Math 138 4/16/09.

Our current goal is:

Thm: If Φ and its Fourier transform $\varphi = \hat{\Phi}$ are integrable and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ then the Φ means of the integral $\int_{\mathbb{R}^n} \hat{f}(t)\, e^{2\pi i t\cdot x}\,dt$ converge to $f(x)$ in the L^1 norm.

(In particular this holds for Abel and Gauss means).

We have the following calculations:

1. For all $\alpha > 0$

$$\int_{\mathbb{R}^n} e^{-\pi\alpha|y|^2} e^{-2\pi i t\cdot y}\,dy = \alpha^{-n/2} e^{-\pi|t|^2/\alpha}$$

(so $e^{-\pi|x|^2}$ is its own F.T).

2. For all $\alpha > 0$

$$\int_{\mathbb{R}^n} e^{-\pi\alpha|y|} e^{-2\pi i t\cdot y}\,dy = c_n \frac{\alpha}{(\alpha^2 + |t|^2)^{\frac{n+1}{2}}}$$

$$c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$$

Some details of these Calculations (1,2) are appended.

We write

(Weierstrass Kernel) $W(t,\alpha) = (4\pi\alpha)^{-n/2} e^{-|t|^2/4\alpha}$

(Poisson kernel) $P(t,\alpha) = c_n \dfrac{\alpha}{(\alpha^2 + |t|^2)^{(n+1)/2}}$

and we have also

3. For all $\alpha > 0$

$$\int_{\mathbb{R}^n} W(x,\alpha)\,dx = 1.$$

4. For all $\epsilon > 0$

$$\int_{\mathbb{R}^n} P(x,\epsilon)\,dx = 1.$$

We will express the Abel and Gauss means of

$$\int \hat{f}(x)\, e^{2\pi i t\cdot x}\,dx$$

in terms of convolutions of f with the Poisson and Weierstrass Kernels.

but to do that we need.

<u>Thm:</u> $f, g \in L^1(\mathbb{R}^n)$

$$\Rightarrow \int_{\mathbb{R}^n} \hat{f}(x) g(x)\, dx = \int_{\mathbb{R}^n} f(x) \hat{g}(x)\, dx.$$

<u>Pf:</u> Use Fubini

$$\int_{\mathbb{R}^n} \hat{f}(x) g(x)\, dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(t) e^{-2\pi i t \cdot x} dt \right) g(x)\, dx$$

$$= \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} g(x) e^{-2\pi i t \cdot x} dx\, dt$$

$$= \int f(t) \hat{g}(t)\, dt. \quad /$$

Consider

$$M_{\epsilon, \Phi}(f) = M_\epsilon(f) = \int_{\mathbb{R}^n} \Phi(\epsilon x) f(x)\, dx.$$

$$(\phi(0) = 1 \; , \; \Phi \in C_0 \,)$$

and suppose Φ is integrable and that $\hat{\Phi} = \varphi$.

Let $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(x/\epsilon)$, $\epsilon > 0$

and recall $(\delta_\epsilon \Phi)(x) = \Phi(\epsilon x)$.

has

$$\widehat{(\delta_\epsilon \Phi)}(x) = \epsilon^{-n}\varphi(x/\epsilon) = \varphi_\epsilon(x)$$

Notice also that

$$e^{2\pi i t\cdot x}\,\delta_\epsilon \Phi(x)$$

has transform $\varphi_\epsilon(x-t)$.

Then our multiplication formula shows

<u>Thm</u> if (f and Φ) $\in L^1(\mathbb{R}^n)$ and $\varphi = \hat{\Phi}$

then

$$\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i t\cdot x}\, \cancel{e^{-2\pi\epsilon|x|}}\, \Phi(\epsilon x)\, dx$$

$$= \int_{\mathbb{R}^n} f(x)\, \varphi_\epsilon(x-t)\, dt.$$

With $\Phi(x) = e^{-4\pi^2 |x|^2}$

we have $\varphi_\epsilon(x) = W(x, \epsilon^2)$

and with $\Phi(x) = e^{-2\pi|x|}$

we have

$$\varphi_\epsilon(x) = P(x, \epsilon).$$

So

$$\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i t\cdot x}\, e^{-2\pi\epsilon|x|}\, dx = \int_{\mathbb{R}^n} f(x)\, P(x-t, \epsilon)\, dx$$

$$\forall \epsilon > 0$$

and

$$\int_{\mathbb{R}^n} \hat{f}(x)\, e^{2\pi i t\cdot x}\, e^{-4\pi^2\alpha|x|^2} dx$$

$$= \int_{\mathbb{R}^n} f(x)\, W(x-t, \alpha)\, dX$$

$$\forall \alpha > 0.$$

Thm: $\varphi \in L^1(\mathbb{R}^n)$

$\int \varphi(x)\, dx = 1$

For $\epsilon > 0$, $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(x/\epsilon)$

