Math 138 4/2/09.

Last time $(\Omega, \mathcal{A}, P)$ $\mathcal{B} \subset \mathcal{A}$ complete σ-fields

1. $\forall f \in L^1 \quad \exists! \ E^{\mathcal{B}}(f) \in L^1(\mathcal{B})$

s.t

$$\int E^{\mathcal{B}}(f) g \, dP = \int f g \, dP \quad \forall g \in L^\infty(\mathcal{B})$$

2. This defines $E^{\mathcal{B}} : L^1 \to L^1(\mathcal{B})$ as a positive, idempotent, linear contraction.

3. $\forall$ real $\mathcal{B}$-msble h

$$E^{\mathcal{B}}(hf) = h E^{\mathcal{B}}(f)$$

if f and hf are integrable.

noticing that $E^{\mathcal{B}}$ could be "relativised" we proved Hölders inequality for conditional expectations.

$p, q \in (1, \infty) \quad \frac{1}{p} + \frac{1}{q} = 1$

$f \in L^p, \ g \in L^q$

$$\Rightarrow \left| E^{\mathcal{B}}(fg) \right| \le E^{\mathcal{B}}(|f|^p)^{1/p} \, E^{\mathcal{B}}(|g|^q)^{1/q}.$$

(pointwise on Ω).

We will now use Hölder to extend our theory of conditional expectations to L^p $p > 1$.

Prop: Let $p \in [1,\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$. $(\Omega, \mathcal{A}, P)$, $\mathcal{B} \subset \mathcal{A}$. as usual.

Then $\forall f \in L^p$, $E^{\mathcal{B}}(f) \in L^p$

$E^{\mathcal{B}} : L^p \to L^p(\mathcal{B})$ is a positive, idempotent linear contraction with $E^{\mathcal{B}}(1) = 1$. and is surjective.

For any $h \in L^q(\mathcal{B})$, $f \in L^p$

$$E^{\mathcal{B}}(hf) = h\, E^{\mathcal{B}}(f).$$

Pf: . Case 1: $p < \infty$.

$L^p \subset L^1 \Rightarrow E^{\mathcal{B}}(f) \in L^1$ is defined as in our earlier proposition. $\forall f \in L^p$.

By Hölder

$$|E^{\mathcal{B}}(f \cdot 1)| \leq |E^{\mathcal{B}}(|f|^p)|^{1/p} |E^{\mathcal{B}}(1^q)|^{1/q}$$

so

$$\int |E^{\mathcal{B}}(f)|^p dP \leq \int E^{\mathcal{B}}(|f|^p)\, dP = \int |f|^p\, dP$$

so $E_\mathcal{B}(f) \in L^p$ and $\|E_\mathcal{B}(f)\|_p \le \|f\|_p$

Case 2 $p = \infty$

$$|E^\mathcal{B}(f)| = |E^\mathcal{B}(f^+) - E^\mathcal{B}(f^-)| \le E^\mathcal{B}(|f|)$$

$$\le \|f\|_\infty E^\mathcal{B}(\mathbb{1})$$

(by positivity)

$$= \|f\|_\infty \cdot \text{ on } \Omega.$$

$\forall f \in L^\infty$.

So $\|E^\mathcal{B}(f)\|_\infty \le \|f\|_\infty$.

all the other properties follow from the previous proposition: Note that $hf \in L^1$ when $f \in L^p, h \in L^q$

Converseley, we may characterize operators on L^p which are conditional expectations.

On L^p $(1 \le p \le \infty)$

a conditional expectation $E^\mathcal{B}$ must have: (with $T = E^\mathcal{B}$)

$$T(f \cdot Tg) = T(f)T(g) \quad \forall f \in L^\infty, g \in L^p.$$

This follows from:

$(L^\infty \subset L^q)$

$$E^\mathcal{B}(fh) = E^\mathcal{B}(f) \cdot h \qquad f \in L^\infty, h \in L^p(\mathcal{B})$$

and the fact that $E^\mathcal{B}$ is surjective.

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If $1 \le p < \infty$, this algebraic property and preservation of the integral are enough to characterize the C.E's.

Prop: $1 \le p < \infty$.

