Math 138. 4/21/09

Last time:

Thm: If Φ and $\varphi = \hat{\Phi}$ are $L^1(\mathbb{R}^n)$

and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$

then $\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Phi(\epsilon t)\hat{f}(t)e^{2\pi i t\cdot x}\,dt$

$\to f(x)$ in $L^1(\mathbb{R}^n)$.

e.g. Φ = Abel or Gauss Kernel.

$\Phi(x) = e^{-|x|}$ $\qquad$ $\Phi(x) = e^{-|x|^2}$

(appropriately normalized).

Corollary: If f and $\hat{f}$ are both in $L^1(\mathbb{R}^n)$

then $f(x) = \int_{\mathbb{R}^n} \hat{f}(t)e^{2\pi i x\cdot t}\,dt$. a.e.

Corollary: If $f_1, f_2 \in L^1(\mathbb{R}^n)$ and

$\hat{f_1} = \hat{f_2}$ $\forall x \in \mathbb{R}^n$ then $f_1 = f_2$ a.e.

Recall that for Borel measures (say on $\mathbb{R}$) we have.

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x}\, d\mu(x).$$

<u>Thm</u>: (Parseval's Formula)

Let μ be (complex) Borel measure on $\mathbb{R}$.

f continuous in $L^1(\mathbb{R})$

s.t $\hat{f} \in L^1(\mathbb{R})$ Then.

$$\int f(x)\, d\mu(x) = \int \hat{f}(\xi)\, \hat{\mu}(-\xi)\, d\xi.$$

<u>Pf</u>:
$$f(x) = \cancel{\tfrac{1}{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

so

$$\int f(x)\, d\mu(x) = \iint \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\mu(x)\, d\xi = \int \hat{f}(\xi)\, \hat{\mu}(-\xi)\, d\xi$$

Now the set of continuous functions f in $L^1(\mathbb{R})$ s.t $\hat{f} \in L^1(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

For:

Consider a C^2 function g with support $[-R, R]$.

$$\hat{g}(t) = \int_{-2R}^{2R} g(x) e^{-2\pi i t \cdot x} \, dx.$$

$$u = g(x) \qquad dv = e^{-2\pi i t x} dx$$

$$du = g'(x)\, dx \qquad v = -\frac{1}{2\pi i t} e^{-2\pi i t x} dx.$$

$$= \frac{1}{(2\pi i t)} \int_{-2R}^{2R} g'(x) e^{-2\pi i t x} \, dx.$$

$$= (\text{similarly}). = \frac{1}{(2\pi i t)^2} \int_{-2R}^{2R} g''(x) e^{-2\pi i t x} dx$$

and so $|\hat{g}(t)| \leq \dfrac{C}{1+t^2} \in L^1(\mathbb{R})$.

So if $\int f(x)\, d\mu(x) = 0 \quad \forall$ such functions

then $\int f(x)\, d\mu(x) = 0 \quad \forall \; f \in C_0(\mathbb{R})$.

Since the space of finite complex Borel measures is the dual space of $C_0(\mathbb{R})$, such a measure must be $\mu \equiv 0$.

$\therefore$

<u>Corollary</u>: if $\hat{\mu}(\xi) = 0 \quad \forall \xi$

then $\mu = 0$.

(Probability)

<u>e.g.</u> The "characteristic function" of a random variable X $(X, \Omega, \mathcal{M}, \mathbb{P})$ is the Fourier transform of its law.

$$P_X(E) = \text{Prob}\{X \in E\}.$$

where $E \subset \mathbb{R}$ is Borel.

and

$$\int e^{iux} P_X(dx) = E\left(e^{iuX}\right)$$

Two random variables with the same characteristic function have the same law.

If X and Y are independent, so are e^{iuX} and e^{ivY}

$\therefore$

$$E\left(e^{i(uX+vY)}\right) = E(e^{iuX})\,E(e^{ivY}).$$

So whe X, Y are independent, the joint characteristic function

$$E\left(e^{i[(u,v)\cdot(X,Y)]}\right)$$

factors into the product of the respective characteristic functions.

The converse also holds.

<u>Prop:</u> If $E\left(e^{i(uX+vY)}\right) = E(e^{iuX})E(e^{ivY})$ $\forall\, u,v$, then X, Y are independent.

Pf: let X' be a random variable with the same law as X

Y' one with same law as Y

s.t X', Y' are independent

(if μ is the law of X

then for $\omega \in [0,1]$, $X'(\omega) = \inf \{ t : \mu((-\infty, t]) \geq \omega \}$.

has $P_{X'}(-\infty, a]) = P(X \leq a) = \mu(-\infty, a]$

$\forall a$

so $\mu = P_{X'}$.).

Let $\Omega = [0,1]^2$ with Lebesgue measure.

X' a function of X

Y' a function of Y

with X', Y' defined as above.

then

$$E\left(e^{i(uX'+vY')}\right) = E(e^{iuX'})\,E(e^{ivY'})$$

Since X, X' have the same law.

$$E(e^{iuX'}) = E(e^{iuX})$$

and similarly for Y, Y'

$\therefore$ (X', Y') has

$$E\left(e^{i(uX'+vY')}\right) = E\left(e^{i(uX+vY)}\right)$$

$\therefore$ (X', Y') has the same joint law as (X, Y).

