Math 138 3/31/09.

Last time $(\Omega, \mathcal{A}, P)$ $\mathcal{B} \subset \mathcal{A}$.

orthogonal projection of L^2 onto $L^2(\mathcal{B})$

$E^{\mathcal{B}}$: has $\forall f \in L^2$

$E^{\mathcal{B}}(f) \in L^2$

$\int E^{\mathcal{B}}(f) g \, dP = \int f g \, dP \quad \forall g \in L^2(\mathcal{B})$.

and we proved:

The projections which are positive and fix $\mathbb{1}$ are exactly the conditional Expectations on L^2.

and: $\forall f \in L^2$ and $h \in L^\infty(\mathcal{B})$

$E^{\mathcal{B}}(hf) = h \, E^{\mathcal{B}}(f)$.

Now we will extend the notion of conditional expectation to L^1.

Prop: $\forall f \in L^1$ $\exists!$ element $E^{\mathcal{B}}(f) \in L^1(\mathcal{B})$

s.t.

$$\int E^{\mathcal{B}}(f)\, g \, dP = \int f g \, dP \quad \forall g \in L^\infty(\mathcal{B})$$

The operator $E^{\mathcal{B}}$, so defined on L^1, is a positive, idempotent, linear contraction with $E^{\mathcal{B}}(1) = 1$.

We have

$$E^{\mathcal{B}}(hf) = h E^{\mathcal{B}}(f)$$

for any real valued $\mathcal{B}$ mble function h s.t.

f and $hf \in L^1$.

We get to this proposition by 1st treating the case of non-negative functions.

<u>Lemma:</u> $\overline{L}_+ =$ msble maps. (equivalence classes). $[\Omega, \mathcal{A}, P] \to \overline{\mathbb{R}}^+ = [0, \infty]$

$\overline{L}_+(\mathcal{B}) = \mathcal{B}$ msble equivalence classes in $\overline{L}_+$

1). $\forall f \in \overline{L}_+ \quad \exists! \quad E^{\mathcal{B}}(f)$ in $\overline{L}_+(\mathcal{B})$ s.t.

$$\int E^{\mathcal{B}}(f)\, g\, dP = \int f g\, dP$$

$\forall g \in \overline{L}_+(\mathcal{B})$

2) $E^{\mathcal{B}}(f)$ is (already) the unique element in $\overline{L}_+(\mathcal{B})$ s.t.

$$\int_B E^{\mathcal{B}}(f)\, dP = \int_B f\, dP \qquad \forall B \in \mathcal{B}.$$

3) $\forall\ h \in \overline{L}_+(\mathcal{B})$

$$E^{\mathcal{B}}(hf) = h\,E^{\mathcal{B}}(f) \qquad \text{if } f \in \overline{L}_+.$$

<u>Pf:</u> If $f \in L^2_+$, $E^{\mathcal{B}}(f) \in L^2_+(\mathcal{B})$

has been defined previously

and satisfies

$$\int E^{\mathcal{B}}(f)\,g\,dP = \int f g\,dP \quad \forall g \in L^2_+(\mathcal{B}).$$

If $g \in \overline{L}_+(\mathcal{B})$, g is the

monotone limit of $g \wedge n$, $n \in \mathbb{N}$.

and $g \wedge n \in L^2_+(\mathcal{B})$.

so the equality holds $\forall\ g \in \overline{L}_+(\mathcal{B})$.

If $f \in \overline{L}_+$ (but not in L^2_+)

then the sequence

$E^{\mathcal{B}}(f \wedge n)$ is defined as before and is increasing with n since

E^B is positive.

Denoting the pointwise limit by $E^B(f)$ we have.

$$\int E^B(f)\, g\, dP = \int f g\, dP \quad \forall\ g \in \bar{L}_+(\mathcal{B})$$

2) <u>Uniqueness</u>

Suppose $h_1, h_2 \in \bar{L}_+(\mathcal{B})$ have.

$$\int_B h_i\, dP = \int_B f\, dP \quad \forall\ B \in \mathcal{B}.$$

for $i = 1, 2$.

Then for any pair of reals

$0 \le a < b$

$$a P\{h_1 \le a < b \le h_2\} \ge \int_{\{h_1 \le a < b \le h_2\}} h_1\, dP = \int_{\{h_1 \le a < b \le h_2} h_2\, dP \ge b P\{h_1 \le a < b \le h_2\}$$

so $\quad P\{h_1 \le a < b \le h_2\} = 0$

Since $\{h_1 < h_2\} = \bigcup_{(a,b)\in \mathbb{Q}\times\mathbb{Q}} \{h_1 \le a < b \le h_2\}$.

we have $P\{h_1 < h_2\} = 0$

and by symmetry $P(h_2 < h_1) = 0$.

so $h_1 = h_2$ a.s.

