Math 138 4/14/09.

Thm: $f \in L^p(\mathbb{R}^n)$ $1 \le p \le \infty$

$g \in L^1(\mathbb{R}^n)$

$\Rightarrow$ $h = f * g$ is well defined

and $h \in L^p(\mathbb{R}^n)$.

In fact,

$$\|h\|_p \le \|f\|_p \|g\|_1.$$

For the proof, we will need

Minkowski's integral inequality

$(X, \mathcal{M}, \mu)$ $(Y, \mathcal{N}, \nu)$

σ finite measure spaces.

f is $\mathcal{M} \otimes \mathcal{N}$ msble on $X \times Y$

a) if $f \ge 0$ and $1 \le p < \infty$ then

$$\left(\int \left(\int f(x,y)\, d\nu(y) \right)^p d\mu(x) \right)^{1/p}$$

$$\le \int \left[\int f(x,y)^p \, d\mu(x) \right]^{1/p} d\nu(y)$$

b) if $\begin{cases} 1 \le p \le \infty \\ f(\cdot, y) \in L^p(\mu) & \text{for a.e. } y \\ y \to \|f(\cdot, y)\|_p & \in L^1(\nu) \end{cases}$

then

$f(x, \cdot) \in L^1(\nu)$ a.e. x

and

$x \to \int f(x, y)\, d\nu \in L^p(\mu)$

and

$$\left\| \int f(\cdot, y)\, d\nu(y) \right\|_p \le \int \|f(\cdot, y)\|_p \, d\nu(y).$$

Pf:

b) follows from a) by putting $f \to |f|$ and using Fubini ($1 \le p < \infty$)

For $p = \infty$, it is just the triangle inequality..

a) if $p = 1$ this is Tonelli's Thm

If $1 < p < \infty$ let q s.t.

$$\frac{1}{p} + \frac{1}{q} = 1.$$

and suppose $g \in L^q(\mu)$.

then

$$\int \left[\int f(x,y)\, d\nu(y)\right] |g(x)|\, d\mu(x)$$

$$= \int\int f(x,y)\, |g(x)|\, d\mu(x)\, d\nu(y)$$

(Tonelli)

$$\leq \|g\|_q \int \left[\int f(x,y)^p\, d\mu(x)\right]^{1/p} d\nu(y).$$

(Hölder).

Now take the sup over $g \in L^q$ with $\|g\|_{L^q} = 1$. and use the L^p, L^q duality

Now back to the theorem on Convolutions.

Pf: $|h(x)| = \left|\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.\right| \le \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy.$

$(\mu = dx \quad , \quad \nu = |g(y)|\,dy.)$.

so Minkowski Integral Inequality gives

$$\int_{\mathbb{R}^n} |h(x)|^p dx \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p dx\right)^{1/p} |g(y)|\, dy.$$

$$= \|f\|_p \|g\|_1.$$

We can extend the definition of convolution:

$h = f * d\mu \qquad (\mu \text{ a finite Borel})$.

$$h(x) = \int_{\mathbb{R}^n} f(x-y)\, d\mu(y).$$

$x \in \mathbb{R}^n, \quad f \in L^p.$

we have (as above).

$$\|h\|_p \le \|f\|_p \int_{\mathbb{R}^n} d|\mu|$$

The connection with Fourier Analysis is through

<u>Thm:</u> $f, g \in L^1(E_n)$

$$\Rightarrow \widehat{(f * g)} = \hat{f}\,\hat{g}$$

Pf:

$$\widehat{f*g}(x) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(t-y) g(y)\, dy \right] e^{-2\pi i x \cdot t}\, dt$$

$$= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(t-y) g(y)\, dy \right] e^{-2\pi i\, x\cdot(t-y)} e^{-2\pi i x \cdot y}\, dt.$$

$$= \overset{\text{by}}{(\text{Fubini})} = \int_{\mathbb{R}^n} g(y) e^{-2\pi i x \cdot y} \int_{\mathbb{R}^n} f(t-y) e^{-2\pi i\, x\cdot(t-y)}\, dt\, dy$$

$$= \hat{f}(x)\, \hat{g}(x).$$

Translations and dilations have simple interactions with $\mathcal{F}$.