$f \in L^p(\mathbb{R}^n)$ $1 \le p < \infty$ or $f \in C_0(\subset L^\infty(\mathbb{R}^n))$

$\Rightarrow \|f * \varphi_\epsilon - f\|_p \to 0$ as $\epsilon \to 0$.

Pf:

$$\int_{\mathbb{R}^n} \varphi_\epsilon(t)\, dt = \int_{\mathbb{R}^n} \epsilon^{-n}\varphi(t/\epsilon)\, dt$$

$$= \int_{\mathbb{R}^n} \varphi(t)\, dt = 1.$$

So

$$f * \varphi_\epsilon(x) - f(x) = \int_{\mathbb{R}^n} f(x-t)\varphi_\epsilon(t)\, dt - \int_{\mathbb{R}^n} f(x)\varphi_\epsilon(t)\, dt$$

$$= \int_{\mathbb{R}^n} \big(f(x-t) - f(x) \big) \varphi_\epsilon(t)\, dt.$$

So

$$\|f*\varphi_\epsilon - f\|_p$$

$$= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x-t) - f(x))\, \varphi_\epsilon(t)\, dt \right|^p dx \right)^{1/p}.$$

$$\le \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-t) - f(x)|\, |\varphi_\epsilon(t)|\, dt \right)^p dx \right)^{1/p}$$

$$\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-t) - f(x)|^p dx \right)^{1/p} \underbrace{|\varphi_\epsilon(t)|}_{\left|\frac{1}{\epsilon^n}\varphi(t/\epsilon)\right|} dt.$$

(by Minkowski's integral inequality)

let $u = t/\epsilon$.

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-\epsilon u) - f(x)|^p dx \right)^{1/p} \varphi(u)\, du.$$

$$\omega(h) = \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \right)^{1/p}$$

is called the L^p modulus of continuity of f.

We have $\omega(h) \leq 2\|f\|_p$

by Minkowski's inequality.

and we claim $\omega(h) \to 0$ as $|h| \to 0$.

The claim ~~that~~ holds when f is Continuous with compact support.

and given $f \in L^p$ and $\epsilon > 0$

we can write

$f(x) = g(x) + k(x)$ where g is cont. with compact support and $\|k\|_p < \epsilon/4$

Then

$$\omega_f(h) \leq \omega_g(h) + \omega_k(h) \leq \omega_g(h) + \epsilon/2$$

and $\exists\, \delta > 0$ s.t. $\omega_g(h) < \epsilon/2$ if $|h| < \delta$.

$\therefore$

$$\| f * \varphi_\epsilon - f \|_p \le \int_{\mathbb{R}^n} \omega_f(-\epsilon t)\, |\varphi(t)|\, dt.$$

$$\to 0 \quad \text{as} \quad \epsilon \to 0.$$

since $\quad \omega_f(-\epsilon t) \le 2\|f\|_p |\varphi(t)| \in L^1.$ //.

__Remarks:__

e.g.

1.) $$u(x, \epsilon) = \int_{\mathbb{R}^n} f(t) P(x-t, \epsilon)\, dt$$

$$\to f \quad \text{in } L^p \quad \text{as } \epsilon \to 0$$

2.) $$s(x, \epsilon) = \int_{\mathbb{R}^n} f(t) W(x-t, \epsilon)\, dt$$

$$\to f \text{ in } L^p \text{ as } \epsilon \to 0.$$

3.) Notice that if $\int_{\mathbb{R}^n} \varphi(t)\, dt = 0$
then the same argument shows
that $\| f * \varphi_\epsilon \|_p \to 0$ as $\epsilon \to 0$.

Combining the previous two theorems we get.

Thm: If Φ and its Fourier Transform $\varphi = \hat{\Phi}$ are integrable and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ then.

the Φ means of $\int_{\mathbb{R}^n} \hat{f}(t)\, e^{2\pi i t\cdot x} dt$ converge to f in L^1 norm.

(so the Abel and Gauss means do).

