3/24/09 Math 138.

$(\Omega, \mathcal{A}, P)$ a probability space.

$L(\Omega, \mathcal{A}, P)$ = space of equiv. classes of finite, real valued msble fcns on $(\Omega, \mathcal{A}, P)$

$L^p(\Omega, \mathcal{A}, P) = L^p \subset L(\Omega, \mathcal{A}, P)$

$1 \le p \le \infty$.

$\mathcal{B} \subset \mathcal{A}$ a sub-σ-field of $\mathcal{A}$

$\mathcal{B}$ is <u>complete</u> in $(\Omega, \mathcal{A}, P)$ if it contains $\{A \in \mathcal{A} : P(A) = 0\}$.

If $\mathcal{B}$ is complete, then any two real valued $\mathcal{A}$-msble fcns which are equal a.s. (a.e) are either both $\mathcal{B}$-msble or both non-$\mathcal{B}$-msble.

If $\mathcal{B}$ is not complete

let $\tilde{\mathcal{B}} = \sigma(\mathcal{B}, \{A \in \mathcal{A} : P(A) = 0\})$.

(the smallest σ-field containing $\mathcal{B}$ and $\{A \in \mathcal{A} : P(A) = 0\}$).

<u>Claim</u>: A real valued ~~$\mathcal{B}$~~ $\mathcal{A}$ msble function f is $\tilde{\mathcal{B}}$-msble iff $\exists$ a $\mathcal{B}$-msble function g s.t. $f = g$ a.s. $(\mathcal{A})$.

<u>Def</u>: Let $L(\mathcal{B}) = L(\tilde{\mathcal{B}})$.

then $L(\mathcal{B}) = \{$ equivalence classes in $L(\Omega, \mathcal{A}, P)$ which contain at least one $\mathcal{B}$-msble fcn $\}$.

Def: $L^p(\mathcal{B}) = L(\mathcal{B}) \cap L^p(\Omega, \mathcal{A}, P)$

$1 \le p \le \infty$.

The spaces $L(\mathcal{B})$, $L^p(\mathcal{B})$ are special subspaces of L and L^p respectively. They can be completely characterized without reference to the σ-fields $\mathcal{B}$).

Considering only complete σ-fields we have for all $p \in [1, \infty]$

Claim: $L(\mathcal{B}_1) \subset L(\mathcal{B}_2) \Leftrightarrow \mathcal{B}_1 \subset \mathcal{B}_2 \Leftrightarrow L^p(\mathcal{B}_1) \subset L^p(\mathcal{B}_2)$

$L(\mathcal{B}_1) = L(\mathcal{B}_2) \Leftrightarrow \mathcal{B}_1 = \mathcal{B}_2 \Leftrightarrow L^p(\mathcal{B}_1) = L^p(\mathcal{B}_2)$

Let $V \subset L$ (or $V \subset L^p$)

<u>Prop</u>: $V = L(\mathcal{B})$ (or $V = L^p(\mathcal{B})$)

for some complete σ-sub-field

$\mathcal{B} \subset \mathcal{A}$

iff the following hold

i) V is lattice ordered

ii) V contains the constant 1

iii) V is closed under monotone limits

i.e. ~~if $f_n \nearrow f$ in $L(\mathcal{B})$ or $L^p(\mathcal{B}$~~)

~~(and $f \in L(\mathcal{B})$ or $f \in L^p(\mathcal{B}$~~)

~~then $f \in L(\mathcal{B})$~~

$\{f_n\} \subset V$ is monotone and

$f_n \to f \in L$ or L^p

then $f \in \cancel{L(\mathcal{B})}\ V$ as $(= L(\mathcal{B})$ or $L^p(\mathcal{B}))$,

If $p \in [1, \infty)$
$V = L^p(\mathcal{B})$ for some complete $\mathcal{B}$

iff V is closed in L^p,
contains $\mathbb{1}$
and is lattice ordered.

<u>Remark</u>: The last statement fails when
$p = \infty$
e.g. $V = C[0,1] \subset L^\infty[0,1]$
(Lebesgue measure).
V is closed in L^∞ (passing to equivalence classes),
but is not $L^p(\mathcal{B})$ for any $\mathcal{B}$.
(consider χ_B with $0 < P(B) < 1$.).

<u>Pf:</u> $L(\mathcal{B})$ and $L^p(\mathcal{B})$

are • closed under monotone limits

• lattice ordered: $\max(f_1, f_2) = f_1 + (f_2 - f_1)^+$

and • contain 1.

$L^p(\mathcal{B})$ is closed in L^p:

if $\{f_n\} \subset L^p(\mathcal{B})$ and $f_n \to f$ in L^p

$\exists$ a subsequence f_{n_k} s.t. $f_{n_k} \to f$ a.s.

so f must be $\mathcal{B}$ msble.

Conversely,

If $V \subset L$ or L^p is

closed under monotone limits

lattice ordered

and contains 1

then let $\mathcal{B} = \{B : B \in \mathcal{A}, \chi_B \in V\}$

Then $\mathcal{B}$ is a complete sub σ field of $\mathcal{A}$:

if $P(B) = 0$ then χ_B is a representative of the zero function in L or L^p, $\therefore \chi_B \in V$.

$$\chi_{B^c} = 1 - \chi_B.$$

$$\chi_{B_1 \cup B_2} = \max(\chi_{B_1}, \chi_{B_2})$$

$$\chi_{\lim_{n\to\infty} \uparrow B_n} = \lim_n \uparrow \chi_{B_n}.$$

Since any non-neg function in $L(\mathcal{B})$ or $L^p(\mathcal{B})$ is a monotone limit of non-neg. simple functions. $\sum_k b_k \chi_B$ with $B \in \mathcal{B}$.

