3/3/09 Math 138

Topological Vector Spaces.

$$(\mathfrak{X}, K, \mathcal{T})$$

vector space, field, topology.

s.t.

$$(x,y) \in \mathfrak{X} \times \mathfrak{X} \longrightarrow x+y \in \mathfrak{X}$$

$$(\lambda, x) \in K \times \mathfrak{X} \longrightarrow \lambda x \in \mathfrak{X}$$

are continuous

The T.V.S. is <u>locally</u> <u>convex</u> if the topology has a base of convex sets.

<u>recall:</u> A is convex iff

$$x, y \in A \implies t \cdot x + (1-t) y \in A \quad \forall\, 0 < t < 1.$$

We mostly deal with locally convex Hausdorff T.V.S's

An important special case is when the topology is defined by a family of semi-norms.

<u>Thm</u>: Let $\{p_\alpha\}_{\alpha\in A}$ be a family of semi-norms on the vector space X.

If $x \in X$, $\alpha \in A$ and $\epsilon > 0$

let

$$U_{x\alpha\epsilon} = \{ y \in X : p_\alpha(y-x) < \epsilon \}$$

and let $\mathcal{T}$ be the topology generated by the sets $U_{x\alpha\epsilon}$.

a) For each $x \in X$, the finite intersections of the sets $U_{x\alpha\epsilon}$ ($\alpha\in A$, $\epsilon>0$) form a neighborhood base at x.

b) If $\langle x_i \rangle_{i\in I}$ is a net in $(X, \mathcal{T})$ then $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ $\forall \alpha \in A$.

c) $(X, \mathcal{T})$ is a locally convex topological vector space.

Pf: a) Any set in $\mathcal{T}$ is a union of sets of the form $\bigcap_1^k U_{x_j \alpha_j \epsilon_j}$.

If $x \in \bigcap_1^k U_{x_j \alpha_j \epsilon_j}$

take $\delta_j = \epsilon_j - p_{\alpha_j}(x - x_j)$

then $U_{x \alpha_j \delta_j} \subset U_{x_j \alpha_j \delta_j}$ $\forall j$

by the triangle inequality

so $\bigcap_1^k U_{x \alpha_j \delta_j} \subset \bigcap_{j=1}^k U_{x_j \alpha_j \delta_j}$.

So the sets $\bigcap_1^k U_{x \alpha_j \delta_j}$ form a nbhd base at x.

b). by a)

$x_i \to x$ iff $\langle x_i \rangle$ is eventually in $U_{x\alpha\epsilon}$ for any $\alpha \in A$ and $\epsilon > 0$.

iff $\quad p_\alpha(x_i - x) \to 0.$

for any $\alpha \in A$.

c). if $x_i \to x$ and $y_i \to y$.

then

$$p_\alpha((x_i + y_i) - (x+y))$$

$$\leq p_\alpha(x_i - x) + p_\alpha(y_i - y) \to 0.$$

so $x_i + y_i \to x + y$.

If $\lambda_i \to \lambda \in K$ and $x_i \to x$.

then $\exists\ i_0$ s.t $\lambda_i \leq |\lambda| + 1$.

$\forall\ i \geq i_0$.

So $\forall\ i \geq i_0$ we have.

$$p_\alpha(\lambda_i x_i - \lambda x) \leq p_\alpha(\lambda_i(x_i - x) + (\lambda_i x - \lambda x))$$

$$\leq (|\lambda|+1)\, p_\alpha(x_i - x) + |\lambda_i - \lambda|\, p_\alpha(x)$$

$\therefore\ \lambda_i x_i \to \lambda x.$

The vector operations are $\therefore$ continuous

The sets $U_{x\alpha\epsilon}$ are convex:

if $y, z \in U_{x\alpha\epsilon}$, then.

$$p_\alpha\left(x - (ty + (1-t)z)\right)$$

$$= p_\alpha\left(tx + (1-t)x - (ty + (1-t)z)\right)$$

$$\leq p_\alpha\left(t(x-y)\right) + p_\alpha\left((1-t)(x-z)\right)$$

$$< t\epsilon + (1-t)\epsilon = \epsilon.$$

Since an intersection of convex sets is convex, the topology is locally convex by a).

Prop. X, Y are T.V.S's.

with $\{p_\alpha\}_{\alpha \in A}$, $\{q_\beta\}_{\beta \in B}$ families of semi norms.

defining the respective topologies.

