Math 138 3/10/09

Hilbert Spaces.

On a complex vector space H, an inner product is a map.

$$(x,y) \longrightarrow \langle x,y\rangle \quad \text{from } H\times H \to \mathbb{C}$$

with the properties

i) $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$
$\forall x,y \in H,\ a,b \in \mathbb{C}$

ii) $\langle y,x\rangle = \overline{\langle x,y\rangle}$

iii) $+\infty > \langle x,x\rangle > 0 \quad \forall x \in H \quad x \neq 0.$

The inner product is "linear in the 1st position" and "conjugate linear in the 2nd".

A complex vector space H, with an inner product is called a pre-Hilbert space.

Let $\quad \|x\| = \sqrt{\langle x, x \rangle}$.

<u>Lemma</u>: $\quad |\langle x, y\rangle| \le \|x\|\,\|y\| \quad \forall x, y \in H$.

Equality holds iff $x = cy$ for some $c \in \mathbb{C}$.

<u>Pf</u>: $\|x-y\|^2 = \langle x-y, x-y\rangle$

$$= \|x\|^2 - \langle y, x\rangle - \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 - 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$$

$$\Rightarrow \quad 2\,\mathrm{Re}\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2.$$

Given $x, y \in H$ with $\|x\| = \|y\| = 1$

choose $\alpha \in \mathbb{R}$ s.t

$$\langle x, e^{i\alpha} y\rangle = e^{-i\alpha}\langle x, y\rangle = |\langle x, y\rangle|.$$

then

$$2|\langle x,y\rangle| = \|x\|^2 + \|e^{i\alpha}y\|^2 - \|x-e^{i\alpha}y\|^2$$
$$\leq \|x\|^2 + \|y\|^2$$

with equality iff $\cancel{x = e^{i\alpha}}$

$$\|x-e^{i\alpha}y\|^2 = 0.$$

iff $x - e^{i\alpha}y = 0$ (by the def. of inner product)

so $|\langle x,y\rangle| \leq 1 = \|x\|\,\|y\|$.

In general, we have

$$\left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle\right| \leq 1$$

so $|\langle x,y\rangle| \leq \|x\|\,\|y\|$.

with equality iff

$$\frac{x}{\|x\|} = e^{i\alpha}\frac{y}{\|y\|}$$

where $e^{i\alpha}\langle x,y\rangle \geq 0$.

Prop $x \to \|x\|$ is a norm on H.

Pf: We just need to verify the triangle inequality.

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2.$$
$$\le \|x\|^2 + 2\,|\langle x,y\rangle| + \|y\|^2.$$
$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$\Rightarrow \quad \|x+y\| \le \|x\| + \|y\|.$

with equality iff $\cancel{\mathrm{Re}\langle x,y\rangle \ge 0}$ $\mathrm{Re}\langle x,y\rangle = |\langle x,y\rangle|$ and x, y are ld.
(i.e. $\langle x,y\rangle \ge 0$).

i.e. $x = ty \quad t \ge 0$.

A Pre-Hilbert space which is complete in the norm $\|x\| = \sqrt{\langle x,x\rangle}$ is called a Hilbert Space.

e.g. $(X, \mathcal{M}, \mu)$ a measure space.

$L^2(\mu)$

$$\left| \int f\bar{g}\, d\mu \right| \le \left(\int |f|^2 d\mu \right)^{1/2} \left(\int |g|^2 d\mu \right)^{1/2}$$

by Cauchy - Schwartz for integrals

$$\langle f, g \rangle = \int f\bar{g}\, d\mu$$

defines an ~~inner~~ ~~product~~.

$$\|f\|^2 = \int f\bar{f}\, d\mu = \int |f|^2 d\mu = \langle f, f \rangle.$$

L^2 is complete in this norm.

(For $1 \le p < \infty$, L^p is a Banach Space)

When μ is counting measure on a non-empty set A.

$L^2(\mu)$ is called $\ell^2(A)$.

$f \in \ell^2(A)$ iff $\sum_{\alpha \in A} |f(\alpha)|^2 < +\infty$.

$\mathcal{H}$ a Hilbert Space

Prop: $x_n \to x$

$y_n \to y$

$\Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle.$

Pf $|\langle x_n, y_n \rangle - \langle x, y \rangle|$

$= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle|$

$= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle|$

$\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\|.$

$\to 0. \quad (\|y_n\| \to \|y\|).$

Prop: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$

Pf: ✓.

Def: $x \perp y \iff \langle x, y \rangle = 0.$

Def: $E \subset \mathcal{H}$, $E^\perp = \{x \in \mathcal{H} : \langle x, y \rangle = 0 \ \forall y \in E\}$

Prop: $x_1, \dots, x_n \in \mathcal{H}$.

