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Tychonoff's Thm.

If $\{X_\alpha\}_{\alpha \in A}$ is any family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact in the product topology.

Pf: The sets $\pi_\alpha^{-1}(U)$, U open in X_α generate the product topology.

By Alexander's Lemma, it suffices to show that any open cover $\mathcal{V}$ ~~by~~ of X by such sets has a finite subcover.

For $\alpha \in A$, let $\mathcal{V}_\alpha = \{U \subset X_\alpha : U \text{ is open}, \pi_\alpha^{-1}(U) \in \mathcal{V}\}$

Claim: $\exists$ $\beta \in A$ s.t $\mathcal{V}_\beta$ covers X_β

if not, then by the axiom of choice $\exists x \in X$ s.t $\pi_\alpha(x) \notin \cup(\mathcal{V}_\alpha)$ for every α.

and such an x has $x \notin \cup \mathcal{V}$.

Since X_β is compact

$\exists\ U_1, \dots, U_n \in \mathcal{V}_\beta$ s.t. $X_\beta = \bigcup_{j=1}^{n} U_j$

Consequently

$$\pi_\beta^{-1}(U_j) \in \mathcal{V}$$

and $\bigcup_1^n \pi_\beta^{-1}(U_j) = \pi_\beta^{-1}(X_\beta) = X.$

Tychonoff's Thm $\Rightarrow$ Axiom of Choice

recall:

<u>Axiom of Choice.</u>

If $\{X_\alpha\}_{\alpha\in A}$ is a non-empty collection of non-empty sets then

$$\prod_{\alpha\in A} X_\alpha \neq \emptyset.$$

Let $\{X_\alpha\}_{\alpha\in A}$ be a non-empty collection of non-empty sets.

Let ω be a a single point set disjoint from every X_α.

Let $X_\alpha^* = X_\alpha \cup \{\omega\}$ and define

$$\mathcal{T}_{X_\alpha^*} = \{\emptyset, X_\alpha, \{\omega\}, X_\alpha\cup\{\omega\}\}.$$

X_α^* is compact in $\mathcal{T}_{X_\alpha^*}$

so by Tychonoff.

$\prod_{\alpha\in A} X_\alpha^*$ is compact.

Let $F_\alpha = \pi_\alpha^{-1}(X_\alpha)$

Then the F_α are ~~compact~~ closed ($\therefore$ compact)

and we claim $\{F_\alpha\}_{\alpha \in A}$ has the finite intersection property.

For this we are allowed to use the axiom of choice for finite collections of sets.

Given a finite set $B \subset A$, we can pick $x_\beta \in X_\beta$ for each $\beta \in B$

so that $\bigcap_{\beta \in B} F_\beta$ contains the point

$x \in X^*$ s.t $\pi_\beta(x) = x_\beta \quad \forall \beta \in B.$

$\pi_\alpha(x) = \omega \quad \forall \alpha \notin B.$

By the F.I.P. characterization of compactness $\bigcap_{\alpha \in A} F_\alpha$ is non-empty.

But $\bigcap_{\alpha \in A} F_\alpha = \prod_{\alpha \in A} X_\alpha$.

Normed Vector Spaces

$\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$

$\mathfrak{X}$ is a vector space over $\mathbb{K}$.

<u>Seminorm:</u> (on $\mathfrak{X}$)

$x \in \mathfrak{X}$, $x \to \|x\| \in [0, \infty)$

$$\|x+y\| \le \|x\| + \|y\| \quad \forall\, x, y \in \mathfrak{X}$$

$$\|\lambda x\| = |\lambda|\,\|x\| \quad \forall\, \lambda \in \mathbb{K},\ x \in \mathfrak{X}$$

if also $\|x\| = 0 \iff x = 0$

then $\|\cdot\|$ is a <u>norm</u>.

$(X, \|\cdot\|)$ where $\|\cdot\|$ is a norm

is called a normed linear space.

$\rho(x,y) = \|x-y\|$ is a metric and defines the <u>norm topology</u> on $\mathfrak{X}$.

Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent norms on X iff.

$\exists\ C_1, C_2 > 0$ s.t.

$$C_1 \|x\|_1 \le \|x\|_2 \le C_2 \|x\|_1 .$$

$$\forall x \in \mathfrak{X}.$$

Equivalent norms define the same Topology and same Cauchy sequences.