Since $S(x,\alpha) = \int_{\mathbb{R}^n} \hat{f}(t)\, e^{2\pi i x\cdot t}\, e^{-4\pi^2 \alpha |t|^2} dt$

$$\to f(x) \text{ in } L^1$$

as $\alpha \to 0$ $(\alpha > 0)$.

we can choose $\alpha_n \searrow 0$ s.t.

$S(x, \alpha_n) \to f(x)$ a.e.

then if $\hat{f}$ is also integrable the dominated convergence theorem gives

<u>Corollary</u>: $f, \hat{f} \in L^1$

$$\Rightarrow \quad f(x) = \int_{\mathbb{R}^n} \hat{f}(t)\, e^{2\pi i x \cdot t}\, dt.$$

a.e. x.

(Right hand side is continuous so f is ~~cont~~ a.e. equal to a continuous function.)

Another important <u>Corollary</u>

If $f_1, f_2 \in L^1(\mathbb{R}^n)$ and

$$\hat{f_1}(x) = \hat{f_2}(x) \quad \forall x \in \mathbb{R}^n$$

then $f_1(t) = f_2(t)$ a.e.

<u>Pf</u>: if $\hat{f}(x) = 0 \ \forall x$ then the abel means of $\int \hat{f}(t) e^{2\pi i x \cdot t} dt$ are all 0. and $f = 0$ a.e.

Calculations

To get to our inversion results we will need the Fourier transforms of.

$$e^{-\epsilon|x|^2} \quad \text{and} \quad e^{-\epsilon|x|}.$$

Recall that

$$\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy.$$

$$= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}dr\,d\theta.$$

$$= 2\pi \int_0^{\infty} re^{-r^2}dr = 2\pi$$

so $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/}dx = 1.$

We claim that.

$$\int_{-\infty}^{\infty} e^{-4\pi^2x^2} e^{-2\pi i x t}dx = \frac{1}{2\sqrt{\pi}}e^{-t^2/4}$$

$$-4\pi^2x^2 - 2\pi i x t$$

$$= -\left(2\pi x + \frac{it}{2}\right)^2 - \frac{t^2}{2}$$

So our integral is

$$e^{-t^2/2}\int_{-\infty}^{\infty} e^{-\left(2\pi x + \frac{it}{2}\right)^2} dx$$

and we just need to show that

$$\int_{-\infty}^{\infty} e^{-\left(2\pi x + \frac{it}{2}\right)^2} dx = \frac{1}{2\sqrt{\pi}}.$$