3). $\forall\ h \in \overline{L}_+^*(\mathcal{B})$ and all $f \in \overline{L}_+$

$h E^{\mathcal{B}}(f) \in \overline{L}_+(\mathcal{B})$

and $\int h E^{\mathcal{B}}(f) g \, dP = \int f h g \, dP \quad \forall g \in \overline{L}_+(\mathcal{B})$

since $hg \in \overline{L}_+(\mathcal{B})$

By the uniqueness we must have

$h E^{\mathcal{B}}(f) = E^{\mathcal{B}}(hf)$. $\blacksquare$

Corollary: $\mathcal{B} \subset \mathcal{A}$, $\{f_n\} \subset \overline{L}_+$

a) if f_n is increasing

$$E^{\mathcal{B}}(\lim_n \uparrow f_n) = \lim_n \uparrow E^{\mathcal{B}}(f_n).$$

since

$$\int_B E^{\mathcal{B}}(\lim_n \uparrow f_n)\, dP = \int_B \lim_n \uparrow f_n \, dP$$

$$= \lim_n \uparrow \int_B f_n \, dP$$

(monotone convergence).

$$= \lim_n \uparrow \int_B E^{\mathcal{B}}(f_n)\, dP.$$

$$= \int_B \lim_n \uparrow E^{\mathcal{B}}(f_n)\, dP.$$

b). for any sequence. ($\overline{L}_+$)

$$E^{\mathcal{B}}\Big(\sum_{\mathbb{N}} f_n\Big) = \sum_{\mathbb{N}} E^{\mathcal{B}}(f_n)$$

by a).

c). for any sequence

$$E^{\mathcal{B}}(\underline{\lim}_n f_n) \leq \underline{\lim}_n E^{\mathcal{B}}(f_n).$$

since

$$\underline{\lim}_n f_n = \lim_j \uparrow g_j$$

where $g_j = \inf_{k \geq j} f_k$

and.

$$E^{\mathcal{B}}(g_j) \leq E^{\mathcal{B}}(f_j)$$

$$\Rightarrow \underline{\lim}\, E^{\mathcal{B}}(g_j) \leq \underline{\lim}\, E^{\mathcal{B}}(f_j)$$

$$\Rightarrow E^{\mathcal{B}}(\liminf f_n) \leq \liminf E^{\mathcal{B}}(f_n).$$

To repeat the earlier proposition

<u>Prop</u>. $\forall f \in L^1$ $\exists!$ $E^{\mathcal{B}}(f) \in L^1(\mathcal{B})$

s.t.

$$\int E^{\mathcal{B}}(f) g \, dP = \int f g \, dP \quad \forall g \in L^\infty(\mathcal{B}).$$

$E^{\mathcal{B}}$ (so defined) is positive, idempotent and a linear contraction w/ $E^{\mathcal{B}}(1) = 1$.

also,

$\forall$ real $\mathcal{B}$-msble h

$$E^{\mathcal{B}}(hf) = h\, E^{\mathcal{B}}(f)$$

holds whenever f and hf are in $L^1(\mathcal{B})$.

<u>Pf:</u> $\forall f \in L^1_+$ $\exists$ (by lemma) $E^{\mathcal{B}}(f)$ in $\overline{L_+}(\mathcal{B})$ s.t.

$$\int E^{\mathcal{B}}(f)\, g\, dP = \int f g\, dP \quad \forall g \in L^\infty_+(\mathcal{B})$$

Putting $g = 1$ we see

$$\int E^{\mathcal{B}}(f)\, dP = \int f\, dP$$

so that $E^{\mathcal{B}}(f) \in L^1_+(\mathcal{B})$ and the equality extends to all $g \in L^\infty(\mathcal{B})$.

Now <u>define</u>

$$E^{\mathcal{B}}(f) = E^{\mathcal{B}}(f^+) - E^{\mathcal{B}}(f^-)$$

for $f \in L^1$

This $E^{\mathcal{B}}(f)$ is in $L^1(\mathcal{B})$ and satisfies $\int E^{\mathcal{B}}(f)\, g\, dP = \int f g\, dP \quad \forall\ g \in L^\infty(\mathcal{B})$.

We have.

$$\int_B E^{\mathcal{B}}(f)\,dP = \int_B f\,dP \qquad \forall B \in \mathcal{B}.$$

and an argument just like the one in the previous lemma shows that $E^{\mathcal{B}}(f)$ is the unique element of $L^1(\mathcal{B})$ for which this is so.