With

$$\tau_h(g)(x) = g(x-h)$$

we have.

$$\widehat{(\tau_h f)}(x) = e^{-2\pi i h \cdot x} \hat{f}(x).$$

$$\widehat{\tau_h f}(x) = \int (\tau_h f)(t)\, e^{-2\pi i x \cdot t}\, dt.$$

$$= \int_{\mathbb{R}^n} f(t-h)\, e^{-2\pi i x \cdot t}\, dt.$$

$$= \int_{\mathbb{R}^n} f(u)\, e^{-2\pi i x \cdot (u+h)}\, du.$$

$$= e^{-2\pi i x \cdot h}\, \hat{f}(x).$$

Also.

$$\widehat{\left(e^{2\pi i t \cdot h} f(t)\right)}(x) = \tau_h \hat{f}(x).$$

since

$$\tau_h \hat{f}(x) = \hat{f}(x-h) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i (x-h)\cdot t}\, dt.$$

$$= \int_{\mathbb{R}^n} \left(e^{+2\pi i h \cdot t} f(t)\right) e^{-2\pi i x \cdot t}\, dt$$

for $a > 0$ let

$$(\delta_a g)(x) = g(ax)$$

then

$$a^n \, (\widehat{\delta_a f})(x) = \hat{f}\left(\tfrac{x}{a}\right)$$

$$\widehat{\delta_a f}(x) = \int_{\mathbb{R}^n} \delta_a f(t)\, e^{-2\pi i x \cdot t}\, dt.$$

$$= \int_{\mathbb{R}^n} f(at)\, e^{-2\pi i x \cdot t}\, dt.$$

$$u = at \qquad du = a^n\, dt.$$

$$= \frac{1}{a^n} \int_{\mathbb{R}^n} f(u)\, e^{-2\pi \frac{x}{a} \cdot u}\, du.$$

$$= \frac{1}{a^n}\, \hat{f}\left(\tfrac{x}{a}\right).$$

There is also a nice relationship with differentiation:

<u>Thm:</u> Suppose $f \in L^1(\mathbb{R}^n)$ and $x_k f(x) \in L^1(\mathbb{R}^n)$ where x_k is the k^{th} coordinate.

then $\hat{f}$ is diff w.r.t x_k and

$$\frac{\partial \hat{f}}{\partial x_k}(x) = \widehat{\left(-2\pi i\, t_k f(t)\right)}(x).$$

<u>Pf:</u> let $h = (0 \dots, h_k, \dots, 0) \neq 0$.

then

$$\frac{\hat{f}(x+h) - \hat{f}(x)}{h_k} = \widehat{\left\{\left(\frac{e^{-2\pi i t\cdot h} - 1}{h_k}\right) f(t)\right\}}(x).$$

and by Lebesgues dominated convergence thm.

this $\rightarrow \widehat{-2\pi i t_k f(t)}\,(x)$ as $h_k \rightarrow 0$.

$$\frac{\hat{f}(x+h_k)-\hat{f}(x)}{h_k} = \widehat{\left\{\frac{e^{-2\pi i t\cdot h}-1}{h_k} f(t)\right\}}(x)$$

$$= \int_{\mathbb{R}^n} \left(\frac{e^{-2\pi i t\cdot h}-1}{h_k}\right) f(t)\, e^{-2\pi i x t}\, dt$$

$$\to \int_{\mathbb{R}^n} -2\pi i t_k f(t)\, e^{-2\pi i x t}\, dt$$

by L.D.C.T.

(i.e. $\frac{\partial \hat{f}}{\partial x_k}(x) = \widehat{-2\pi i t_k f(t)}(x).$)

when $f \in L^1$ and $x_k f(x) \in L^1$)

Def: When $f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that

$$\left(\int_{\mathbb{R}^n} \left| \frac{f(x+h)-f(x)}{h_k} - g(x) \right|^p dx \right)^{1/p} \to 0$$

as $h_k \to 0$

the function g is called the partial derivative of f with respect to x_k in the L^p norm.

Thm: If $f \in L^1(\mathbb{R}^n)$ and g is the partial derivative of f with respect to x_k in the L^1 norm then

$$\hat{g}(x) = 2\pi i x_k \hat{f}(x)$$

Pf:

$$\left| \hat{f}(x)\frac{(e^{2\pi i h\cdot x}-1)}{h_k} - \hat{g}(x)\right| \leq \left|\int \left(\frac{f(t+h)-f(t)}{h_k} - g(t)\right)e^{-2\pi i t\cdot x}\,dt\right|$$

$$\leq \int \left|\frac{f(t+h)-f(t)}{h_k} - g(t)\right| dt.$$

$$\to 0 \quad h_k \to 0$$

$$\therefore \quad \hat{g}(x) = 2\pi i x_k \hat{f}(x).$$

Further L^p derivatives are defined in the obvious way and, under the natural integrability assumptions on f and $P(x)f(x)$ and differentiability assumptions on f, we have.

i) $P(D)\hat{f}(x) = \left(P(-2\pi i t) f(t)\right)^{\wedge}(x)$

ii) $\widehat{(P(D)f)}(x) = P(2\pi i x)\hat{f}(x)$.

notation: multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} \to P(x)$$

P a polynomial in n variables $\to P(D)$

Given the Fourier transform ~~of~~, $\hat{f}$ of an integrable function f, How do we recover f?