$T : X \to Y$ is a linear map.

T is continuous iff for each $\beta \in B$.

$\exists$ $\alpha_1, \dots \alpha_n \in A$ and $C > 0$ s.t.

$$q_\beta(Tx) \le C \sum_{1}^{k} p_{\alpha_j}(x)$$

Pf: $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ $\forall \alpha \in A$.

so the 2nd condition ~~shows that~~

$\Rightarrow q_\beta(Tx_i - Tx) \to 0$ $\forall \beta$ if $x_i \to x$.

so $Tx_i \to Tx$. and T is continuous

Conversely,
If T is continuous

if T is continuous and $\beta \in B$

$\exists$ a nbhd U of $0 \in \mathfrak{X}$ s.t.

$q_\beta(Tx) < 1 \quad \forall \; x \in U.$

W.M.A. $\quad U = \bigcap_{j=1}^{k} U_{0 \alpha_j \epsilon_j}$

Let $\epsilon = \min \{\epsilon_1, \dots \epsilon_k\}$.

then $\quad p_{\alpha_j}(x) < \epsilon \quad \forall \; j = 1, \dots k$

$$\Rightarrow q_\beta(Tx) < 1.$$

<u>Let $x \in X$:</u>

<u>Case I</u> $\quad p_{\alpha_j}(x) > 0$ for some $j \in \{1, \dots k\}$

then $\quad y = \dfrac{\epsilon \, x}{\sum_1^k p_{\alpha_j}(x)}$ has $p_{\alpha_j}(y) < \epsilon. \; \forall j.$

so $q_\beta(Ty) < 1$ and $\therefore$

$$q_\beta(Tx) = \frac{1}{\epsilon} \sum_{j=1}^{k} p_{\alpha_j}(x) \, q_\beta(Ty) \le \frac{1}{\epsilon} \sum_{j=1}^{k} p_{\alpha_j}(x).$$

<u>Case II</u> $p_{\alpha_j}(x) = 0 \quad \forall j.$

Then $p_{\alpha_j}(rx) = 0 \quad \forall j \quad \forall r > 0.$

so $q_\beta(T(rx)) < 1 \quad \forall r > 0$

and $\therefore$ $r\, q_\beta(Tx) < 1 \quad \forall r > 0.$

$\therefore$ $q_\beta(Tx) = 0.$

and $q_\beta(Tx) \leq \frac{1}{\epsilon} \sum_1^k p_{\alpha_j}(x).$

in this case also.

$\boxtimes$.

<u>Prop</u>: X a T.V.S. with $\mathcal{T}$ determined by $\{p_\alpha\}_{\alpha \in A}$

a) X is Hausdorff iff for each $x \neq 0$. $\exists\, \alpha \in A$ s.t. $p_\alpha(x) \neq 0$.

b) if X is Hausdorff and A is countable, then X is metrizable with a translation invariant metric $\left(\begin{array}{l} \rho(x,y) \\ = \rho(x+z, y+z) \end{array} \quad x,y,z \in X.\right.$

Pf: a). Suppose that each $x \neq 0$ $\exists\, \alpha \in A$ s.t. $p_\alpha(x) \neq 0$.

If $x_0 \neq y_0$, choose $\alpha \in A$ s.t.

$$\epsilon_0 \equiv p_\alpha(x_0 - y_0) > 0.$$

Claim:
$$\{x : p_\alpha(X - X_0) < \epsilon_0/4\}$$
$$\{y : p_\alpha(y - y_0) < \epsilon_0/4\}.$$
are disjoint.

If $p_\alpha(x - x_0) < \epsilon_0/4$. then.

$\cancel{p_\alpha(x-y_0) = p_\alpha(x)}$

$$\epsilon_0 = p_\alpha(X_0 - y_0) \leq p_\alpha(X_0 - x) + p_\alpha(x - y_0)$$

and ∴
$$p_\alpha(x - y_0) \geq \epsilon_0 - p_\alpha(X_0 - x)$$
$$\geq \epsilon_0 - \frac{\epsilon_0}{4} = \frac{3\epsilon_0}{4}.$$

∴ $x \notin \{y : p_\alpha(y - y_0) < \epsilon_0/4\}$.

and similarly $p_\alpha(y - y_0) < \epsilon_0/4 \Rightarrow y \notin \{x : p_\alpha(x - x_0) < \epsilon_0/4\}$

So X is Hausdorff

If $\mathfrak{X}$ is Hausdorff.

and ~~x~~ $x \neq 0$.

let U be a nbhd of x disjoint from 0.

