty.$

$$L(\varphi) = L_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx.$$

$$|L(\varphi)| \le \left(\int_{\mathbb{R}^n} |f|^p dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |\varphi(x)|^q dx\right)^{1/q}.$$

$$(\forall \varphi \in \mathcal{S}.)$$

$$\le \|f\|_p \left(\|\varphi\|_\infty \left(\frac{\omega_{n-1}}{n}\right)^{1/q} + \rho_{2n,0}(\varphi)\left(\frac{\omega_{n-1}}{n(2q-1)}\right)^{\frac{1}{q}}\right)$$

2). μ a finite Borel measure

$$L = L_\mu$$

$$L(\varphi) = L_\mu(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)$$

$$|L(\varphi)| \le \|\mu\|\,\|\varphi\|_\infty.$$

3). A measurable function f s.t.

$$\frac{f(x)}{(1+|x|^2)^k} \in L^p(\mathbb{R}^n) \quad (1 \le p \le \infty)$$

for some k.

is called a tempered L^p function (or slowly increasing if $p=\infty$).

$$L(\varphi) = \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx \qquad \varphi \in \mathcal{S}.$$

is in $\mathcal{S}'$.

since

$$L(\varphi) = \int_{\mathbb{R}^n} \left((1+|x|^2)^k \varphi(x)\right) \left(\frac{f(x)}{(1+|x|^2)^k}\right) dx$$

and $\varphi(x) \to (1+|x|^2)^k \varphi(x)$

is continuous in $\mathcal{S}$.

4) A tempered measure is a μ s.t.

$\int_{\mathbb{R}^n} (1+|x|^2)^{-k} |d\mu| < \infty$ for some k

$L(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$, $L \in \mathcal{S}'$!

5) $x_0 \in \mathbb{R}^n$ fixed.

$$L(\varphi) = D^\beta \varphi(x_0)$$

defines a tempered distribution.

For $\beta = 0$ this is the same as $\mu = \delta_0$

The distributions of examples 1, 3 are called functions

Those of 2, 4 are called measures.

The topology on $\mathcal{S}'$ is the weakest topology s.t. all the functionals (on $\mathcal{S}'$)

$L \to L(\varphi)$ are cont.

$(\varphi \in \mathcal{S})$.