Math 138 3/26/09

remark: if $\mathcal{B}_n$ is a decreasing sequence of sub-σ-fields in $\mathcal{A}$.

then $\forall p \in [1,\infty]$

$$\bigcap_{\mathbb{N}} L^p(\mathcal{B}_n) = L^p\left(\bigcap_{N} \mathcal{B}_n\right).$$

Another characterization based on algebraic (rather than lattice) structure

Prop: $p \in [1,\infty)$

$M \subset L^p$ a closed subspace containing the constants

If M contains a sub-algebra $\mathcal{A}$ of L^∞ which is dense in M in the L^p norm

then $M = L^p(\mathcal{B})$ for some complete sub-σ-field $\mathcal{B}$ of $\mathcal{A}$.

Remarks: 1) If $\mathcal{B}$ is a complete sub σ-field of $\mathcal{A}$ then the space $L^p(\mathcal{B})$ contains the algebra $L^\infty(\mathcal{B})$ (a sub-algebra of L^∞) as an L^p-dense subset. So the converse of the proposition holds.

2) The proposition is false for $p = +\infty$.

$\exists$ L^∞ closed subalgebras of L^∞ which are not $L^\infty(\mathcal{B})$ for any complete sub-σ-field $\mathcal{B}$.

e.g. $C[0,1] \subset L^\infty[0,1]$.

Lemma: Let $\Lambda \subset L^\infty$ be a subalgebra containing the constants.

Let $\mathcal{B}$ be the complete sub-σ-field of $\mathcal{A}$ generated by Λ.

Then,
$\forall p \in [1,\infty)$, $\overline{\Lambda}$ (L^p closure) $= L^p(\mathcal{B})$

Notice that the lemma is what we need to prove the proposition:
If the sub-algebra $\Lambda \subset L^\infty$ contains the constants, the lemma shows that $M = \overline{\Lambda} = L^p(\mathcal{B})$)
If Λ does not contain the constants, let Λ' be the vectorspace generated by Λ and 1

Then Λ' is a sub-algebra of L^∞,
dense in M (in L^p norm)
and by the lemma again, $M = L^p(\mathcal{B})$.

Proof of lemma: Let $p \in [1,\infty)$ be fixed.

It is enough to show that $\bar{\Lambda}$ is lattice ordered ~~and~~ in L^p and for this it suffices to show that $f \in \bar{\Lambda} \Rightarrow f^+ \in \bar{\Lambda}$

since $\max(f_1, f_2) = f_1 + (f_2 - f_1)^+$.

Now, $f^+(x) = g(f)(x)$ where $g(x) = x^+$

By the Weierstrass approximation thm,
given $f \in \Lambda$

$\exists$ polynomials $(P_n, n \in \mathbb{N})$ converging uniformly to $g(x) = x^+$ on $\{x : |x| \leq \|f\|_\infty\}$.

Now,
$P_n(f) \in \Lambda$ since Λ is an algebra
containing the constants and f

$P_n(f) \to f^+$ uniformly, $\therefore$ in L^p.

$\therefore\ f^+ \in \bar{\Lambda}$ when $f \in \Lambda$

Since $f \to f^+$ is a continuous mapping ~~on~~ L^p, we have

$$f \in \bar{\Lambda} \Rightarrow f^+ \in \bar{\Lambda}.$$

So

$$\bar{\Lambda} = L^p(\mathcal{B}')$$ for some complete sub-σ-field $\mathcal{B}' \subset \mathcal{A}$.

Claim: $\mathcal{B} = \mathcal{B}'$

$$\Lambda \subset L^p(\mathcal{B}) \Rightarrow \bar{\Lambda} = L^p(\mathcal{B}') \subset L^p(\mathcal{B})$$

so $\mathcal{B}' \subset \mathcal{B}$

$$\Omega \subset L^p(\mathcal{B}')$$

$\Rightarrow$ functions in Ω (equivalence classes) are $\mathcal{B}'$ msble

$$\Rightarrow \mathcal{B} \subset \mathcal{B}'.$$

$$\therefore\ \bar{\Lambda} = L^p(\mathcal{B}).$$

Def: The orthogonal projection of L^2 onto the closed vector subspace $L^2(\mathcal{B})$ is the <u>conditional Expectation</u> w.r.t $\mathcal{B} \subset \mathcal{A}$.

$\to E^{\mathcal{B}}$.

For $f \in L^2$

a) $E^{\mathcal{B}}(f) \in L^2(\mathcal{B})$

and b) $\int E^{\mathcal{B}}(f)\, g \, dP = \int f g \, dP \quad \forall g \in L^2(\mathcal{B})$

and these characterize $E^{\mathcal{B}}(f)$.

b) holds for all $g \in L^2(\mathcal{B})$
iff it holds for all g in a generating subset of $L^2(\mathcal{B})$

e.g. $E^{\mathcal{B}}(f)$ is the unique element of L^2 s.t.

$$\int_B E^{\mathcal{B}}(f)\, dP = \int_B f \, dP \quad \forall B \in \mathcal{B}.$$

Prop An orth. proj. on L^2 is a conditional expectation

iff

it is positive and leaves 1 invariant

Pf: $E^{\mathcal{B}}(1) = 1.$

and we claim $E^{\mathcal{B}}(f) \geq 0$ when $f \geq 0.$

Let $g = \chi_{\{E^{\mathcal{B}}(f) < 0\}} \in L^2(\mathcal{B})$

then

$$\int E^{\mathcal{B}}(f)\, g \, dP = \int f g \, dP$$

$$\text{so } 0 \geq \int_{(E^{\mathcal{B}}(f)<0)} E^{\mathcal{B}}(f)\, dP = \int_{(E^{\mathcal{B}}(f)<0)} f \, dP \geq 0.$$

$\therefore \quad P(E^{\mathcal{B}}(f) < 0) = 0.$

and $E^{\mathcal{B}}(f) \geq 0$ in L^2.

Conversely,

Suppose U is a positive orth. proj. on L^2 s.t. $U\mathbb{1} = \mathbb{1}$.

Then our corollary from last time shows

$$\{f : Uf = f\} = L^2(\mathcal{B})$$

for some complete $\mathcal{B} \subset \mathcal{A}$.

U is the orth. projection onto $\{f : Uf = f\}$ so $U = E^{\mathcal{B}}$. $\bullet$

Prop $\forall f \in L^2$

$$E^{\mathcal{B}}(hf) = h\,E^{\mathcal{B}}(f) \qquad \text{if } h \in L^\infty(\mathcal{B}).$$

Pf: The properties a) b) characterize $E^{\mathcal{B}}(hf)$ so we verify them For $h\,E^{\mathcal{B}}(f)$

a) $h\,E^{\mathcal{B}}(f) \in L^2(\mathcal{B})$

b) $\forall g \in L^2$, $hg \in L^2$ so $\int E^{\mathcal{B}}(f)\,hg\,dP = \int (fh)g\,dP$.

So $h\,E^{\mathcal{B}}(f)$ is the C: Exp. of fh.

(i.e. $$\int \left((E^{\mathcal{B}} f) h - fh \right) g \, dP = 0$$
$\forall g \in L^2(\mathcal{B})$.)