2/19/09 Math 138

Hahn-Banach Theorem.

X is a real Vector space.

Def: A Minkowski Functional is a map $p : X \to \mathbb{R}$.

s.t

$$p(x+y) \le p(x) + p(y) \quad \forall x, y \in X$$

(sub additive).

and $p(\lambda x) = \lambda p(x) \quad \forall x \in X \quad \forall \lambda \ge 0.$

(positive homogeneous).

Thm: ~~X~~ a real vector space.

p a Mink functional on X.

$\mathcal{M} \subset X$ is a subspace.

f a linear functional on $\mathcal{M}$

s.t $f(x) \le p(x) \quad \forall x \in \mathcal{M}$.

Then

$\exists$ a linear fcn'l F on X s.t

$F(x) \le p(x) \quad \forall x \in X$ and $F|_{\mathcal{M}} = f$

and $F|_{M} = f$.

Pf: We first extend to one extra dimension (while retaining the property of being dominated by p).

if $x \in X \setminus M$ then any linear extension of f from M to $M + \mathbb{R}x$ will be determined by

$$\tilde{f}(y + sx) = f(y) + s\alpha$$

$$s \in \mathbb{R}, \quad y \in M.$$

We want to choose α s.t.

$$f(y) + s\alpha \leq p(y + sx).$$

Since f and p are both positive homogenous, its enough to check this when $s=1$ and $s=-1$.

$$\left(\begin{array}{l} \underline{s>0.} \quad f\left(\frac{y}{s}\right) + \alpha \leq p\left(\frac{y}{s}+x\right). \\ \underline{s<0.} \quad f\left(\frac{y}{s}\right)+\alpha \geq -p\left(-\frac{y}{s}-x\right). \\ \qquad \Leftrightarrow f\left(-\frac{y}{s}\right)-\alpha \leq p\left(-\frac{y}{s}-x\right). \end{array}\right.$$

So we demand that.

$$f(y) - p(y-x) \leq \alpha \leq -f(z) + p(z+x)$$

$$\forall y, z \in M.$$

We have by assumption

$$-f(z) + p(z+x) - f(y) + p(y-x)$$

$$= p(y-x) + p(z+x) - f(y+z) \geq p(y+z) - f(y+z)$$

$$\geq 0.$$

So we may choose

$$\alpha \in \left[\sup\{ f(y) - p(y-x) : y \in M \}, \inf_{z \in M} \{ -f(z) + p(z+x) : \right]$$

$(\neq \emptyset)$.

So a 1-dim extension exists.

Consider pairs $(\mathfrak{Z}, \psi)$ where $\mathfrak{Z}$ is a subspace of X containing M and ψ is a functional on $\mathfrak{Z}$ dominated by p s.t.

$$\psi|_M = f.$$

Order such pairs by

$$(\mathfrak{Z}_1, \varphi_1) \le (\mathfrak{Z}_2, \varphi_2)$$

$$\Longleftrightarrow \mathfrak{Z}_1 \subset \mathfrak{Z}_2 \text{ and } \varphi_2|_{\mathfrak{Z}_1} = \varphi_1.$$

If $\{(\mathfrak{Z}_\beta, \psi_\beta) : \beta \in B\}$

is a totally ordered collection

of such pairs, let

$$\mathfrak{Z} = \bigcup_{\beta \in B} \mathfrak{Z}_\beta$$

and define ψ on $\mathfrak{Z}$ by

$$\psi(z) = \psi_\beta(z) \quad \text{if} \quad z \in \mathfrak{Z}_\beta.$$

Since the collection is totally ordered.

$\mathfrak{Z}$ is a vector subspace of X containing

all the $\mathfrak{Z}_\beta$ and ψ is well defined

on $\mathfrak{Z}$ and extends all the ψ_β's.

So $(\mathfrak{Z}, \psi)$ is a majorant of

the given totally ordered collection

By Zorn's lemma, there is a maximal extension $(\tilde{\mathcal{M}}, \tilde{f})$ of $(\mathcal{M}, f)$ dominated by p which must have $\tilde{\mathcal{M}} = X$ by the 1st part of the proof. $\blacksquare$

<u>Thm</u>: (Complex Version).