note that

$$\int_{-\infty}^{\infty} e^{-4\pi^2x^2} dx$$

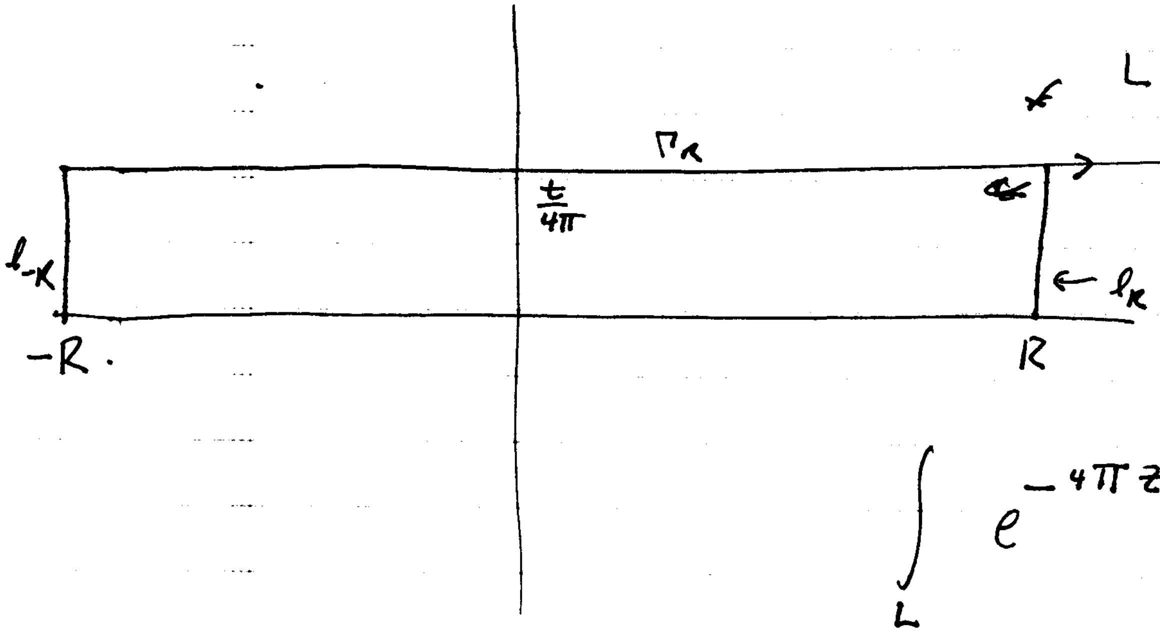
$4\pi^2x^2 = u^2/2.$

$u = \cancel{2\pi x}\ 2\sqrt{2}\,\pi x.$

$du = \cancel{2\pi dx}\ .\ 2\sqrt{2}\,\pi\, x.$

$$= \frac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \frac{\sqrt{2\pi}}{2\sqrt{2}\,\pi} = \frac{1}{2\sqrt{\pi}}.$$

$$e^{-4\pi^2\left(x+\frac{it}{4\pi}\right)^2}$$



$z = x + \frac{it}{4\pi}$

$dz = dx$.

$$\int_L e^{-4\pi z^2}\,dz$$

$$\int_{\Gamma_R} e^{-4\pi z^2}\,dz = 0.$$

So our result will be proved if we show that.

$$I = \int_{l_R} e^{-4\pi z^2}\,dz \to 0 \quad R \to \infty$$

$$II = \text{and} \int_{l_{-R}} e^{-4\pi z^2}\,dz \to 0 \quad R \to \infty.$$

$$|I| = \left| \int_0^{\frac{t}{4\pi}} e^{-4\pi(R+iy)^2}\, i\,dy. \right|$$

$z = R + iy$
$dz = i\,dy.$

$$\leq e^{-4\pi R^2} \int_0^{t/4\pi} e^{y^2}\,dy$$

$$|II| = \left| \int_0^{\frac{t}{4\pi}} e^{-4\pi(-R+iy)^2}\, i\,dy \right| \leq e^{-4\pi R^2}\int_0^{t/4\pi} e^{y^2}\,dy$$

$$\longrightarrow 0.$$

So we have.

$$\int_{\mathbb{R}^n} e^{-4\pi^2|x|^2} e^{-2\pi i x\cdot t}\,dx = \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-4\pi^2 x_j^2} e^{-2\pi i x_j t_j}\,dx_j$$

$$= \prod_{j=1}^{n} \frac{1}{2\sqrt{\pi}} e^{-t_j^2/4} = \frac{1}{2^n}\cdot\frac{1}{\pi^{n/2}}\cdot e^{-|t|^2/4}.$$

and changing variables

$$\left(\frac{\sqrt{\alpha}}{2\sqrt{\pi}}\right) y = x \qquad \left(dx = \frac{\alpha^{n/2}}{2^n \pi^{n/2}} dy \right)$$

we have

<u>Thm:</u> for all $\alpha > 0$.

$$\int_{\mathbb{R}^n} e^{-\pi\alpha|y|^2} e^{-2\pi i t\cdot y} = \frac{1}{\alpha^{n/2}} e^{-\pi|t|^2/\alpha}.$$

($e^{-\pi|x|^2}$ is its own Fourier transform).

We also have (for the Abel means results).

Thm: $\forall \ \alpha > 0$

$$\int_{\mathbb{R}^n} e^{-2\pi |y| \alpha} e^{-2\pi i t \cdot y} \, dy = \frac{c_n \, \alpha}{(\alpha^2 + |t|^2)^{n+1/2}}$$

with $c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{(n+1)/2}$

Pf: If we know.

$$\int_{\mathbb{R}^n} e^{-2\pi |u|} e^{-2\pi i t \cdot u} \, du = \frac{c_n}{(1+|t|^2)^{n+1/2}}.$$

then taking $u = \alpha y$.

$$\alpha^n \int_{\mathbb{R}^n} e^{-2\pi \alpha |y|} e^{-2\pi i t \cdot \alpha y} \, dy = \frac{c_n}{(1+|t|^2)^{n+1/2}}$$

So
$$\int_{\mathbb{R}^n} e^{-2\pi \alpha |y|} e^{-2\pi i t \cdot y} \, dy = \frac{c_n}{\alpha^n \left(1 + \left|\frac{t}{\alpha}\right|^2\right)^{\frac{n+1}{2}}}$$

$$= \frac{c_n \, \alpha}{(\alpha^2 + |t|^2)^{\frac{n+1}{2}}}$$

So we take $\alpha = 1$.