Pf: if $V \subset \mathcal{X}$ is $\| \, \|_2$ open

and $x \in V$, $\exists$ $\epsilon > 0$ s.t.

$$\{y : \|y-x\|_2 < \epsilon\} \subset V.$$

$$\therefore \{y : C_1\|y-x\|_1 < \epsilon\} \subset V.$$

and $\{y : \|y-x\|_1 < \epsilon/C\} \subset V.$

so V is $\|\cdot\|_1$ - open.

if $\{x_n\}$ is norm $\|\cdot\|_1$ Cauchy

then given $\epsilon > 0$ $\exists$ N s.t.

$\forall$ $m, n \geq N$

$$\|x_m - x_n\|_1 < \epsilon/C_2.$$

So $\forall$ $m, n \geq N$,

$$\|x_m - x_n\|_2 < C_2\|x_m - x_n\|_1 < \epsilon.$$

__Def:__ A Banach Space is a normed vector space which is complete in the norm topology.

~~Recall~~: If $\{x_n\}$ is a sequence in $\mathfrak{X}$

$\sum_1^\infty x_n$ __converges to__ x iff

$\sum_1^N x_n \to x$ as $N \to \infty$.

and $\sum_1^\infty x_n$ is absolutely convergent

iff $\sum_1^\infty \|x_n\| < +\infty$.

__Thm:__ A normed vector space is complete iff every absolutely convergent series in $\mathfrak{X}$ converges.

Pf: Suppose every absolutely convergent series in $\mathcal{H}$ converges and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy-Sequence.

For each k, choose n_k s.t. $\forall\, m, n \geq n_k$.

$$\|x_m - x_n\| < 2^{-k}$$

then
$$\sum_{k=2}^{N} (x_{n_k} - x_{n_{k-1}}) = x_{n_N} - x_{n_1}$$

and $\|x_{n_k} - x_{n_{k-1}}\| < 2^{-(k-1)}$.

So
$$\sum_{k=2}^{\infty} \|x_{n_k} - x_{k-1}\| < +\infty.$$

and $\therefore\ x = \lim_{N \to \infty} x_{n_N} = x_{n_1} + \sum_{k=2}^{\infty} (x_{n_k} - x_{n_{k-1}})$

exists.

Since $\{x_n\}$ is Cauchy, $\lim_{n \to \infty} x_n = x$.

Conversely, if $\mathcal{X}$ is complete and $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ then the partial sums

$$S_N = \sum_{n=1}^{N} x_n$$

have $\|S_N - S_M\| = \|\sum_{n=M+1}^{N} x_n\|$
$(N > M)$

$$\leq \sum_{n=M+1}^{N} \|x_n\|$$

$$< \epsilon$$

if N, M are sufficiently large. ▨

e.g. $B(X)$, $BC(X)$, $L^1(\mu)$
$(X, \mathcal{M}, \mu)$.

are Banach Spaces.

X, Y Normed Vector Spaces.

$T : X \to Y$ linear

T is called <u>bounded</u> iff $\exists\, C > 0$

s.t.

$$\|Tx\| \leq C\|x\| \qquad \forall\, x \in X.$$

<u>Prop:</u> X, Y N.V.S's.

$T : X \to Y$ linear $(\cancel{T \in L(X,Y)})$

T.F.A.E.

i) T is continuous

ii) T is continuous at 0

iii) T is bounded.

<u>Pf</u> $i \Rightarrow ii)$ ✓

$ii) \Rightarrow iii)$

If T is continuous at 0

$\exists\, \delta > 0$ s.t $\|x\| < \delta \Rightarrow \|Tx\| < 1.$

then $\forall\ x \in X$

$$\left\| T\left(\frac{\delta}{\|x\|}\ x \right) \right\| < 1.$$

so $\|Tx\| < \frac{1}{\delta}\|x\|$

and T is bounded.

iii) $\Rightarrow$ i).

If T is bounded, $\exists\ C > 0$ s.t for any $x_1, x_2 \in X$.

$$\|T(x_1 - x_2)\| \leq C\|x_1 - x_2\|$$

so $\|Tx_1 - Tx_2\| \leq C\|x_1 - x_2\|$.

