3/5/09 Math 138.

Def: A <u>Frechet Space</u>
is a complete Hausdorff T.V.S.
with topology from a countable
family of semi-norms.

e.g. Let X be a locally compact
Hausdorff space (Every point has
a compact nbhd).

$$\mathbb{C}^X = \{\text{complex valued functions on } X\}$$

Let $$p_K(f) = \sup_{x \in K} |f(x)|$$

where $K \subset X$ is a fixed
compact set.

These semi-norms define the
"topology of uniform convergence on
compact sets".

If X is σ-compact (a countable union of compact sets)
$\mathbb{C}^X$ is a Frechet space:

~~Ideal~~

e.g. $L^1_{loc}(\mathbb{R}^n) = \{ f \text{ msble in } \mathbb{R}^n : \int_B |f| dx < +\infty \}$ for and ball B

The seminorms

$$p_n(f) = \int_{|x| \le n} |f(x)|\, dx$$

make L^1_{loc} a Frechet Space.

e.g. $p_k(f) = \sup_{0 \le x \le 1} |f^{(k)}(x)|$ $k = 0, 1, 2, \dots$

make $C^\infty[0,1]$ into a Frechet Space.

exercise 9.

$C^k[0,1]$ = continuous derivatives up to order k on $[0,1]$, one sided at endpoints.

a) If $f \in C[0,1]$.

then $f \in C^k([0,1])$ iff f is

k times cont. diff on $(0,1)$

and $\lim_{x \searrow 0} f^{j}(x)$ and $\lim_{x \nearrow 1} f^{(j)}(x)$

exist $\forall j \leq k$.

assume. $f \in C[0,1]$.

<u>Pf:</u> Clearly if $f \in C^k[0,1]$ then the 2nd condition holds.

For the converse, we need to check that the one sided derivatives exist and equal the given limits.

for $x > 0$.

$$\frac{f^{j-1}(x) - f^{j-1}(0)}{x - 0}$$

put y here and let $y \to 0$

$$= \frac{f^{j}(x')\,(x-0)}{(x-0)} = f^{j}(x')$$

for some x' with $0 < x' < x$.

So $\lim_{x \to 0} \frac{f^{j-1}(x) - f^{j-1}(0)}{x - 0} = \lim_{x' \to 0} f^{j}(x')$.

and similarly at $x = 1$.

b). $\|f\| = \sum_{0}^{k} \|f^{(j)}\|_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach Space.

<u>Pf:</u> (It's clearly a norm.)
Suppose

$$\sum_{n=1}^{\infty} \|f_n\| < +\infty$$

i.e. $$\sum_{n=1}^{\infty} \sum_{j=0}^{k} \|f_n^{(j)}\|_u < +\infty.$$

When $k = 1$ $\sum_{n=1}^{\infty} \|f_n\|_u < +\infty$ and $\sum_{n=1}^{\infty} \|f_n'\|_u < +\infty$.

so $$\sum_{n=1}^{\infty} f_n = F \in C[0,1]$$
$$\sum_{n=1}^{\infty} f_n' = \cancel{G}.$$

and we claim that $F \in C'[0,1]$ and $F' = \cancel{g}\ G$.

let $F_N = \sum_{n=1}^{N} f_n$

$G_N = \sum_{n=1}^{N} f_n'$

then $F_N' = G_N$

and $F_N(x) - F_N(0) = \int_0^x G_N(t)\,dt.$

The convergence is uniform
(to a bdd limit).

so $F(x) - F(0) = \int_0^x G(t)\,dt.$

and $F'(x) = G(x).$

Clm the previous argument we may replace f_n by $f_n^{(k-1)}$ and f_n' by $f_n^{(k)}$ and proceed by induction.

Clm the case of $C^\infty[0,1]$.
we replace the norm above with

$$\|f\| = \sum_{k=0}^{\infty} 2^{-k} \| f^{(k)} \|_u$$

and $\frac{d}{dx}$: becomes a continuous operator on this space.

since $p_k(f') = p_{k+1}(f)$. [2 propositions back].

<u>Weak topologies</u>

X a vector space, Y a normed linear space

$\{T_\alpha\}_{\alpha \in A}$ a collection of linear maps from X to Y.

The weak topology $\mathcal{T}$ generated by the T_α.

makes X a locally convex T.V.S.

In fact

$\mathcal{T}$ is the same as the topology $\mathcal{T}'$ ~~generated~~ defined by the semi-norms

$$p_\alpha(x) = \| T_\alpha x \|.$$

$\mathcal{T}$ is generated by sets of form

$\{x : \|T_\alpha x - y_0\| < \epsilon\}$ where $y_0 \in Y$.

