Math 138 2/10/09

Def: A __net__ in a space X is a pair

(Λ, i)

where Λ is an partially ordered set in which any pair of els. has a common majorant.

and $i: \Lambda \to X$ is a map.

We will write

$$\{x_\lambda\}_{\lambda \in \Lambda} \quad (\text{where } x_\lambda = i(\lambda).)$$

to denote a net.

$(X, \mathcal{T})$ a t.p. space. $E \subset X$

__Vocabulary:__

$\langle x_\alpha \rangle_{\alpha \in A}$ is __eventually__ in E

if $\exists\ \alpha_0 \in A$ s.t $x_\alpha \in E$ $\forall\ \alpha \geq \alpha_0$.

$\langle x_\alpha \rangle_{\alpha \in A}$ is __frequently__ in E

if for each $\alpha \in A$, $\exists\ \beta \geq \alpha$ s.t $x_\beta \in E$

A net $(x_\lambda)_{\lambda \in \Omega}$ <u>converges to $x \in X$</u>

iff x_λ is eventually in any $A \in \mathcal{N}(x)$.

A net $(x_\lambda)_{\lambda \in \Omega}$ has a cluster point at $x \in X$

iff x_λ is frequently in any $A \in \mathcal{N}(x)$.

<u>Prop</u>: $(X, \mathcal{T})$ a top space. , $E \subset X$
$x \in X$

x is an accumulation point of E
iff there is a net in $E \setminus \{x\}$ which
converges to x.
and $x \in \bar{E}$ iff there is a net in E
which converges to x.

Pf: If x is an accumulation pt of E let $N(x)$ be the set of nbhds of x ordered by reverse inclusion.

For each $U \in N(x)$ pick $x_U \in (U \setminus \{x\}) \cap E$

then $\{x_U\}_{U \in N(x)}$ has $x_U \to x$.

Conversely, if $x_\alpha \in E \setminus \{x\}$ and $x_\alpha \to x$ then every deleted nbhd of x contains some x_α

so x is an accumulation pt. of E.

Recall that we showed $\overline{E} = E \cup C(E)$ where $C(E) =$ accumulation pts of E.

If $x_\alpha \to x$ where $x_\alpha \in E$ then $x \in \overline{E}$

If $x \in \overline{E}$ then either $x \in E$ and put $x_\alpha = x$ $\forall \alpha$ or $x \in C(E)$ and we use the above remark

Prop: If X, Y are top spaces and $f : X \to Y$

then f is cont at $x \in X$ iff for every net $\langle x_\alpha \rangle$ converging to x, $\langle f(x_\alpha) \rangle \to f(x)$.

Pf: If f is cont. at $x \in X$ and $\langle x_\alpha \rangle \to x$, let V be a nbhd of $f(x)$. Then $f^{-1}(V)$ is a nbhd of x and $\exists\, \alpha_0$ s.t. $x_\alpha \in f^{-1}(V)$ $\forall\, \alpha \geq \alpha_0$.

$\therefore$ $f(x_\alpha) \in V$ $\forall\, \alpha \geq \alpha_0$.

so $\langle f(x_\alpha) \rangle \to \langle f(x) \rangle$.

If f is <u>not</u> continuous at x, $\exists$ a nbhd V of $f(x)$ s.t. $f^{-1}(V)$ is <u>not</u> a nbhd of x.

i.e. $x \in \left(f^{-1}(V)\right)^{\circ}$

or $x \in \overline{f^{-1}(V^c)}$,

By the last prop. there is a net $\langle x_\alpha \rangle$ in $f^{-1}(V^c)$ converging to x

But then $f(x_\alpha) \in V^c$

so $f(x_\alpha) \not\to f(x)$.

<u>Def</u>: A subnet of a net $\langle x_\alpha \rangle_{\alpha \in A}$ is a net $\langle y_\beta \rangle_{\beta \in B}$ together with a map $\beta \to \alpha_\beta$ from B to A

$h: B \to A \qquad h(\beta) = \alpha(\beta)$

s.t. i) $y_\beta = x_{\alpha_\beta}$

and

ii) for every $\alpha_0 \in A$ $\exists$ $\beta_0 \in B$ s.t. if $\beta \geq \beta_0$ then $\alpha_\beta \geq \alpha_0$.

<u>Prop.</u> If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in $(X, \mathcal{T})$ then $x \in X$ is a cluster point of $\langle x_\alpha \rangle$ iff $\langle x_\alpha \rangle$ has a subnet which converges to x.

<u>Pf:</u> Suppose $\langle y_\beta \rangle = \langle x_{\alpha_\beta} \rangle$ is a subnet of $\langle x_\alpha \rangle$ converging to x and let U be a nbhd of x.

$\exists \; \beta_1 \;$ s.t. $\; \forall \beta \geq \beta_1 \;\; y_\beta \in U$.

Given $\alpha \in A \quad \exists \;\; \beta_2 \in B \quad$ s.t. $\forall \beta \geq \beta_2$

$\alpha_\beta \geq \alpha$.

Let β be common majorant of β_1, β_2.

then $x_{\alpha_\beta} \in U$ and $\alpha_\beta \geq \alpha$.

so $\langle x_\alpha \rangle$ is frequently in U $\therefore x$ is a cluster point.

Conversely, Suppose $x \in X$ is a cluster point of $\langle x_\alpha \rangle$.

Let $\mathcal{N}(x)$ be the collection of nbhds of x and order $\mathcal{N}(x) \times A$ by.

$$(U_1, \alpha_1) \leq (U_2, \alpha_2)$$

iff $U_2 \subset U_1$ and $\alpha_1 \leq \alpha_2$

For each $(U, \gamma) \in \mathcal{N}(x) \times A$ we can choose $\alpha_{(U,\gamma)} \in A$

s.t $\alpha_{(U,\gamma)} \geq \gamma$ and $x_{\alpha(U,\gamma)} \in U$.

So if $(U', \gamma') \geq (U, \gamma)$ $\left(\begin{matrix} U' \subset U \\ \gamma' \geq \gamma \end{matrix}\right)$

then $\alpha_{U', \gamma'} \geq \gamma' \geq \gamma$
and $x_{\alpha(U',\gamma')} \subset U' \subset U$.

so

$\langle x_{\alpha(u,\gamma)} \rangle$ is a subnet of $\langle x_\alpha \rangle$ which converges to x.

Prop. $(X, \mathcal{T})$ a top. space is Hausdorff iff each net converges to at most one point.

Pf: If $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to x in the Hausdorff space (X, τ) and $y \neq x$ choose disjoint nbhds A (of x) and B (of y).

Then $\{x_\lambda\}$ is eventually in A $\therefore$ not eventually in B. so (x_λ) does not converge to y.

If each net has at most one convergence point and $x \neq y$, Consider the family $N(x)$ of nbhds of x and $N(y)$ of nbhds of y.

Let $N(x) \times N(y)$ have the partial ordering

$(A_1, B_1) \leq (A_2, B_2)$ iff $A_2 \subset A_1$ and $B_2 \subset B_1$.

If for any $(A, B) \in N(x) \times N(y)$

we could find $x_{A,B}$ in $A \cap B$.

then $(x_{A,B})_{(A,B) \in N(x) \times N(y)}$ would

be a net converging to both x and y.

Consequently, $\exists$ $(A, B) \in N(x) \times N(y)$

s.t $A \cap B = \emptyset$.