Math 138 4/23/09.

<u>Last time</u>

<u>Thm:</u> $f \in L^1 \cap L^2 \Rightarrow \|\hat{f}\|_2 = \|f\|_2$.

Since $L^1 \cap L^2$ is dense in L^2, the Fourier operator has a unique continuous extension to all of L^2 and we continue to call this extension $\mathcal{F}$.

More concretely:

For any $f \in L^2$ let $h_k \to f$ in L^2 with $h_k \in L^1 \cap L^2$ then

$$\mathcal{F} h_k \to \mathcal{F} f$$

We can take $h_k(x) = \begin{cases} f(x) & |x| \le k \\ 0 & \text{else.} \end{cases}$

and then

$$\hat{f}(x) = \lim_{\substack{k \to \infty \\ (L^2)}} \int_{|t| \le k} f(t) e^{-2\pi i x \cdot t} \, dt$$

$$= \lim_{\substack{k \to \infty \\ (L^2)}} \int_{\mathbb{R}^n} h_k(t) e^{-2\pi i x \cdot t} \, dt.$$

Theorem: $\mathcal{F}: L^2 \to L^2$ as defined above is onto.

Pf: • The range of $\mathcal{F}$ is a closed subspace of L^2.

Suppose $g_n = \mathcal{F} f_n$ has.

$$g_n \to g \quad \text{in } L^2.$$

Then $\|g_m - g_n\| = \|f_m - f_n\|$

$\forall\, m,n$ since $\mathcal{F}$ is an isometry.

The Cauchy sequence $\{f_n\}$ $\therefore$ has an L^2 limit f and by continuity

$$\mathcal{F} f = g.$$

If the range of $\mathcal{F}$ is not all of L^2 we can choose $g \in L^2$, $g \neq 0$ s.t.

$$\langle \hat{f}, g \rangle = 0 \quad \forall f \in L^2$$

From earlier $\int f \hat{g}\, dx = \int f \hat{g}\, dx$

when f, g are in L^1

Since $f, g \in L^2 \Rightarrow \hat{f}, \hat{g} \in L^2$ this multiplication formula extends to L^2.

So we have $\int f \hat{g}\, dx = 0 \quad \forall f \in L^2$.

$\therefore \quad \hat{g} = 0 \qquad \therefore \; g = 0. \quad (\text{since } \|g\|_2 = \|\hat{g}\|_2).$

$\rightarrow \leftarrow$

A linear operator on a Hilbert Space which is isometric and onto is called a <u>unitary operator</u>.

<u>A digression on adjoints and unitaries:</u>

if $T : H \to H \quad \in \mathcal{L}(H)$.
and $g \in H$ is fixed.

then $\quad l(f) = \langle Tf, g \rangle$

is a bdd linear functional on H.

$$|l(f)| \leq \|T\| \, \|f\| \, \|g\|.$$

So $\exists!$ element of H, denoted T^*g.

s.t. $\quad l(f) = \langle f, T^*g \rangle.$

$$\langle Tf, g_1 \rangle + \langle Tf, g_2 \rangle = \langle Tf, g_1 + g_2 \rangle = \langle f, T^*(g_1+g_2) \rangle$$

$$= \langle f, T^*g_1 \rangle + \langle f, T^*g_2 \rangle \quad (\forall f).$$

$$= \langle f, T^*g_1 + T^*g_2 \rangle$$

So T^* is linear.

$$|\langle T^* g, T^* g\rangle| = |\ell(T^* g)| \le \|T\| \|T^* g\| \|g\|$$

$$= \|T^* g\|^2$$

$\therefore \quad \|T^* g\| \le \|T\| \|g\|$ and $\|T^*\| \le \|T\|.$

$$\langle T^* f, g\rangle = \overline{\langle g, T^* f\rangle} = \overline{\langle Tg, f\rangle}$$

