Math 152 3/12/09

linear functionals on H.

By the Cauchy - Schwarz inequality, each $y \in H$ defines a bounded linear functional.

$$\ell(x) = \langle x, y \rangle$$

$$|\ell(x)| \leq \|y\| \, \|x\|. \quad \left(\begin{array}{c} \text{and since } \ell(y) = \|y\|^2 \\ \|\ell\| = \|y\|. \end{array} \right).$$

In fact, each element of H^* is represented in this way.

Thm: If $f \in H^*$, there is a unique $y \in H$ s.t $f(x) = \langle x, y \rangle \quad \forall x \in H$.

Pf: if $\langle x, y_1 \rangle = \langle x, y_2 \rangle \quad \forall x \in H$

then $y_2 - y_1 \in H^{\perp} = \{0\}$.

So uniqueness is clear.

$Ker(f) = \{x \in H : f(x) = 0\}$ is a closed subspace of H and w.m.a. $Ker(f) \neq H$.

Let $z \in Ker(f)^{\perp}$ s.t $\|z\| = 1$.

then for any $x \in H$.

$$f(z)x - z f(x) \in Ker(f)$$

so $\langle f(z)x - z f(x), z \rangle = 0$.

$$\Rightarrow f(z)\langle x, z\rangle = f(x)\|z\|^2 = f(x).$$

So $f(x) = \langle x, \overline{f(z)}\, z\rangle$.

<u>Orthonormal sets</u>

$\{u_\alpha\}_{\alpha \in A} \subset H$ is orthonormal

iff $u_\alpha \perp u_\beta$ $\alpha \neq \beta$ and $\|u_\alpha\| = 1$ $\forall \alpha \in A$.

<u>Gramm-Schmidt</u>

If $\{x_n\}_{n=1}^{\infty}$ is linearly independent

(i.e each finite subset is ~~lf~~).

$\exists$ an orthonormal sequence

$\{u_n\}_{n=1}^{\infty}$ s.t $span(\{u_n\}_{n=1}^{N}) = span(\{x_n\}_{n=1}^{N})$

for each N.

let $u_1 = \frac{x_1}{\|x_1\|}$

then $\langle x_2 - \langle x_2, u_1 \rangle u_1 \; , \; u_1 \rangle$

$= 0$

and $x_2 - \langle x_2, u_1 \rangle u_1 \neq 0$ since

$x_2 \notin \text{span}\{x_1\}$.

let $u_2 = \frac{x_2 - \langle x, u_1 \rangle u_1}{\| x_2 - \langle x, u_1 \rangle u_1 \|}$

Having defined $u_1, \dots, u_{N-1}$

let $v_N = x_N - \sum_{j=1}^{N-1} \langle x_N, u_j \rangle u_j$

then $v_N \neq 0$ since $v_N \notin \text{span}\{x_1, \dots x_{N-1}\}$

for $j' < N$.

and $\langle v_N, u_{j'} \rangle = \langle x_N, u_{j'} \rangle - \langle x_N, u_{j'} \rangle$

$= 0$.

since $u_{j'} \perp u_j$ $j \neq j'$ $1 \leq j \leq N-1$.

let $u_N = \frac{v_N}{\|v_N\|}$.

Bessel's Inequality

If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in H, then for any $x \in H$.

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2.$$

This shows that

$\{\alpha : \langle x, u_\alpha\rangle \neq 0\}$ is countable.

Pf: It is enough to show that
$$\sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$$ when

F is any finite subset of A.

If A is countable $\sum_{\alpha\in A}$ is a limit of such finite sums and if A is uncountable, all but countably many $\alpha\in A$ give $\langle x, u_\alpha\rangle = 0$, otherwise $\exists\, n$ s.t. $\frac{1}{n+1} \leq |\langle x, u_\alpha\rangle| < \frac{1}{n}$ for ∞'ly many α.
and $\sum_{\alpha\in F} > \|x\|^2$ for a finite subset.

$$0 \leq \Big\| x - \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\|^2$$

$$= \|x\|^2 - 2\,\mathrm{Re}\Big\langle x, \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\rangle + \Big\| \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\|^2$$

$$= \|x\|^2 - 2 \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 + \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2$$ Pythagorean Thm.

$$= \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2$$

$$\therefore \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.$$

__Thm__: $\{u_\alpha\}_{\alpha \in A}$ orthonormal in $\mathcal{H}$.

T.F.A.E. (equivalent conditions for an o.n. basis).