If $T : L^p \to L^p$ continuous

s.t $\int Tf \, dP = \int f \, dP \quad \forall f \in L^p$

and $T(f \cdot Tg) = T(f) T(g)$

$\forall f \in L^\infty, g \in L^p$

then $T = E^{\mathcal{B}}$ for some complete $\mathcal{B} \subset \mathcal{A}$.

Pf: We will need the following

Claim: $T(L^\infty) \subset L^\infty$

For each $f \in L^\infty$ let $f_1 = f$

and

$$f_{n+1} = f \cdot T(f_n) \qquad n \ge 1.$$

By induction, $f_n \in L^p$. and

$$T f_{n+1} = T[f \cdot T(f_n)] = T(f) T(f_n) \quad \forall n \ge 1$$

so $$Tf_n = (T(f))^n \in L^p \quad \forall n.$$

$\therefore$ $Tf \in L^r \quad \forall r < \infty.$

and since $Tf_{n+1} = T(f \cdot Tf_n)$

$$\|Tf_{n+1}\|_p \le \|T\| \, \|f \cdot Tf_n\|_p \le \|T\| \, \|f\|_\infty \, \|Tf_n\|_p.$$

Since $\|Tf_1\|_p \le \|T\| \, \|f\|_p \le \|T\| \, \|f\|_\infty$

induction shows that.

$$\|(Tf)^n\|_p = \|Tf_n\|_p \le (\|T\| \, \|f\|_\infty)^n$$

$$= \left(\int (Tf)^{np} \right)^{1/p} = \|Tf\|_{np}^n$$

so $$\|Tf\|_{np} \le \|T\| \, \|f\|_\infty. \quad \forall n.$$

<u>Fact:</u> if $g \in \bigcap_{p<\infty} L^r$ then $\|g\|_r \nearrow \min(\|g\|_\infty, +\infty)$

so $$\|Tf\|_\infty \le \|T\| \, \|f\|_\infty$$

and $\therefore$ $T(L^\infty) \subset L^\infty$.

Let $\Lambda =$

$$\{h : h \in L^\infty, \ T(fh) = T(f)h \quad \forall f \in L^\infty\}.$$

Λ is a sub-algebra of L^∞ containing the constants.

We know from an earlier lemma that.

$\bar{\Lambda} \subset L^p$ (L^p closure)

is $L^p(\mathcal{B})$ for some complete $\mathcal{B} \subset \mathcal{A}$.

<u>Claim:</u>
for $f \in L^\infty$, $T(fh) = T(f)h$ holds for $h \in \bar{\Lambda}$.

<u>Pf.</u> if $h \in L^p(\mathcal{B})$

let $\{h_n\} \subset \Lambda$ converge to h in L^p.

T is continuous, so $T(fh_n) \to T(fh)$ in L^p.

Then $\exists$ a subsequence n_j s.t

$h_{n_j} \to h$ a.s. and $T(fh_{n_j}) \to T(fh)$ a.e.

Since $T(fh_{n_j}) = h_{n_j} T(f)$
we have $T(fh) = hT(f)$.

Since T preserves the integral we have

$$\int fh\, dP = \int T(fh)\, dP$$

by previous claim $\quad = \int T(f)h\, dP \quad \forall f \in L^\infty$

if $h \in L^p(\mathcal{B})$.

$\therefore$ (taking $h = \chi_B \quad B \in \mathcal{B}$)

$$E^{\mathcal{B}}(f) = E^{\mathcal{B}}(T(f)). \qquad \forall f \in L^\infty.$$

Since $T(f \cdot Tg) = T(f)T(g)$

$\forall f \in L^\infty, g \in P.$

and $Tg \in L^\infty$ if $g \in L^\infty$.

we have.

$$Tg \in \Lambda \text{ for } g \in L^\infty.$$

Since $Tg \in \Lambda \subset L^p(\mathcal{B})$

Tg is $\mathcal{B}$ msble and $\therefore$ $E^{\mathcal{B}}(Tg) = Tg$.

$\forall g \in L^\infty$.

$$\therefore \quad Tf = E^{\mathcal{B}}(f) \quad \forall f \in L^\infty$$

and by continuity in L^p $\quad Tf = E^{\mathcal{B}}(f) \quad \forall f \in L^p$.

