2/26/09 Math 138

<u>Thm:</u> (Uniform Boundedness Principle).

X, Y N.V.S's.

$\mathcal{A} \subset \mathcal{L}(X, Y)$.

a) $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ $\forall x$ in a $\underbrace{\text{non-meager}}$ subset of X.

(<u>not</u> a countable union of nowhere dense sets).

$\Rightarrow$ $\sup_{T \in \mathcal{A}} \|T\| < \infty$.

b) If X is a Banach Space.

and $\sup_{T \in \mathcal{A}} \|Tx\| < +\infty$ $\forall x \in X$.

then $\sup_{T \in \mathcal{A}} \|T\| < +\infty$.

<u>Pf:</u> b) follows from a) since a Banach Space X is non-meager. (Baire Category).

Pf:

a) Let $E_n = \{x \in X : \sup_{T \in \mathcal{A}} \|Tx\| \le n\}$.

$$= \bigcap_{T \in \mathcal{A}} \{x \in X : \|Tx\| \le n\}.$$

Then each E_n is closed.

One of the E_n fails to be nowhere dense. So E_n contains a closed ball $\overline{B(r, x_0)}$ for some n some $x_0 \in E_n$ and some $r > 0$.

$\therefore$, if $\|x\| \le r$, $x_0 + x \in E_n$ and

$$\|Tx\| = \|T(x_0 + x - x_0)\|$$

$$\le \|T(x_0 + x)\| + \|Tx_0\| \le 2n.$$

for any $T \in \mathcal{A}$.

$\therefore$ $\|x\| \le 1 \Rightarrow \|rx\| \le r \Rightarrow \|T(rx)\| \le 2n.$

$$\Rightarrow \|Tx\| \le \frac{2n}{r}.$$

$\forall T \in \mathcal{A}$ and $\sup_{T \in \mathcal{A}} \|T\| \le 2n/r$ ∎

#12. $\mathcal{X}$ a N.V.S. $\mathcal{M} \subset X$

a proper closed subspace.

a) $\|x+\mathcal{M}\| = \inf\{\|x+y\| : y \in \mathcal{M}\}$ is a norm

on $X/\mathcal{M}$.

$$\boxed{\text{if } \|x+\mathcal{M}\| = 0 \quad \exists\ y_n \in \mathcal{M} \text{ s.t } \|x-y_n\| \to 0. \Rightarrow x \in \mathcal{M}.}$$

$\lambda \neq 0. \Rightarrow$

$$\|\lambda x + \mathcal{M}\| = \inf\{\|\lambda x + y\| : y \in \mathcal{M}\}$$

$$= \inf\{|\lambda|\ \|x + \tfrac{y}{\lambda}\| : y \in \mathcal{M}$$

$$= |\lambda| \inf\{\|x+u\| : u \in \mathcal{M}\}.$$

$$= |\lambda|\ \|x+\mathcal{M}\|.$$

$$\|x_1 + x_2 + \mathcal{M}\| = \inf\{\|x_1 + x_2 + y\|\ y \in \mathcal{M}\}.$$

$$= \inf\{\|x_1 + y$$

given $\epsilon > 0$

Choose $y_1 \in M$ s.t.

$$\|x_1 - y_1\| < \|x_1 + M_1\| + \epsilon/2.$$

and Choose $y_2 \in M$ s.t.

$$\|x_2 - y_2\| < \|x_2 + M_1\| + \epsilon/2.$$

then $\|x_1 + x_2 - (y_1 + y_2)\| \le \|x_1 - y_1\| + \|x_2 - y_2\|$

$$\le \|x_1 + M_1\| + \|x_2 + M_1\| + \epsilon$$

and $\inf\{\|x_1 + x_2 + y\| : y \in M\} \le \|x_1 + x_2 - (y_1 + y_2)\|.$ $\square$

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b) for any $\epsilon > 0$ $\exists\, x \in X$ s.t. $\|x\| = 1$

and $\|x + M\| \ge 1 - \epsilon$.

Pf: Suppose not. Then $\exists\, \epsilon_0 > 0$ s.t.

$\|x + M\| < 1 - \epsilon_0$ $\forall$ x with $\|x\| = 1$.

So for each such x $\exists$ $y_1 \in M$ s.t.