So (X, Y) are independent.

Pointwise convergence

Last semester we saw that if $f \in L^1_{loc}(\mathbb{R}^n)$ then

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(u)\,du \to f(x).$$

a.e.

In fact,

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(u)-f(x)|\,du \to 0$$

at each point of the "Lebesgue set of f". (which is a.e.).

Since $\frac{1}{|B(x,r)|}\int_{B(x,r)} f(u)\,du$.

$$= f * K_r(x)$$

where $K_r(x) = \frac{\chi_{B(0,r)}}{|B(0,r)|}$

The following is Plausible

$\forall f \in L^p(\mathbb{R}^n) \quad 1 \le p \le \infty.$

$$\int_{\mathbb{R}^n} f(t)\, P(x-t, \epsilon)\, dt \to f(x)$$

as $\epsilon \to 0$.

for $x \in$ Lebesgue set of f.

and similarly for

$$\int_{\mathbb{R}^n} f(t)\, W(x-t, \epsilon)\, dt \to f(x)$$

taking this as a fact,

note that any point of continuity of f is in the Lebesgue set of f.

So if f is continuous at 0

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x)\, e^{-2\pi \epsilon |x|}\, dx = f(0).$$

$$\left(\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i t \cdot x} e^{-2\pi \epsilon |x|} dx = \int_{\mathbb{R}^n} f(x)\, P(x-t, \epsilon)\, dx. \right)$$

If $\hat{f} \geq 0$ also, then by Fatou's lemma.

$$\int \varliminf_{\epsilon \to 0} \hat{f}(x) e^{-2\pi\epsilon|x|}\, dx \leq \varliminf_{\epsilon \to 0} \int \hat{f}(x) e^{-2\pi\epsilon|x|}\, dx$$

$$= f(0).$$

so $\hat{f} \in L^1(\mathbb{R}^n)$.

and $\int \hat{f}(t) e^{2\pi i t x}\, d\cancel{t}$ defines a cont. fcn of x which equals $f(x)$ a.e.

<u>Corollary</u>: (to the improved statement)

$f \in L^1$, $\hat{f} \geq 0$, f cont at 0

$\Rightarrow \hat{f} \in L^1$ and

$$f(\cancel{x}) = \int_{\mathbb{R}^n} \hat{f}(\cancel{t})\, e^{2\pi i t x}\, d\cancel{t}. \quad \text{a.e. } x.$$

and in particular $f(0) = \int_{\mathbb{R}^n} \hat{f}(t)\, dt.$

L^2 Theory of $\mathcal{F}$

For a general $f \in L^2(\mathbb{R}^n)$, $\hat{f}(t)$ is not defined by $\int f(x) e^{-2\pi i t x} dx$.

but, <u>Theorem:</u> if $f \in L^1 \cap L^2$ then

$$\|\hat{f}\|_2 = \|f\|_2$$

<u>Pf:</u>

let $g(x) = \overline{f(-x)}$

and $h = f * g$.

Earlier we proved that if $f \in L^p$, $1 \le p \le \infty$ and $g \in L^1$

then $h = f * g \in L^p$

has $\|h\|_p \le \|f\|_p \|g\|_1$.

So $h \in L^1(\mathbb{R}^n)$ and.

$$\hat{h} = \hat{f}\hat{g}$$

But $\hat{g}(t) = \int \overline{f(-x)}\, e^{-2\pi i t x} dx$

$$= \overline{\int f(-x)\, e^{2\pi i t x} dx} = \overline{\int f(u) e^{-2\pi i t u} du}$$

$$= \overline{\hat{f}(t)}.$$

So $\hat{h} = |\hat{f}|^2$

Now h is uniformly continuous:

$$h(x) = \int_{\mathbb{R}^n} f(u)\, g(x-u)\, du.$$

$$|h(x+\epsilon) - h(x)| = \left| \int_{\mathbb{R}^n} f(u) \left(g(x+\epsilon-u) - g(x-u) \right) du \right|$$

$$\leq \left(\int_{\mathbb{R}^n} |f(u)|^2 \right)^{1/2} \underbrace{\left(\int_{\mathbb{R}^n} |g(x-u+\epsilon) - g(x-u)|^2 du \right)^{1/2}}_{\to 0 \text{ as } \epsilon \to 0}$$

by continuity of translation in L^2.

From our discussion of the pointwise theory.

$\hat{h} \in L^1(\mathbb{R}^n)$ and $h(0) = \int_{\mathbb{R}^n} \hat{h}(x)\, dx$.

$$\therefore \int_{\mathbb{R}^n} |\hat{f}|^2 dx = \int_{\mathbb{R}^n} \hat{h}\, dx = h(0)$$

$$= \int f(x)\, g(0-x)\, dx$$

$$= \int f(x)\, \overline{f(x)}\, dx$$

$$= \int_{\mathbb{R}^n} |f|^2 dx.$$