Prop: $p \in [1,\infty)$, $p \neq 2$

If T is an idempotent linear contraction on $L^p(\Omega, \mathcal{A}, P)$

s.t. $T1 = 1$

Then $T = E^{\mathcal{B}}$ for some complete $\mathcal{B} \subseteq \mathcal{A}$.

Remark: The proposition is false for $p=2$

Not all closed subspaces of $L^2(\Omega, \mathcal{A}, P)$ are $L^2(\mathcal{B})$ for some $\mathcal{B} \subset \mathcal{A}$.

Take any function $h \in L^2$ which has more than two values.

Then the subspace generated by $(1, h)$ is not $L^2(\mathcal{B})$ for any $\mathcal{B}$, since $\dim L^2(\mathcal{B}) > 2$ for and $\mathcal{B} \subseteq \mathcal{A}$ making $1, h$ msble.

Though true in general, we shall prove an easier version, assuming also that T is positive and preserves the integral.

In this case, the prop. holds also for $p=2$.

We then show that in the case $p=1$.
every idempotent linear contraction T
s.t $T(\mathbb{1}) = \mathbb{1}$
must be positive and preserve the integral.

<u>Pf:</u> If T is positive then we know from earlier (Positive contractions $\mathcal{U}$).

that

$\{f: Tf=f\} \subseteq L^p$ must be $L^p(\mathcal{B})$

for some complete $\mathcal{B} \subseteq \mathcal{A}$.

<u>Claim:</u> $\forall\, B \in \mathcal{B}$ and mble $f: \Omega \to [0,1]$.

$T(f \cdot \chi_B) = T(f)\,\chi_B$.

Pf of claim:

$$0 \le T(f\chi_B) \le T(\chi_B) = \chi_B$$

and $\quad 0 \le T(f\chi_{B^c}) \le T(\chi_{B^c}) = \chi_{B^c}$

by positivity of T.

So $\quad T(f\chi_B) = 0 \quad$ on B^c

and $T(f\chi_B^c) = 0 \quad$ on B.

Since $\quad T(f\chi_B) + T(f\chi_{B^c}) = T(f)$

we get $\quad T(f\chi_B) = T(f)\chi_B$.

Taking linear combns. and monotone limits we have.

$$T(f\chi_B) = T(f)\chi_B \qquad \forall f \in L^p$$

and $\forall B \in \mathcal{B}$.

$\therefore$

$$\int_B f\,dP = \cancel{\int_B T(f)\,dP} \int f\chi_B\,dP = \int T(f\chi_B)\,dP$$

$$= \int T(f)\chi_B\,dP = \int_B T(f)\,dP.$$

so that

$$E^{\mathcal{B}}(f) = E^{\mathcal{B}}(Tf). \quad \forall f \in L^p$$

Now since $T(Tf) = Tf \in \mathcal{M}$,

we have $Tf \in L^p(\mathcal{B})$. (fixed points of T)

and $\therefore$

$$E^{\mathcal{B}}(f) = Tf \qquad \forall f \in L^p.$$

Now we show that if $p = 1$.

$f \geq 0 \implies Tf \geq 0$ $\cancel{\&}$

and $\int Tf \, dP = \int f \, dP$.

under the original assumptions.

It suffices to do this for $f: \Omega \to [0,1]$ measurable.

<u>note:</u> $a \in \mathbb{R} \implies |a| + |1-a| \geq 1$.

with equality iff $0 \leq a \leq 1$.

When $f : \Omega \to [0,1]$ is msble

then

$$\int (|Tf| + |1 - Tf|)\, dP = \int |Tf| + |T(1-f)|\, dP$$

$$\leq \int |f| + |1-f|\, dP = 1$$

since T is a contraction and $T(1) = 1$.

Since $|Tf| + |1 - Tf| \geq 1$.

we must have.

$|Tf| + |1 - Tf| = 1$ a.s.

$\iff 0 \leq Tf \leq 1$ a.s.

On the other hand, if $\int |Tf|\,dP < \int |f|\,dP$.

then since $\int |T(1-f)|\, dP \leq \int |1-f|\, dP$.

equality can't hold.

so $\int |Tf|\, dP = \int |f|\, dP$.

i.e. $\int Tf\, dP = \int f\, dP$ since $f \geq 0$ and $Tf \geq 0$.