$\|x + y_1\| < 1 - \epsilon_0$.

and $y_2 \in \mathcal{M}$ s.t.

$$\left\| \frac{x+y_1}{\|x+y_1\|} + y_2 \right\| < 1-\epsilon_0 .$$

$$\therefore \quad \| x + \underbrace{y_1 + y_2\|x+y_1\|}_{\in \mathcal{M}.} \| < (1-\epsilon_0)\|x+y_1\| < (1-\epsilon_0)^2$$

Continuing inductively we find $y_n^* \in \mathcal{M}$.

with $y_n^* \to x$. $\therefore$ $x \in \mathcal{M}$.

since $\mathcal{M}$ is closed.

c) $\pi(x) = x + \mathcal{M}$ ~~from~~ X to $X/\mathcal{M}$

has norm $\underline{1}$.

Since $0 \in \mathcal{M}$

$$\|x+\mathcal{M}\| \le \|x\|.$$

and since for each $\epsilon > 0$ $\exists$.

x with $\|x\| = 1$.

s.t.

$$\|\pi(x)\| = \|x+\mathcal{M}\| \ge (1-\epsilon)\|x\|.$$

we have $\|\pi\| = 1$

d). $\mathcal{X}$ complete $\Rightarrow$ $\mathcal{X}/\mathcal{M}$ is complete.

if $\quad \sum \|x_n + \mathcal{M}\| < +\infty$.

Choose $y_n \in \mathcal{M}$ s.t.

$$\|x_n + M\| < \|x_n + y_n\| < \|x_n + \mathcal{M}\| + \frac{1}{2^n}.$$

then $\sum \|x_n + y_n\| < +\infty$.

so $\sum (x_n + y_n)$ converges

by completeness of $\mathcal{X}$.

to z.

$\cancel{\|y_n\| = \|y_n + x_n - x_n\|}$

$$\sum (x_n + y_n) = z.$$

$$\sum_{n=1}^{N} (x_n + \mathcal{M}) = \sum_{n=1}^{N} (x_n + y_n) + \mathcal{M}$$

$$\longrightarrow z + \mathcal{M}.$$

e). —

#19. a) $\cancel{\exists \{}$ X ∞ dim n. v. s.

$\exists \; \{x_j\}$ s.t $\|x_j\| = 1 \;\; \forall j$

$$\|x_j - x_n\| \geq 1/2.$$

b) X is not locally compact.

#33. There is no $\{a_n\}$ with $a_n > 0$

and s.t $\sum a_n |c_n| < +\infty$

iff ~~Σa_n~~ $\{c_n\}$ is bdd.

$B(\mathbb{N})$ = bdd fns on $\mathbb{N}$.

$L^1(\mu)$ = absolutely summable sequences on $\mathbb{N}$.

(integrable w.r.t. counting measure)

If such an $\{a_n\}$ exists.

$T : B(\mathbb{N}) \to L^1(\mu)$

defining by $(Tf)(n) = a_n f(n)$

gives a bijection

if $\{x_n\} \in L^1(\mu)$ then.

$$\sum a_n \left| \frac{x_n}{a_n} \right| = \sum |x_n| < +\infty.$$

so $\frac{x_n}{a_n}$ is bdd.

and $\{x_n\} = T\{\frac{x_n}{a_n}\}$.

if $Tf_1 = Tf_2$ then $a_n(f_1(n) - f_2(n)) = 0 \ \forall n$

so $f_1 = f_2$.

but $A = \{ f \in L^1(\mu)$ s.t $f(n) = 0$ $\forall$ but finitely many $n \}$.

is dense in $L^1(\mu)$

and $T^{-1}\{ A \} =$

$\{ g \in B(\mathbb{N})$ s.t $a_n g(n) = 0$ $\forall$ but finitely many n

$= \{ g \in B(\mathbb{N}) :$ $g(n) = 0$ $\forall$ but finitely many $g \}$

is not dense in $B(\mathbb{N})$.

[redacted]

#40 (Condensation of singularities). ~~Suppose that~~

~~for each x $\exists$ k~~ s.t.

$\cancel{\sup\{\|T_{jk}x\| : j \in \mathbb{N}\}} < +\infty$

let $E_k = \{x : \sup\{\|T_{jk}x\| : j \in \mathbb{N}\} < +\infty\}$.

~~Then $X = \bigcup E_k$~~

If $\overline{E_k}$ contains a ball, for some k.

then by U.b.P.

$$M = \sup\{\|T_{jk}\| : j \in \mathbb{N}\} < +\infty.$$

then for each $x \in X$.

$$\|T_{jk}x\| \leq M\|x\| \quad \forall j.$$

So if for each ~~x~~ k, $\exists x \in X$ s.t.

$$\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = +\infty.$$

then each E_k is nowhere dense.
and the set $\bigcup E_k$ is meager.