Completeness a) $\langle x, u_\alpha \rangle = 0 \quad \forall\, \alpha \in A \implies x = 0.$

Parseval's Formula b) $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \quad \forall\, x \in \mathcal{H}.$

c) $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ for each $x \in \mathcal{H}$.

- countably many non-zero terms in the sum
- sum converges in $\|\cdot\|$, no matter how ordered

Pf:

b) $\Rightarrow$ a) is clear

c) $\Rightarrow$ b) Let $\alpha_1, \alpha_2, \dots$ enumerate the α's s.t. $\langle x, u_\alpha \rangle \neq 0$.

then

$$x = \sum_{j} \langle u, u_{\alpha_j} \rangle u_{\alpha_j}$$

If there are finitely many such α's then

$$x = \sum_{j=1}^{N} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$$

and

$$\|x\|^2 = \sum_{j=1}^{N} |\langle x, u_{\alpha_j} \rangle|^2$$

by Pythagoras.

otherwise (as in proof of Bessel).

$$\|x\|^2 - \sum_{1}^{n} |\langle x, u_{\alpha_j} \rangle|^2 = \|x - \sum_{1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}\|^2$$

$$\to 0$$

as $n \to \infty$.

a) $\Rightarrow$ c).

a) $\Rightarrow$ c) For $x \in H$ let $\alpha_1, \alpha_2, \ldots$ be an enumeration of α's s.t $\langle x, u_\alpha \rangle \neq 0$.

Then $\sum_{j=1} |\langle x, u_{\alpha_j} \rangle|^2 \leq \|x\|^2$

by Bessel so $\sum_j |\langle x, u_{\alpha_j} \rangle|^2$

converges.

$\therefore$ $\left\| \sum_n^m \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2 = \sum_n^m |\langle x, u_{\alpha_j} \rangle|^2$

$\to 0$

as $m, n \to \infty$

$\therefore$ $\sum_j \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$ converges ~~six~~ since H is complete.

Now

$\langle x - \sum_j \langle x, u_{\alpha_j} \rangle u_{\alpha_j}, u_\alpha \rangle$

$= \langle x, u_{\alpha_j} \rangle - \langle x, u_{\alpha_j} \rangle = 0$ if $\alpha = \alpha_j$ some j

and $= \langle x, u_\alpha \rangle = 0$ if $\alpha \neq \alpha_j$ for any j.

Note that the enumeration was arbitrary

e.g. $H = \ell^2(A)$

$e_\alpha \in \ell^2(A)$ $\qquad e_\alpha(\beta) = 1 \quad \beta = \alpha.$

$\qquad e_\alpha(\beta) = 0 \quad \beta \neq \alpha.$

$$\langle e_\alpha, e_\beta \rangle = \sum_{\gamma \in A} e_\alpha(\gamma)\overline{e_\beta(\gamma)}$$

$$= \cancel{e_\alpha(\alpha)}\,\cancel{\overline{e_\beta(\alpha)}} + \cancel{e_\alpha(\beta)}\,\cancel{\overline{e_\beta(\beta)}}$$

$= 1 \quad \alpha = \beta$

$= 0$ else.

$$\langle f, e_\alpha \rangle = \sum_{\beta \in A} f(\beta)\overline{e_\alpha(\beta)} = f(\alpha).$$

So if $\langle f, e_\alpha \rangle = 0 \ \forall \alpha$ then $f \equiv 0$.

and $\{e_\alpha\}_{\alpha \in A}$ is an o.n. basis.

Prop: Every Hilbert Space has an O.N. basis.

PF: Zorn's lemma. on orthonormal sets ordered by inclusion.

$\Longrightarrow \exists$ a maximal orthonormal set in H.

Say $\{u_\alpha\}_{\alpha \in A}$, so if $\langle x, u_\alpha \rangle = 0 \ \forall \alpha \in A$ then $x = 0$.

Unitary maps.

