2/24/09 Math 188

<u>Recall</u>: <u>Baire Category Thm:</u>

X a complete Metric Space.

a) if U_n is a sequence of dense open subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.

b) X is not a countable union of nowhere dense sets.

(E is nowhere dense if $(\overline{E})^\circ = \emptyset$.).

<u>Pf:</u> b) if $\{E_n\}$ is a sequence of nowhere dense sets then each $(\overline{E_n})^c$ is dense and open

so by a) $\bigcap_{n=1}^\infty (\overline{E_n})^c$ is densen in X.

$\therefore \left(\bigcup_{n=1}^\infty \overline{E_n}\right)^c$ is non-empty

$\therefore \left(\bigcup_{n=1}^\infty E_n\right)^c$ is non-empty.

$\therefore \bigcup_{n=1}^\infty E_n \neq X$.

Def: $(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$ top spaces.

$f: X \to Y$ is open (an open mapping)

iff $f(U) \in \mathcal{T}_Y$ $\forall U \in \mathcal{T}_X$.

For metric spaces X, Y this means.

~~$\forall x$~~ if B is a ball in X centered at x then $f(B)$ contains a ball in Y centered at $f(y)$.

If X, Y are N.L.S. then a linear map f is open iff $f(B)$ contains a ball centered at $0 \in Y$ when $B = \{x \in X : \|x\| < 1\}$.

Thm: (Open Mapping Theorem)
X, Y Banach Spaces

$T \in \mathcal{L}(X,Y)$ surjective
$\Rightarrow T$ is open.

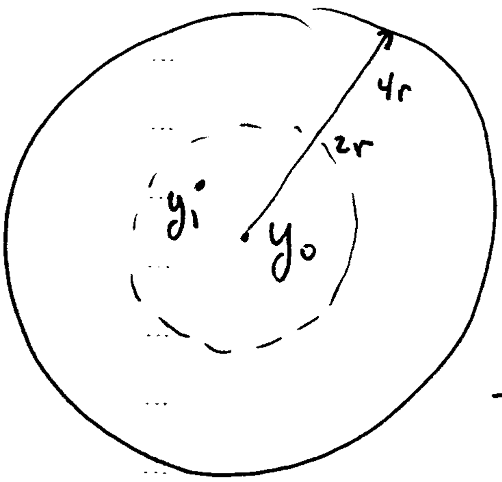
Pf: Let $B_r = \{x \in X : \|x\| < r\}$.
It suffices to show that
$T(B_1)$ contains a ball centered
at $0 \in Y$.

Since T is surjective
$$Y = \bigcup_{n=1}^{\infty} T(B_n).$$

Since Y is complete $T(B_m)$ is not
nowhere dense for some m
but $T(B_m) = m(T(B_1))$.
So $T(B_1)$ is not nowhere dense.

$\therefore$ $\exists\, y_0 \in Y$ and $r > 0$ s.t.

$$B(4r, y_0) \subset \overline{T(B_1)}$$



Choose $y_1 \in T(B_1)$ s.t.

$y_1 = Tx_1$ has $\|y_1 - y_0\| < 2r$.
$(x_1 \in B_1)$.

Then $B(y_1, 2r) \subset B(y_0, 4r)$
$\subset \overline{T(B_1)}$

If $\|y\| < 2r$ then

$$y_1 + y \in B(y_1, 2r) \subset \overline{T(B_1)}$$

so

$$y = -Tx_1 + (y + y_1) \in \overline{T(-x_1 + B_1)} \subset \overline{T(B_2)}$$

and $y/2 \in \overline{T(B_1)}$.

So $\|y\| < r \Rightarrow y \in \overline{T(B_1)}$

and by linearity

$$\|y\| < r \cdot 2^{-n} \Rightarrow y \in \overline{T(B_{2^{-n}})}$$

So we have shown (e.g.) that.

if $\|y\| < \frac{r}{2}$

$\exists \; x_1 \in B_{1/2}$ s.t.

$\|y - Tx_1\| < r/4.$

likewise

$\exists \; x_2 \in B_{1/4}$ s.t.

$\|y - Tx_1 - Tx_2\| < r/8.$

and continuing by induction

$\exists \; x_n \in B_{2^{-n}}$ s.t.

$\|y - \sum_{j=1}^{n} Tx_j\| < \frac{r}{2^{n+1}}.$

Since X is complete and $\|x_n\| < 2^{-n}$

$\sum_{n=1}^{\infty} x_n$ converges (say) to x.

and $\|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} 2^{-n} = 1.$

Now, $y = Tx$.

So $\|y\| < r/2 \Rightarrow y \in T(B_1)$ ▨

Corollary: X, Y Banach Spaces.

$T \in \mathcal{L}(X,Y)$ Bijective.

then T is an isomorphism.

i.e. $T^{-1} \in \mathcal{L}(Y,X)$.

Pf: T is surjective $\therefore$ open.

Since T is injective T^{-1} exists and

Since $T(U)$ is open in Y $\forall U$ open in X

T^{-1} is continuous.

Thm: (Closed Graph Theorem).

If X, Y are Banach Spaces

and $T: X \to Y$ is a linear map

s.t

$\Gamma(T) = \{(x, Tx) : x \in X\}$ is a

closed subspace of $X \times Y$ (product topology) then T is bounded.

<u>Pf:</u> Let $\pi_1(x, Tx) = x$.

$\pi_2(x, Tx) = Tx$.

Since X, Y are complete so is $X \times Y$

and $\therefore$ $\Gamma(T)$ (being closed)

is also complete;

$\pi_1 \in \mathcal{L}(\Gamma(T), X)$

$\pi_2 \in \mathcal{L}(\Gamma(T), Y)$.

(e.g. $\|(x, Tx)\| = \max(\|x\|, \|Tx\|)$

$\|\pi_1(x, Tx)\| = \|x\| \leq \|(x, Tx)\|$

π_1 is a Bijection from $\Gamma(T)$ to X.

so π_1^{-1} is bounded. (by Previous Corollary).

$\therefore$ $T = \pi_2 \circ \pi_1^{-1}$ is bounded.

The closed Graph Thm says that if a linear T has a closed graph

$$(x_n \to x \quad \text{and } Tx_n \to y \quad \Rightarrow y = Tx)$$

then T is continuous

$$(x_n \to x \quad \Rightarrow \quad Tx_n \to Tx).$$

So to verify that $Tx_n \to Tx$
when $x_n \to x$

we may assume that Tx_n converges (to y say) and we just need to show that $y = Tx$.