W.m.a. $U = \bigcap_{j=1}^{k} U_{x\alpha_j\epsilon_j}$.

So $0 \notin U$, $\exists\ j \in \{1, \dots, n\}$ s.t.

$$0 \notin U_{x\alpha_j\epsilon_j}.$$

$$\therefore \quad p_{\alpha_j}(x-0) \geq \epsilon_j.$$

b). Define

$$\rho(x,y) = \sum_{k=1}^{\infty} 2^{-k} p_k(x-y).$$

Then $\rho(x,y)=0 \iff p_k(x-y)=0\ \forall k.$

$\iff x=y$ by part a).

The other properties of a metric are easily verified.

If $\mathcal{X}$ is a T.V.S.

with $\mathcal{T}$ induced by $\{p_\alpha\}_{\alpha \in A}$.

and

$f: \mathcal{X} \to K (= \mathbb{R} \text{ or } \mathbb{C})$.

is linear then

f is continuous iff $\exists\ \alpha_1, \dots, \alpha_n \in A$
and $C > 0$

s.t

$$|f(x)| \leq C \sum_1^n p_{\alpha_j}(x) \qquad \forall x \in X.$$

$\sum_1^n p_{\alpha_j}$ is also a semi-norm

so Hahn Banach

$\Rightarrow \exists$ ~~bdd.~~ continuous linear functionals

if $\mathcal{X}$ is Hausdorff and $x \neq y$ $\exists\ \alpha$

s.t $p_\alpha(x-y) \neq 0$.

and by H.B. $\exists$ a ~~bont.~~ Cont. linear fcnl on X

s.t $f(x-y) = p_\alpha(x-y)$.

so the cont linear functionals separate points.

Let X^* denote the set of continuous linear functionals.

Some other terminology

X a T.V.S.

$\langle x_i \rangle_{i \in I}$ a net in X

is Cauchy iff $\langle x_i - x_j \rangle_{(i,j) \in I \times I}$

converges to zero.

($(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$)

X is complete if every Cauchy net converges.

If X is first countable then completeness is equivalent to the convergence of Cauchy Sequences

Pf:

Let $\{U_j\}_{j=1}^{\infty}$ be a countable nbhd base at 0 and assume, as we may by replacing with finite intersections, that $U_{j+1} \subset U_j \quad \forall j$.

If $\langle x_\alpha \rangle_{\alpha \in I}$ is a Cauchy net.

$\exists \ \langle \alpha_j, \beta_j \rangle$ s.t $\quad x_{\alpha} - x_\beta \in U_j$

$\forall$ ~~$\alpha \geq \alpha_j$~~ su. $(\alpha, \beta) \geq (\alpha_j, \beta_j)$.

With i_j a common majorant of α_j and β_j we have. $\quad x_\alpha - x_\beta \in U_j \quad \forall \ (\alpha, \beta) \geq (i_j, i_j)$.

That is. $\{x_{i_j}\}_{j=1}^{\infty}$ is a Cauchy <u>sequence</u>

which is $\therefore$ convergent to some $x \in X$.

If ~~V is a nbhd of x, $\exists$ j_0 s.th $x_{i_j} \in V$ $\forall$ $j \geq j_0$~~ ~~$x + U_{j_0} \subset V$~~

(continuity of vector space operations)

So $\quad x - x_{i_j} \to 0 \quad$ as $j = 1, 2, \ldots \to +\infty$.

and $\langle x_\alpha - x_\beta \rangle_{(\alpha,\beta) \in I \times I} \to 0$

so

$$x - x_\alpha = (x - x_{i_j}) + (x_{i_j} - x_\alpha) \to 0.$$

by continuity of vector operations.

~~$j \to +\infty$~~ $\alpha \geq i_j$.

mean

If X is Hausdorff and

$\mathcal{T}$ is induced by $\{p_n\}_{n=1}^{\infty}$

(countable family of semi-norms)

Then $\mathcal{T}$ is 1st countable

(by our 1st theorem).

and is given by a translation invariant metric by last proposition.

A sequence is Cauchy in this situation iff it is Cauchy w.r.t. the metric ρ.