$U : H_1 \to H_2$ (Hilbert spaces)

is unitary iff U is invertible

and $\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1}$

$\forall\ x, y \in H_1$.

If $x = y$ this says.

$$\|Ux\|_{H_2} = \|x\|_{H_1}$$

i.e a unitary is an isometry.

Conversely, Every surjective isometry is unitary.

By the Polarization identity

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \|x+i^k y\|^2$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \langle x+i^k y, x+i^k y\rangle$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \left(\|x\|^2 + \langle x, i^k y\rangle + \overline{\langle x, i^k y\rangle} + \|y\|^2\right)$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \left(i^{-k}\langle x,y\rangle + i^k\overline{\langle x,y\rangle}\right)$$

(since $1+i+i^2+i^3=0$)

$$= \frac{1}{4}\sum_{k=0}^{3}\langle x,y\rangle + \frac{1}{4}\sum_{k=0}^{3} i^{2k}\overline{\langle x,y\rangle}$$

$$= \langle x,y\rangle \checkmark.$$

A surjective isometry is invertible and.

$$\langle Ux, Uy\rangle_{H_2} = \cancel{\tfrac{1}{4}\|U(x+y)\|^2_{H_2}} \quad \frac{1}{4}\sum_{k=0}^{3} i^k \|U(x+i^k y)\|^2_{H_2}$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \|x+i^k y\|^2_{H_1}$$

$$= \langle x,y\rangle_{H_1}.$$

Prop: $\mathcal{H}$ (a Hilbert Space) is separable iff it has a countable orthonormal basis.

If $\mathcal{H}$ is separable, then every o.n. basis is countable

Pf: Suppose $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of $\mathcal{H}$.

~~Discarding elements as needed, we may assume~~

~~$\operatorname{span}\{x_n\}_{n=1}^{N} \subsetneq \operatorname{span}\{x_n\}_{n=1}^{N+1}$~~

discarding x_n if $x_n \in \operatorname{span}\{x_1, \dots, x_{n-1}\}$.

obtain $y_1 \dots y_n \dots$

s.th ~~let~~ with dense linear span.

Applying Gramm-Schmidt to $\{y_j\}_{j=1}^{\infty}$ obtain an o.n. set $\{u_j\}_{j=1}^{\infty}$ whose linear span is dense.

by c) from last theorem

this implies $\{u_j\}_{j=1}^{\infty}$ is o.n.

basis.

Conversely, if $\{u_n\}_{n=1}^{\infty}$ is an o.n.

basis, finite linear combinations

with coefficients in $\mathbb{Q} + i\mathbb{Q}$

form a countable dense subset of H.

If $\{v_\alpha\}_{\alpha \in A}$ is an orthonormal

basis in a separable Hilbert space

let $\{u_j\}_{j=1}^{\infty}$ be a countable

o.n. basis then

$A_n = \{\alpha \in A : \langle v_\alpha, u_n \rangle \neq 0\}$ is countable.

For each $\alpha \in A$ $\exists$ n s.t. $\langle v_\alpha, u_n \rangle \neq 0$.

So $A = \cup A_n$ and A is countable.

Prop: Let $\{e_\alpha\}_{\alpha \in A}$ be an orthonormal basis for X.

$x \to \hat{x}$ defined by $\hat{x}(\alpha) = \langle x, e_\alpha \rangle$

is a unitary map from H to $l^2(A)$.

Pf: We show that $x \to \hat{x}$ is a surjective isometry.

It is linear ✓ and

by Parseval

$$\|x\|^2 = \sum |\langle x, e_\alpha \rangle|^2 = \sum |\hat{x}(\alpha)|^2$$

$$= \|\hat{x}\|^2_{l^2(A)}.$$

so $x \to \hat{x}$ is isometric.

If $f \in l^2(A)$ then $\sum_{\alpha \in A} |f(\alpha)|^2 < +\infty$.

(countably many terms are non-zero)

So the partial sums of $\sum_{\alpha} f(\alpha) u_\alpha$ are Cauchy.

so $x = \sum f(\alpha) u_\alpha$ exists and $\hat{x}(\alpha) = f(\alpha)$ $\forall \alpha \in A$

$\therefore$ $x \to \hat{x}$ is surjective.