X a complex vector space.

p a semi-norm on X.

$\mathcal{M} \subset X$ a subspace.

f a complex linear functional

s.t. $|f(x)| \le p(x) \quad \forall x \in \mathcal{M}$.

Then $\exists$ ~~Fe~~ a linear fcn'l F on X s.t. $|F(x)| \le p(x) \quad \forall x \in X$ and $F|_{\mathcal{M}} = f$.

note that in this situation

if $u: X \to \mathbb{R}$ is real.
$|U(x)| \leq p(x)$
$\Leftrightarrow U(x) \leq p(x)$
since $|U(x)| = \pm U(x) = U(\pm x)$
and $p(-x) = p(x)$.

Pf: Let $u = \operatorname{Re} f$.

By the real version

$\exists$ a real linear extension U of u to X s.t. $|U(x)| \leq p(x)$ $\forall x \in X$.

Take $F(x) = U(x) - iU(ix)$

then F is a complex linear extension of f. ($f(x) = u(x) - iu(ix)$) from earlier).

With $F(x) = e^{i\theta}|F(x)|$

we have.

$$|F(x)| = e^{-i\theta}F(x) = F(e^{-i\theta}x) = U(e^{-i\theta}x)$$
$$\leq p(e^{-i\theta}x) = p(x).$$

$\blacksquare$

Applications of H-B.

Thm: $\mathcal{X}$ a normed vector space.

a) If $x \neq 0$, $x \in \mathcal{X}$

$\exists\, f \in \mathcal{X}^*$ s.t $\|f\| = 1$ and $f(x) = \|x\|$.

Pf: Let $\mathcal{M} = \mathbb{C}x$

and define f on $\mathcal{M}$ by

$f(\lambda x) = \lambda \|x\|$.

With $p(y) = \|y\|$, H-B gives the desired extension.

b) $\mathcal{X}^*$ separates the points of X.

pf: if $x \neq y$, then by a)

$\exists\, f \in \mathcal{X}^*$ s.t $f(x-y) = \|x-y\|$.

i.e $f(x) \neq f(y)$.

c)

c) If $\mathcal{M}$ is a closed subspace of X and $x \in X \setminus \mathcal{M}$, $\exists\ f \in X^*$ s.t $f(x) \neq 0$ and $f = 0$ on $\mathcal{M}$.

If $\delta = \inf \{ \|x-y\| : y \in \mathcal{M} \}$ we can take $\|f\| = 1$ and $f(x) = \delta$.

Pf: Define

f on $\mathcal{M} + \mathbb{C}x$ as.

$$f(y+\lambda x) = \lambda \delta \qquad y \in \mathcal{M},\ \lambda \in \mathbb{C}.$$

then

$$|f(y+\lambda x)| = |\lambda| \delta \leq |\lambda| \, \| x + \tfrac{1}{\lambda} y \|.$$

$$= \| y + \lambda x \|.$$

So $\|\cdot\|$ dominates f on $\mathcal{M} + \mathbb{C}x$ and H-B applies.

d) for $x \in X$

define $\hat{x} : X^* \to \mathbb{C}$

by $\hat{x}(f) = f(x)$.

The map $x \to \hat{x}$ is a linear isometry from X into $X^{**} = (X^*)^*$.

Pf: $\hat{x}$ is linear on X^* ✓.

$x \to \hat{x}$ is a linear map.

$|\hat{x}(f)| = |f(x)| \leq \|f\| \, \|x\|$

$\Rightarrow \|\hat{x}\| \leq \|x\|$.

by a) $\exists \; f \in X^*$ s.t $\|f\| = 1$ and $f(x) = \|x\|$.

so $\hat{x}(f) = \|x\| = \|x\| \, \|f\|$.

$\therefore \|\hat{x}\| \geq \|x\|$

$\blacksquare$