Math 138 4/9/09.

$f \in L^1(\mathbb{R}^n)$

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i x \cdot t}\, dt$$

Thm: a) $\mathcal{F}: f \to \hat{f} \in \mathcal{L}(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$.

and $\|\hat{f}\|_\infty \le \|f\|_1$.

b) $f \in L^1(\mathbb{R}^n) \Rightarrow \hat{f}$ is uniformly continuous

Pf:

a) $$|\hat{f}(x)| \le \int_{\mathbb{R}^n} |f(t)\, e^{-2\pi i x \cdot t}|\, dt = \|f\|_1.$$

b) $$|\hat{f}(y) - \hat{f}(x)| \le \int_{\mathbb{R}^n} |f(t)|\, |e^{-2\pi i y \cdot t} - e^{-2\pi i x \cdot t}|\, dt$$

$$= \int_{\mathbb{R}^n} |f(t)|\, |e^{-2\pi i (y-x) \cdot t} - 1|\, dt.$$

Given $\epsilon > 0$,

Choose a large ball B_R centered at the origin

s.t.

$$\int_{\mathbb{R}^n / B} |f(t)|\, dt < \epsilon/4$$

then

$$|\hat{f}(y) - \hat{f}(x)| = \int_{B_R} |f(t)|\, |e^{-2\pi i (y-x)\cdot t} - 1|\, dt$$

$$+ \int_{\mathbb{R}^n \setminus B_R} |f(t)|\, |e^{-2\pi i (y-x)\cdot t} - 1|\, dt.$$

$$\leq \int_{B_R} |f(t)|\, |e^{-2\pi i (y-x)\cdot t} - 1|\, dt$$

$$+ \epsilon/2.$$

On B_R, we have

$$|(y-x)\cdot t| \leq \|y-x\| \cdot R$$

and $\exists\ \delta$ s.t. if $|a| < \delta$ then

$$|e^{-2\pi i a} - 1| < \frac{\epsilon}{2\|f\|_1}$$

$\therefore$ if $|y-x| < \frac{\delta}{R}$, we have.

$$|\hat{f}(y) - \hat{f}(x)| < \epsilon.$$

and $\hat{f}$ is uniformly continuous.

<u>Thm:</u> (Riemann-Lebesgue)

$f \in L^1(\mathbb{R}^n) \to \hat{f}(x) \to 0$ as $|x| \to \infty$.

$\therefore$ (from last result)

$\hat{f} \in C_0 =$ continuous fcns vanishing at ∞.

<u>Pf:</u> Suppose 1st that f is χ_I

with $I = \{x \in \mathbb{R}^n : a_1 \le x_1 \le b_1, \ldots a_n \le x_n \le b_n\}$

then

$$\hat{f}(x) = \int_{a_n}^{b_n} \cdots e^{-2\pi i x_2 t_2} \left(\int_{a_1}^{b_1} e^{-2\pi i x_1 t_1} dt_1 \right) dt_2$$

(Fubini)

$$= \int_{a_n}^{b_n} e^{-2\pi i x_n t_n} dt_n \int_{a_{n-1}}^{b_{n-1}} e^{-2\pi i x_{n-1} t_{n-1}} dt_{n-1} \cdots \int_{a_1}^{b_1} e^{-2\pi i x_1 t_1} dt_1$$

Now $\left| \int_{a_1}^{b_1} e^{-2\pi i x_1 t_1} \, dt_1 \right| = \left| \frac{1}{-2\pi i x_1} e^{-2\pi i x_1 t_1} \Big/_{a_1}^{b_1} \right|$

$$= \left| \frac{1}{2\pi i x_1} \; e^{-2\pi i a_1 x_1} - e^{-2\pi i b_1 x_1} \right|$$

$$\leq \frac{1}{\pi} \cdot \frac{1}{|x_1|}.$$

and it is the same for the other integrals

so $\hat{f}(x) \to 0$ as $|x| \to \infty$.

and $\therefore$ $\hat{f}(x) \to 0$ when f is a finite linear comb. of characteristic functions of intervals.

Since such combinations are dense in L^1 (why?) we can write, for a given $\epsilon > 0$. and a general $f \in L^1(\mathbb{R}^n)$.

$f = g + h$ where g has $\hat{g}(x) \to 0$.

and $\|h\|_{L^1} < \epsilon/2$

Clearly, $\exists R > 0$ s.t $|\hat{f}(x)| = |\hat{g}(x) + \hat{h}(x)| < \epsilon$ if $|x| \geq R$.

For finite Borel measures μ on $\mathbb{R}^n$ we define

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot t} \, d\mu(t)$$

We have.

$$|\hat{\mu}(x)| \leq \int_{\mathbb{R}^n} d|\mu|(t)$$

and $\hat{\mu}(x)$ is uniformly continuous (exactly as before).

e.g. if X is a $\mathbb{R}$ valued r.v. defined a probability space $(\Omega, \mathcal{A}, P)$. then X gives rise to a Borel probability measure on $\mathbb{R}$ by.

$$\mu_X(E) = \text{Prob}\{X \in E\}$$

Sadly, the function

$$\hat{\mu}_X\left(\frac{-x}{2\pi}\right) = \int_{\mathbb{R}} e^{ixt} \, d\mu_X(t)$$

is called the "characteristic function" of X by probabilists note that

$$\int_{\mathbb{R}} e^{iut}\, d\mu_X(t) = E\left(e^{iuX}\right)$$

(by our change of variables rule from last semester).

Convolution

If $f, g \in L^1(\mathbb{R}^n)$

$h = f * g$ is defined by

$$h(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.$$

Now $f(x-y)g(y)$ is a msble function of (x,y). So by Fubini's thm

$$\int_{\mathbb{R}^n} |h(x)|\, dx \le \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \right| dx$$

$$\le \int_{\mathbb{R}^n} |f(u)|\, du \int_{\mathbb{R}^n} |g(y)|\, dy = \|f\|_1 \|g\|_1.$$

$f * g(x) = \int f(x-y) g(y) dy.$

$g * f(x) = \int g(x-y) f(y) dy.$

putting $u = x - y.$

$du = (-1)^n dy.$

$$\int f(x-y) g(y) dy = \int_{+\infty}^{-\infty} \cdots \int_{+\infty}^{-\infty} f(u) g(x-u) (-1)^n \underline{du}$$

$$= \int g(x-u) f(u) du.$$

Another application of Fubini's theorem shows that $*$ is also associative.

(the multiplication $*$ makes $L^1(\mathbb{R}^n)$ a Banach Algebra).

In fact, $f * g$ is defined whenever $f \in L^p$ $1 \leq p \leq \infty$ and $g \in L^1$.