$x_i \perp x_j \quad i \neq j$

$\Rightarrow \|\sum_1^n x_j\|^2 = \sum_1^n \|x_j\|^2$.

Pf.

$$\|\sum_1^n x_j\|^2 = \langle \sum_1^n x_j , \sum_1^n x_j \rangle$$

$$= \sum_{j=1}^n \sum_{k=1}^n \langle x_j , x_k \rangle$$

$$= \sum_{j=1}^n \|x_j\|^2.$$

Thm: $\mathcal{M} \subset \mathcal{H}$ a closed subspace.

$\Rightarrow \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$

i.e. $x \in \mathcal{H} \Rightarrow x = y + z \quad y \in \mathcal{M}, z \in \mathcal{M}^\perp$

and if $x = y_1 + z_1 = y_2 + z_2 \quad y_1, y_2 \in \mathcal{M}, z_1, z_2 \in \mathcal{M}^\perp$

then $y_1 = y_2 , z_1 = z_2$

also,

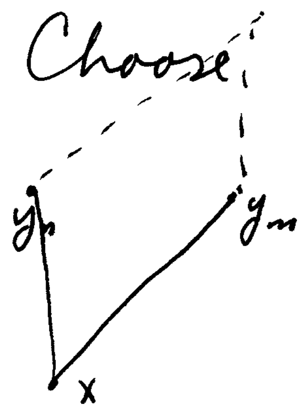
$\|x - y\| = \inf\{\|x-u\| : u \in \mathcal{M}\}$ and y is the unique element of $\mathcal{M}$ for which the inf is attained similarly

$$\|x-z\| = \inf\{\|x-v\| : v \in M^{\perp}\}$$

and z is the unique element of $M^{\perp}$ for which the inf is attained.

<u>Pf:</u> Given $x \in H$, let $\delta = \inf\{\|x-y\| : y \in M\}$

Choose $y_n \in M$ s.t. $\|y_n - x\| \to \delta$.



Then $2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right)$

$$= \|(y_n-x)+(y_m-x)\|^2 + \|(y_n-x)-(y_m-x)\|^2$$

$$= \|y_n + y_m - 2x\|^2 + \|y_n - y_m\|^2.$$

So,

$$\|y_n - y_m\|^2 = 2\left(\|y_n-x\|^2 + \|y_m-x\|^2\right) - 4\left\|\frac{y_n+y_m}{2} - x\right\|^2$$

$$\leq 2\left(\|y_n-x\|^2 + \|y_m-x\|^2\right) - 4\delta^2$$

$$\longrightarrow 0 \quad \text{as } m, n \to \infty.$$

So $\{y_n\}$ is Cauchy

Let $y = \lim_{n\to\infty} y_n$ and let $z = x - y$.

Since M is closed $y \in M$ and

$\|y - x\| = \delta$.

If $y' \in M$ then $z + ty' = x - y + ty'$

$$= x - (y - ty')$$

so $\|z + ty'\|^2 \geq \delta^2$

with equality at $t = 0$.

$\therefore$ $f(t) = \|z\|^2 + 2t \operatorname{Re}\langle y', z\rangle + t^2 \|y'\|^2$

is minimized when $t = 0$.

$\therefore$ $f'(t)\big|_{t=0} = 2\operatorname{Re}\langle y', z\rangle = 0$.

$\therefore$ $\operatorname{Re}\langle z, y'\rangle = 0 \quad \forall y' \in M$.

$\therefore$ $\langle z, y'\rangle = 0 \quad \forall y' \in M$.

since $\operatorname{Re}\langle z, y'\rangle = e^{i\alpha}\langle z, y'\rangle$

for some $\alpha \in \mathbb{R}$.

So $z \in \mathcal{M}^\perp$ and if $z' \in \mathcal{M}^\perp$ is another element then

$$\|x - z'\|^2 = \|x - z\|^2 + \|z - z'\|^2 \geq \|x - z\|^2.$$

(by the Pythagorean thm).

with equality iff $z = z'$.

Similarly if $y' \neq y$ ~~&~~ and $y' \in \mathcal{M}$.

$$\|x - y'\| = \|x - y\|^2 + \|y - y'\|^2 \geq \|x - y\|^2.$$

If $x = y' + z'$ with $y' \in \mathcal{M}$, $z' \in \mathcal{M}^\perp$

then $x = y + z = y' + z'$

so $y - y' = z' - z. \in \mathcal{M} \cap \mathcal{M}^\perp$

and are $\therefore$ orthogonal to themselves

i.e. $\|y - y'\|^2 = \|z' - z\|^2 = 0$,

so the representation is unique.