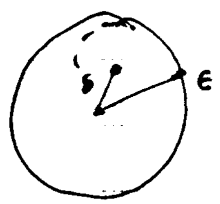
$\mathcal{T}'$ is generated by sets of the form

$\{x : \|T_\alpha x - T_\alpha x_0\| < \epsilon$ with $x_0 \in X.\}$

- each set of the 2nd type is of the 1st type.
- with y_0 fixed.

suppose $\|T_\alpha x - y_0\| = \delta < \epsilon$ for some x.

then $x \in \{u \in X : \|T_\alpha u - T_\alpha(x)\| < \frac{\epsilon - \delta}{2}\}$



$\subset \{x' : \|T_\alpha x' - y_0\| < \epsilon\}$.

So each set of the 1st type is a union of sets of the second type.

X a normed vector space

The weak topology generated by X^* is called *the weak topology* on X.

convergence in this topology is known as *weak convergence*.

$\langle x_\alpha \rangle$ a net in X

$x_\alpha \to x$ weakly iff $f(x_\alpha) \to f(x)$ $\forall f \in X^*$

The weak topology on X^* is the topology generated by X^{**}.

but we are usually more interested in the topology on X^* generated by $X \subset X^{**}$

("The weak-$*$ topology".) on X^*.

$$\boxed{x \cancel{\in} X^{\cancel{*}} \longrightarrow \hat{x} \in X^{**} \quad \text{by } \hat{x}(f) = f(x)}$$

$\{f_\alpha\}_{\alpha \in A} \subset X^*$ $\quad f_\alpha \to f$ weak-$*$

iff $\hat{x}(f_\alpha) \to \hat{x}(f)$ $\forall \hat{x}$ iff $f_\alpha(x) \to f(x)$ $\forall x$.

This is the topology of pointwise convergence.

~~I f~~

Strong Operator Topology

X, Y Banach Spaces

~~Strong~~ Operator Top on $\mathcal{L}(X,Y)$:

$T_\alpha \to T$ strongly iff $T_\alpha x \to Tx$ in norm on Y. for each $x \in X$.

i.e. generated by the maps.

$$T \to Tx \qquad x \in X.$$

Weak Operator Top on $\mathcal{L}(X,Y)$.

$T_\alpha \to T$ weakly iff.

$T_\alpha x \to Tx$ in the weak topology of Y for each $x \in X$.

i.e. generated by the linear functionals
$$T \to f(Tx) \qquad x \in X,\ f \in Y^*\}$$

Prop: Suppose $\{T_n\}_{n=1}^{\infty} \subset \mathcal{L}(X, Y)$

$$\sup_n \|T_n\| < +\infty$$

$$T \in \mathcal{L}(X, Y)$$

If $\|T_n x - Tx\| \to 0$

$\forall\, x \in D$, D a dense subset of X

then $T_n \to T$ strongly.

Pf: For any $x \in X$, $x' \in D$ we have

$$\|T_n x - Tx\| \leq \|T_n x - T_n x'\| + \|T_n x' - Tx'\| + \|Tx' - Tx\|.$$

$$\leq \sup_n \|T_n\| \, \|x - x'\| + \|T_n x' - Tx'\| + \|T\| \, \|x' - x\|.$$

Since D is dense
We can choose x' s.t.

$$\sup_n \|T_n\| \, \|x - x'\| < \epsilon/3$$

and $\|T\| \, \|x - x'\| < \epsilon/3$

and then if n is suff. large we have

$$\|T_n x' - Tx'\| < \epsilon/3.$$

$\Rightarrow \|T_n x - Tx\| < \epsilon \quad \forall n$ suff large.

Alaoglu's Thm:

If X is a normed vector space, the closed unit ball

$$B^* = \{ f \in X^* : \|f\| \le 1 \} \text{ in } X^*$$

is compact in the weak-$*$ topology.

remark: X^* is not locally compact in the weak-$*$ topology. (balls are not nbhds).

Pf: given $x \in X$
let $D_x = \{ z \in \mathbb{C} : |z| \le \|x\| \}$.

$$D = \prod_{x \in X} D_x$$

D is compact (product topology) by Tychonoff.

D consists of all complex valued fcns ϕ on X s.t. $|\phi(x)| \le \|x\| \quad \forall x \in X$

$B^* \subset D$. consists of the linear elements of D.

<u>B^* is closed in D.</u>

If $\langle f_\alpha \rangle$ is a net in B^* which converges to $f \in D$.

then for any $x, y \in X$, $a, b \in \mathbb{C}$

$$
\begin{aligned}
f(ax+by) &= \lim f_\alpha(ax+by) \\
&= \lim \left(a f_\alpha(x) + b f_\alpha(y)\right) \\
&= a f(x) + b f(y).
\end{aligned}
$$

so $f \in B^*$.

The relative topology on B^* from the product topology of D is the topology of pointwise convergence which is also the topology induced on B^* by the weak* topology on X^*.