Given $\epsilon > 0$, if $\|x_1 - x_2\| < \frac{\epsilon}{C}$ then

$\|Tx_1 - Tx_2\| < \epsilon$.

so T is cont. at the arb. point x_1.

$\blacksquare$

X, Y normed vector spaces

$\mathcal{L}(X,Y) = \{$ bounded linear maps from X to $Y \}$.

$L(X,Y)$ is a vector space and for $T \in L(X,Y)$

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$$

$$= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\}.$$

gives a norm on $L(X,Y)$.

The Operator norm.

<u>Prop:</u> Y is complete $\Rightarrow$ $L(X,Y)$ is complete.

<u>Pf:</u> If $\{T_n\}$ is Cauchy in $L(X,Y)$ and $x \in X$ then

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\|.$$

So $\{T_n x\}$ is a Cauchy sequence in Y.

Denote its limit by Tx and define $T: X \to Y$ by.

$$Tx = \lim_{n\to\infty}(T_n x).$$

T is clearly linear and.

$$\| Tx \| \leq \| T_n x + (T - T_n) x \|$$

$$\leq \| T_n \| \| x \| + \| (T - T_n) x \|$$

$$\leq (\| T_n \| + \epsilon) \| x \| \quad \text{for suff large } n.$$

So $T \in L(X, Y)$.

To see that $\| T - T_n \| \to 0$.

Let $\epsilon > 0$ be given and

Choose N s.t $\| T_n - T_m \| < \epsilon/4 \ \forall m, n \geq N$.

Given $n > N$,

Choose x with $\|x\| = 1$ s.t.

$$\| T - T_n \| \leq 2 \, \| (T - T_n) x \|.$$

then. $\forall \; m > n$.

$$\|T - T_n\| \le 2\left(\|(T - T_m)x\| + \|(T_m - T_n)x\| \right).$$

So ~~if~~ taking m is suff. large we have.

$$\|T - T_n\| \le 2\left(\epsilon/4 + \epsilon/4 \right) = \epsilon.$$

<u>Linear Functionals</u>: X a N.V.S. over $\mathbb{R}$ or $\mathbb{C}$, $l : X \to \mathbb{R}$, $l : X \to \mathbb{C}$

l linear, bdd.

The set of such l is X^*, the <u>dual space of X</u>.

By previous proposition X^* is a Banach Space with the operator norm.

A vector space X over $\mathbb{C}$ is also a vector space over $\mathbb{R}$.

Real and complex linear functionals are related as follows.

<u>Prop:</u> $\mathfrak{X}$ a (normed) vector space over $\mathbb{C}$

~~Iff~~ f is complex linear ~~iff~~ then $u = \mathrm{Re}\, f$ is real linear

and $f(x) = u(x) - i\,u(ix) \quad \forall x \in X$.

If u is real linear and

$f(x) = u(x) - i\,u(ix)$ then

f is complex linear.

and (if $\mathfrak{X}$ has a norm).

$\|u\| = \|f\|$.

Pf: Suppose f is complex linear and let $u = \mathrm{Re}\, f$.

Then u is real linear

and $\mathrm{Im}\,(f(x)) = \mathrm{Re}(-i\, f(x)) = u(-i f(x)) =$ [illegible]

$$= -u(ix).$$

So $f(x) = u(x) - i\, u(ix)$.

If u is real linear

and $f(x) = u(x) - i u(ix)$

then f is linear over $\mathbb{R}$ and

$$f(ix) = u(ix) - i u(-x)$$

$$= i u(x) + u(ix) = i(u(x) - i u(ix))$$

$$= i f(x).$$

So f is also linear over $\mathbb{C}$.

If X is normed then

$$|u(x)| = |\mathrm{Re}\, f(x)| \le |f(x)|.$$

so $\|u\| \le \|f\|$.

If $f(x) \neq 0$ then

$$f(x) = e^{i\theta} |f(x)| \quad \text{for some } \theta.$$

and
$$|f(x)| = e^{-i\theta} f(x) = f(e^{-i\theta} x)$$
$$= u(e^{-i\theta} x).$$

(since $|f(x)| \in \mathbb{R}$).

so $|f(x)| \le \|u\| \, \|e^{-i\theta} x\| = \|u\| \, \|x\|$

$\therefore \quad \|f\| \le \|u\|$.

$\blacksquare$