Claim: For $\beta > 0$
$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u}\, du.$$

Given the claim, we have

$$\int_{\mathbb{R}^n} e^{-2\pi|y|} e^{-2\pi i t\cdot y}\, dy$$

$$= \int_{\mathbb{R}^n} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-4\pi^2|y|^2/4u}\, du \; e^{-2\pi i t\cdot y}\, dy.$$

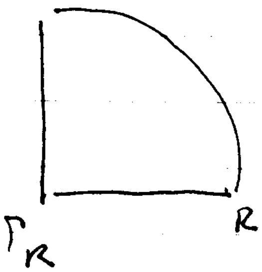
$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \underbrace{\int_{\mathbb{R}^n} \underset{e^{-(\frac{\pi}{u})\pi|y|^2}\cdots}{e^{-4\pi^2|y|^2/4u}}\; e^{-2\pi i t\cdot y}\, dy}\; du$$

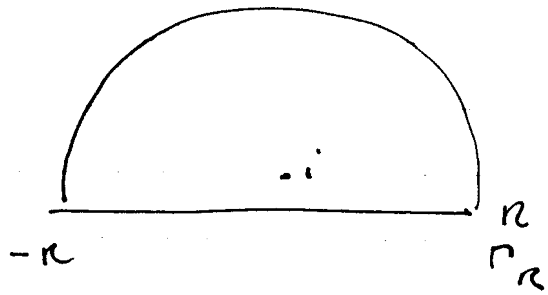
$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left\{ \left(\frac{u}{\pi}\right)^{n/2} e^{-u|t|^2} \right\} du.$$

$$= \frac{1}{\pi^{\frac{n+1}{2}}} \int_0^\infty e^{-u} u^{\frac{n-1}{2}} e^{-u|t|^2}\, du.$$

$s = (1+|t|^2)u.$
$ds = (1+|t|^2)^n du.$

$$= \frac{1}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|t|^2)^{\frac{n+1}{2}}} \int_0^\infty e^{-s} s^{(n-1)/2}\, ds = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|t|^2)^{\frac{n+1}{2}}}$$





$$2\int_0^\infty \frac{\cos \beta x}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos\beta x}{1+x^2}\,dx.$$

$$\int_{\Gamma_R} \frac{e^{i\beta z}}{1+z^2}\,dz = 2\pi i \ \mathrm{Res}\left[\frac{e^{i\beta z}}{1+z^2},\ z=i\right]$$

$$= 2\pi i\ \frac{e^{-\beta}}{2i} = \pi e^{-\beta}.$$

$$= \int_{-R}^{R} \frac{e^{i\beta x}}{1+x^2}\,dx + \underbrace{\int_0^\pi \frac{e^{i\beta R e^{i\theta}}}{1+R^2e^{i2\theta}}\, iRe^{i\theta}\,d\theta}.$$

$\cos\theta + i\sin\theta.$

$$\left| \quad \right| \le \int_0^\pi \frac{e^{-\beta R\sin\theta}\,R}{R^2-1}\,d\theta.$$

$$\le \frac{C}{R}.$$

$\therefore \quad e^{-\beta} = \frac{2}{\pi} \int_0^\infty \frac{\cos \beta x}{1+x^2}\, dx .$

and it's clear that

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u}\, du .$$

Now

$$e^{-\beta} = \frac{2}{\pi} \int_0^\infty \frac{\cos \beta x}{1+x^2}\, dx = \frac{2}{\pi} \int_0^\infty \cos \beta x \left\{ \int_0^\infty e^{-u} e^{-ux^2} du \right\} dx$$

$$= \frac{2}{\pi} \int_0^\infty e^{-u} \left\{ \int_0^\infty e^{-ux^2} \cos \beta x\, dx \right\} du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-u} \left\{ \frac{1}{2} \int_{-\infty}^\infty e^{-ux^2} e^{i\beta x} dx \right\} du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-u} \left\{ \pi \int_{-\infty}^\infty e^{-4\pi^2 u y^2} e^{-2\pi i \beta y} dy \right\} du$$

$$= \frac{2}{\pi} \int_0^\infty e^{-u} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{u}}\, e^{-\beta^2/4u} \right\} du .$$

which was our claim.