This shows that

$f \to E^{\mathcal{B}}(f)$ is linear.

from L^1 to $L^1(\mathcal{B})$.

and ~~the an~~ $E^{\mathcal{B}}$ is positive and fixes 1 by definition.

To see that $E^{\mathcal{B}}$ is a contraction note that

$$|E^{\mathcal{B}}(f)| = |E^{\mathcal{B}}(f^+) - E^{\mathcal{B}}(f^-)|$$
$$\leq E^{\mathcal{B}}(f^+) + E^{\mathcal{B}}(f^-)$$
$$= E^{\mathcal{B}}(f^+ + f^-) = E^{\mathcal{B}}(|f|)$$

so that

$$\|E^{\mathcal{B}}(f)\|_1 = \int |E^{\mathcal{B}}(f)|\,dP \leq \int E^{\mathcal{B}}(|f|)\,dP = \int |f|\,dP = \|f\|_1.$$

It is also clear that

if $f \in L^1(\mathcal{B})$

$$E^{\mathcal{B}}(f) = f.$$

By our lemma, if h is $\mathcal{B}$ mble then

$$E^{\mathcal{B}}(hf) = h\, E^{\mathcal{B}}(f)$$ whenever

h and f are ≥ 0.

If hf and f are both integrable we have.

$$E^{\mathcal{B}}(hf) = E^{\mathcal{B}}((h^+ - h^-)(f^+ - f^-))$$

$$= E^{\mathcal{B}}(h^+f^+ + h^-f^- - (h^-f^+ + h^+f^-))$$

$$= E^{\mathcal{B}}(h^+f^+ + h^-f^-) - E^{\mathcal{B}}(h^-f^+ + h^+f^-)$$

etc...

So $$E^{\mathcal{B}}(hf) = h\, E^{\mathcal{B}}(f)$$

if hf and f are integrable.

"relativising" Conditional Expectation

If $f_1 = f_2$ on $B \in \mathcal{B}$ then

$$E^{\mathcal{B}}(f_1) = E^{\mathcal{B}}(f_2) \quad \text{on } B.$$

because

$$\chi_B E^{\mathcal{B}}(f_1) = E^{\mathcal{B}}(\chi_B f_1) = E^{\mathcal{B}}(\chi_B f_2)$$

$$= \chi_B E^{\mathcal{B}}(f_2)$$

$\therefore$ It makes sense to talk about $E^{\mathcal{B}}(f)$ for a real valued function (positive or integrable on B) whose values are only defined on B.

$E^{\mathcal{B}}(f)$ is just the common value for all fcns f^* which are positive and integrable and agree with f on B.

Hölders inequality (for Conditional expectations)

Let $1 < p < \infty$, $1 < q < \infty$

s.t $\frac{1}{p} + \frac{1}{q} = 1$.

then.

$$|E^{B}(fg)| \leq E^{B}(|f|^{p})^{1/p} E^{B}(|g|^{q})^{1/q}$$

on Ω $\forall$ $f \in L^{p}$ and $g \in L^{q}$.

Pf: We have

$$xy \leq \frac{x^{p}}{p} + \frac{y^{q}}{q} \quad \forall\, x, y \in \mathbb{R}^{+}.$$

since for fixed $y > 0$.

$g(x) = xy - \frac{x^{p}}{p}$ has

$g'(x) = y - x^{p-1}$

$g''(x) = -(p-1)\, x^{p-2} < 0$ for $x > 0$.

$g'(x) = 0$ at $x = y^{\frac{1}{p-1}}$ so $xy - \frac{x^{p}}{p} \leq y^{\frac{1}{p-1}} y - \frac{y^{\frac{p}{p-1}}}{p}$

$$\frac{E^{B}(|fg|)}{E^{B}(|f|^{p})^{1/p}\,E^{B}(|g|^{q})^{1/q}} \le \frac{1}{p}+\frac{1}{q} = 1 \quad \text{on } B.$$

so,

$$|E^{B}(fg)| \le E^{B}(|fg|) \le E^{B}(|f|^{p})^{1/p}\,E^{B}(|g|^{q})^{1/q}$$

on B.

On the set $B_f = \{E^{B}(|f|^{p}) = 0\}$

we have

$$\int_{B_f} |f|^{p}\,dP = \int_{B_f} E^{B}(|f|^{p})\,dP = 0.$$

so $|f| = 0$ a.s

so $E^{B}(fg) = 0$ a.s. on $B_f \in \mathcal{B}$.

Similarly $E^{B}(fg) = 0$ on $B_g = \{E^{B}(|g|^{q}) = 0\}$.

and since $\Omega = B \cup B_f \cup B_g$

Holder's inequality holds on Ω.