$$= \langle f, Tg\rangle$$

so $(T^*)^* = T.$

$\therefore \quad \|T\| \le \|T^*\|$ also and $\|T\| = \|T^*\|.$

$$\langle T_1 T_2 f, g\rangle = \langle T_2 f, T_1^* g\rangle$$

$$= \langle f, T_2^* T_1^* g\rangle.$$

$$\therefore \quad (T_1 T_2)^* = (T_2^* T_1^*).$$

$U \in \mathcal{L}(H)$ is unitary iff $UU^* = I$. $= U^*U = I$

~~(since $I^* = I$, always iff $UU^* = I$)~~ $U^*U = I$

if U is isometric ~~and onto~~ then by polarization

$$\langle x, y\rangle = \langle Ux, Uy\rangle = \langle U^*U x, U^*U y\rangle$$

$$\therefore (U^*Uy - y) \perp H. \quad \therefore U^*Uy = y$$

$$\langle Ux, y \rangle = \langle x, U^{*}y \rangle.$$

$$= \langle Ux, UU^{*}y \rangle.$$

$$\Rightarrow (UU^{*}y - y) \perp Ux \quad \forall x.$$

Since U is onto $UU^{*}y = y \quad \forall y.$

if (conversely)

$$UU^{*} = U^{*}U = I$$

then $\langle x, y \rangle = \langle U^{*}U x, y \rangle = \langle Ux, Uy \rangle$

$\forall x, y.$

so U is an isometry

and if $x \perp \text{Range}(U)$ then.

$\cancel{\langle x, Uy \rangle = 0 \ \forall y.}$

$\cancel{\text{so } \langle U^{*}x, y \rangle = 0 \ \forall y.}$

$$\langle x, U(U^{*}x) \rangle = 0. \Rightarrow \langle x, x \rangle = 0$$

$$\Rightarrow x = 0.$$

Thm: $\mathcal{F}^{-1}$ is well defined on L^2 and

$$(\mathcal{F}^{-1} g)(x) = (\mathcal{F} g)(-x) \qquad \forall g \in L^2(\mathbb{R}^n)$$

Pf: We claim that

$\mathcal{F}^{-1} \hat{f}$ is the L^2 limit of

$$f_n(t) = \int_{|x| \le n} \hat{f}(x) e^{2\pi i t \cdot x} dx.$$

If $\hat{f} \in L^1 \cap L^2$ we can put

$$\tilde{f}(t) = \int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i x \cdot t} dx.$$

$$= \lim_{\substack{n \to \infty \\ (L^2)}} \int_{|x| \le n} \hat{f}(x) e^{2\pi i t \cdot x} dx.$$

and $(\forall g \in L^1 \cap L^2)$

$$\langle g, \tilde{f} \rangle = \int_{\mathbb{R}^n} g(t) \left(\overline{\int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i x t} dx} \right) dt$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(t) e^{-2\pi i x t} dt \right) \overline{\hat{f}(x)} \, dx$$

$$= \langle \mathcal{F} g, \hat{f} \rangle$$

$$= \langle g, \mathcal{F}^* \hat{f} \rangle. = \langle g, \mathcal{F}^{-1} \hat{f} \rangle.$$

$$= \langle g, f \rangle.$$

$\therefore \quad \tilde{f} = f.$

We get the general case by using the continuity of the extension of $\mathcal{F}, \mathcal{F}^*$.

We have shown that

$$(\mathcal{F}^{-1} \hat{f})(x) = (\mathcal{F} f)(-x) \qquad \forall f \in L^2.$$

Since $\mathcal{F}$ is unitary, we are done.

There are corresponding pointwise convergence results for Abel and Gauss means.

(The summability kernels are in L^2).

We can extend the definition of $\mathcal{F}$ to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ by linearity

if $f = f_1 + f_2 \qquad f_1 \in L^1, f_2 \in L^2$

$$\widehat{(f_1 + f_2)} \equiv \hat{f}_1 + \hat{f}_2$$

and if $\quad f = f_1 + f_2 = g_1 + g_2 \quad g_1 \in L^1, g_2 \in L^2$ also.

then $\quad f_1 - g_1 = g_2 - f_2 . \in L^1 \cap L^2$

$$\therefore \widehat{f_1 - g_1} = \widehat{g_2 - f_2}$$

$$\therefore \hat{f_1} + \hat{f_2} = \hat{g_1} + \hat{g_2} .$$

note:

$\forall p \quad 1 \leq p \leq 2 \qquad L^p \subseteq L^1 + L^2 .$

and the positivio method can be used to solve the inversion problem.

Thm: (proof omitted).

$$f \in L^1 , \; g \in L^p \qquad 1 \leq p \leq 2$$

$$\Rightarrow h = f * g \in L^p$$

and $\hat{h}(x) = \hat{f}(x) \hat{g}(x) \quad \forall$ a.e. x.

on the other hand
See 4.13 in Stein and Weiss.