We will see that when f and $\hat{f}$ are both in $L^1(\mathbb{R}^n)$ then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(t)\, e^{2\pi i x\cdot t}\, dt.$$

and this is the situation usually treated in elementary courses on Fourier integrals

In general, $\hat{f}$ is not integrable when $f \in L^1(\mathbb{R}^n)$

e.g. $\quad f(x) = \chi_{[-1/2,\, 1/2]}(x) \quad$ in $\mathbb{R}^1$.

$$\hat{f}(x) = \int_{-1/2}^{1/2} e^{-2\pi i x\cdot t}\, dt = \frac{1}{-2\pi i x}\, e^{-2\pi i x\cdot t}\Big|_{-1/2}^{1/2}$$

$$= -\frac{1}{2\pi i x}\left(e^{-2\pi i \frac{x}{2}} - e^{2\pi i \frac{x}{2}}\right)$$

$$= \frac{\sin(\pi x)}{\pi x}.$$

Abel ~~Poisson~~ summation

$$\lim_{r\to 1} \sum_{n=0}^{\infty} a_n r^n = \lim_{n=0}^{\infty} \sum_{n=0}^{\infty} a_n \, e^{-n(\log \frac{1}{r})}$$

e.g.

$a_n = 1$ n even

$a_n = -1$ n odd.

$\sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent for each $r < 1$.

$$= \sum_{k=0}^{\infty} r^{2k} - \sum_{k=0}^{\infty} r^{(2k+1)}$$

$$= \frac{1}{1-r^2} - \frac{r}{1-r^2} = \frac{1-r}{1-r^2} = \frac{1}{1+r} \to \frac{1}{2}$$

as $r \to 1$.

When $\sum_{n=0}^{\infty} a_n$ does exist the limits agree.

The analog for integrals is:

for $\epsilon > 0$ let

$$A_\epsilon(f) = \int_{\mathbb{R}^n} f(x)\, e^{-\epsilon |x|}\, dx$$

If $f \in L^1(\mathbb{R}^n)$ then $\lim_{\epsilon \to 0} A_\epsilon(f) = \int_{\mathbb{R}^n} f(x)\, dx$.

But the $A_\epsilon(f)$ are defined for more general f, e.g. if f is bounded $A_\epsilon(f)$ exists.

and $$\lim_{\epsilon\to 0} A_\epsilon(f) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(x)\, e^{-\epsilon|x|}\, dx$$

may exist when $f \notin L^1(\mathbb{R}^n)$

e.g. $(n=1)$ $$\int_{\mathbb{R}} \frac{\sin x}{x}\, e^{-\epsilon|x|}\, dx = 2\int_0^\infty e^{-\epsilon x} \cdot \frac{\sin x}{x} \cdot dx.$$

- $\dfrac{e^{-\epsilon x}\sin x}{x} \in L^1$ for each $\epsilon > 0$.
- $\displaystyle\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\, dx = l$ exists

(though $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx = +\infty$)

given $\delta > 0$ Choose N_1 s.t.

$$\left| \int_0^R \frac{\sin x}{x}\, dx - l \right| < \delta/2. \quad \forall R > N_1.$$

~~Choose N_2 s.t.~~

$$\int_0^R e^{-\epsilon x} \frac{\sin x}{x}\, dx + \int_R^\infty e^{-\epsilon x} \frac{\sin x}{x}\, dx.$$

When

$$L = \lim_{\epsilon \to 0} A_\epsilon(f) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(x)\, e^{-\epsilon |x|} dx$$

exists, we call $\int_{\mathbb{R}^n} f\,dx$ Abel summable to L.

Gauss summability

$$G_\epsilon(f) = \int_{\mathbb{R}^n} f(x)\, e^{-\epsilon |x|^2} dx$$

$\int_{\mathbb{R}^n} f\,dx$ is Gauss summable to L

if $L = \lim_{\epsilon \to 0} G_\epsilon(f)$.

In general, with $\Phi \in C_0$ and $\Phi(0) = 1$.

$$M_\epsilon(f) = \int_{\mathbb{R}^n} f(x)\, \Phi(\epsilon x)\, dx$$

$\int_{\mathbb{R}} f\,dx$ is Φ summable to L if

$$\lim_{\epsilon \to 0} M_\epsilon(f) = L$$