$L(\mathcal{B})$ and $L^p(\mathcal{B})$ are generated by $\{\chi_B : B \in \mathcal{B}\}$ by forming linear combinations and monotone limits.

$\therefore$ $L(\mathcal{B})$ or $L^p(\mathcal{B}) \subset V$.

If $f \in V$ then

$$\chi_{\{f>0\}} = \lim_n \uparrow \min(nf^+, 1).$$

so $\chi_{\{f>0\}} \in L(\mathcal{B})$ or $L^p(\mathcal{B})$

and $\therefore \{f>0\} \in \mathcal{B}$.

$\therefore \{f>a\} = \{f-a>0\} \in \mathcal{B}$
$\forall a \in \mathbb{R}$.

and $\therefore V \subset L(\mathcal{B})$ or $L^p(\mathcal{B})$.

Finally,
For $p \in [1, \infty)$,

any closed subset of L^p is closed under monotone limits in L^p.

If $\{f_n\} \subset L^p$ converges monotonically to $f \in L^p$ pointwise then
$|f_n - f|^p \le (|f_n| + |f|)^p \le (2|f|)^p$
so $\int |f_n - f|^p \searrow 0$ by dominated convergence

<u>Corollary</u>: $p \in [1,\infty)$ fixed

$U : L^p \to L^p$ a

positive linear contraction

(i.e. $f \geq 0 \Rightarrow Uf \geq 0$

$\|Uf\|_p \leq \|f\|_p$)

s.t. $U(1) = 1$

Then $\{f \in L^p : Uf = f\}$ is $L^p(\mathcal{B})$

for some complete sub σ field

$\mathcal{B} \subset \mathcal{A}$.

<u>Pf</u>: $M = \{f \in L^p : Uf = f\}$ is a

closed subspace of L^p containing 1

so we just need to show that M is lattice

ordered.

Since U is positive and $f^+ - f \geq 0$

we have $U(f^+) \geq U(f)$ $\forall f \in L^p$.

and $U(f^+) \geq 0$

$\therefore U(f^+) \geq U(f)^+$ $\forall f \in L^p$

For $f \in M$ this is

$$U(f^+) \geq f^+ \geq 0.$$

Since $\|Uf\|_p \leq \|f\|_p$, we must have

$$U(f^+) = f^+.$$

$\therefore \quad f \in M \implies f^+ \in M.$

Since $\max(f_1, f_2) = f_1 + (f_2 - f_1)^+$,

M is lattice ordered. $\blacksquare$

<u>Corollary</u>: $(\mathcal{B}_n, n \in \mathbb{N})$ is an increasing sequence of sub-σ-fields in $(\Omega, \mathcal{A}, P)$

$$\mathcal{B}_\infty = \sigma\left(\bigcup_{\mathbb{N}} \mathcal{B}_n\right)$$

Then $\forall p \in [1, \infty)$

$$L^p(\mathcal{B}_\infty) = \overline{\bigcup_{\mathbb{N}} L^p(\mathcal{B}_n)} \text{ in } L$$

Pf: Since the $\mathcal{B}_n$ increase, the subspaces $L^p(\mathcal{B}_n)$ increase

$\therefore$ $\bigcup_{\mathbb{N}} L^p(\mathcal{B}_n)$ is a vector subspace of L^p containing $\underline{1}$ which is lattice ordered.

$\overline{\bigcup_{\mathbb{N}} L^p(\mathcal{B}_n)}$ has the same properties

since

$$f \to af$$
$$(f,g) \to f+g$$
$$(f,g) \to f \vee g$$
$$(f,g) \to f \wedge g$$

are all continuous mappings.

(note that $|a' \vee b' - a \vee b| \le |a'-a| + |b'-b|$ $\forall\, a', b', a, b \in \mathbb{R}$.

$\therefore$ $\|f' \vee g' - f \vee g\|_{L^p} \le \|f - f'\|_{L^p} + \|\cancel{g} - g'\|_p$.

$\forall\, f', g', f, g \in L^p$).

$$\therefore \quad \overline{\bigcup_{\mathbb{N}} L^p(\mathcal{B}_n)} = L^p(\mathcal{B})$$

for some complete sub-σ-field $\mathcal{B} \subset \mathcal{A}$.

and we claim that $L^p(\mathcal{B}) = L^p(\mathcal{B}_\infty)$:

$$L^p(\mathcal{B}_n) \subset L^p(\mathcal{B}) \qquad \text{for each } n$$

$$\Rightarrow \quad \mathcal{B}_n \subset \tilde{\mathcal{B}}_n \subset \mathcal{B} \qquad (\mathcal{B} \text{ is complete})$$

$$\Rightarrow \quad \tilde{\mathcal{B}}_\infty \subset \mathcal{B}.$$

$$\Rightarrow \quad L^p(\mathcal{B}_\infty) \subset L^p(\mathcal{B}).$$

Simil $\mathcal{B}_n \subset \mathcal{B}_\infty$

we have $L^p(\mathcal{B}_n) \subset L^p(\mathcal{B}_\infty) \quad \forall n.$

$$\therefore \quad \overline{\bigcup_{\mathbb{N}} L^p(\mathcal{B}_n)} \subset L^p(\mathcal{B}_\infty)$$

and $\therefore \quad \tilde{\mathcal{B}}_\infty = \mathcal{B}.$