Math 138 2/12/09

Compactness

Let $(X, \mathcal{T})$ be a topological space.

T.F.A.E.

i) Every open covering of X has a finite subcovering

ii) If $\Delta = \{F_\alpha\}_{\alpha \in A}$ is a family of closed subsets of X such that no intersection of finitely many elements from Δ is empty, then $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.

iii) Every net in X has a cluster point

iv) Every net in X has a convergent subnet.

Pf: i) $\Rightarrow$ ii)

If $\bigcap_{\alpha \in A} F_\alpha = \emptyset$ then $\{F_\alpha^c\}$ is an open cover of X. By i) $\exists\ \alpha_1, \dots, \alpha_n$ $n \in \mathbb{Z}^+$ s.t $X = \bigcup_{j=1}^{n} F_{\alpha_j}^c$.

But then $\bigcap_{j=1}^{n} F_{\alpha_j} = \emptyset$.

So $\{F_\alpha\}_{\alpha \in A}$ fails to have the finite intersection property.

ii) $\Rightarrow$ iii)

If $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net in X,

let $F_\lambda = \overline{\{x_\mu \mid \lambda \leq \mu\}}$

Given $\lambda_1, \ldots, \lambda_n \in \Lambda$ there is a common majorant λ of all the λ_j, $j = 1, \ldots, n$ so that

$$F_\lambda \subset F_{\lambda_j} \quad \forall j = 1, \ldots, n.$$

and $\therefore \bigcap_{j=1}^{n} F_{\lambda_j} \neq \emptyset$.

By ii) $\exists\, x \in \bigcap_{\lambda \in \Lambda} F_\lambda$.

Consequently, If A is any nbhd of x then for any $\lambda \in \Lambda$ $\exists$ $x_\mu \in A$ with $\mu \geq \lambda$.

So $\{x_\lambda\}_{\lambda \in \Lambda}$ is frequently in A.

As A was an arbitrary nbhd of x, x is a cluster point.

iii) $\Rightarrow$ (iv)

We showed earlier that x is a cluster point for $\{x_\lambda\}_{\lambda\in\Lambda}$ iff $\{x_\lambda\}_{\lambda\in\Lambda}$ has a subnet converging to x.

This also shows that (iv) $\Rightarrow$ (iii).

iii) $\Rightarrow$ i).

($\sim$ i) $\Rightarrow$ $\sim$ iii).

Let $\{U_\beta\}_{\beta\in B}$ be an open cover of X with no finite subcover. Let $\mathcal{A}$ be the collection of finite subsets of B partially ordered by inclusion, and for each $A\in\mathcal{A}$ let $x_A \in \left(\bigcup_{\beta\in A} U_\beta\right)^c$.

<u>Claim</u>: Then $\langle x_A\rangle_{A\in\mathcal{A}}$ is a net with no cluster point.

<u>Pf</u>: if $x\in X$ choose $\beta\in B$ s.t. $x\in U_\beta$.

If $\beta\in A$ (i.e. $A \gtrsim \{\beta\}$), then $x_A\notin U_\beta$

$\therefore$ x is not a cluster point of $\langle x_A\rangle$ ∎

Zorn's Lemma

An order on a set X is a binary relation ($\leq$) s.t.

$$x \leq y \text{ and } y \leq z \Rightarrow x \leq z$$

$$x \leq x \quad \forall x$$

$$x \leq y \text{ and } y \leq x \Rightarrow x = y.$$

$(X, \leq)$ is called totally ordered if for every pair (x, y) either $x \leq y$ or $y \leq x$.

Zorn's lemma states that

> if every totally ordered subset of X has a majorant in X then X has a maximal element.

Zorn's lemma is equivalent to the axiom of choice.

<u>Alexander's Lemma</u>

$(X, \mathcal{T})$, $\mathcal{T}$ generated by $\mathcal{E}$.

If every cover of X by members of $\mathcal{E}$ has a finite subcover, then X is compact.

<u>Pf:</u> If X is not compact, let $\mathcal{A}$ be the collection of all open covers of X with no finite subcover.

$\mathcal{A}$ is partially ordered by inclusion

If $\{\mathcal{A}_\beta\}_{\beta \in B}$ is a totally ordered subcollection of $\mathcal{A}$ and $U_1, \dots, U_n \in \bigcup_{\beta \in B} \mathcal{A}_\beta$

then $U_1, \dots, U_n \in A_{\beta_0}$ for some $\beta_0 \in B$

so $\bigcup_{j=1}^{n} U_n \neq X$

$\therefore \bigcup_{\beta \in B} A_\beta \in \mathcal{A}$.

We have shown that every linearly ordered subset of $\mathcal{a}$ has a majorant in $\mathcal{a}$ so Zorn's lemma
$\Rightarrow \exists$ a maximal element $\mathcal{A} \in \mathcal{a}$.
i.e. $\mathcal{A}$ is an open cover with no finite subcover and if
U is open, $U \notin \mathcal{A}$ then
$\mathcal{A} \cup \{U\}$ has a finite subcover.

Let $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$.

<u>Claim:</u> $\mathcal{B}$ covers X.

Since no finite subfamily of $\mathcal{B}$ can cover X, the claim shows the existence of a cover by els. of $\mathcal{E}$ which has not finite subcover, thus proving the lemma.

Pf of Claim:

Suppose there is some $x \in X \setminus (\bigcup \mathcal{B})$

$$\left(\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B \right).$$

Choose $U \in \mathcal{A}$ with $x \in U$.

Since $\mathcal{T}$ is generated by $\mathcal{E}$, $\exists$ $V_1, \ldots, V_n \in \mathcal{E}$ s.t.

$$x \in \bigcap_{j=1}^{n} V_j \subset U.$$

Since $x \notin \bigcup \mathcal{B}$, no V_j can be in $\mathcal{A}$.

By maximality of $\mathcal{A}$, for each j there is a set W_j which is a finite union of els of $\mathcal{A}$ s.t.

$$V_j \cup W_j = X.$$

$$\therefore \quad V_j \cup \left(\bigcup_{k=1}^{n} W_k\right) = X \qquad \forall j.$$

$$\therefore \quad \bigcap_{j=1}^{n} V_j \cup \left(\bigcup_{k=1}^{n} W_k\right) = X.$$ But $\bigcup_{k=1}^{n} W_k \cup \left(\bigcap_{j=1}^{n} V_j\right) \subseteq \bigcup_{k=1}^{n} W_k \cup U$

which gives a finite subcover from $\mathcal{A}$.

This contradiction proves the claim.