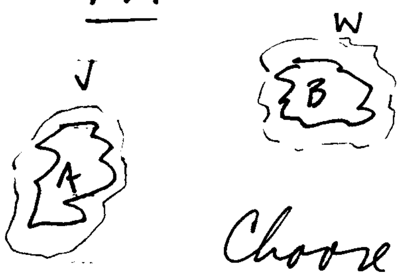
Math 138 2/5/09

(suggested study problems appended)

- Lemma: Let X be a normal space. If A and B are disjoint closed sets in X, for each dyadic rational $k \cdot 2^{-n} \in (0,1]$ there is an open set U_r such that $A \subset U_r \subset B^c$ and $\overline{U_r} \subset U_s$ for $r < s$.

Pf:



Take:

$$U_1 = B^c$$

Choose V, W open s.t. $V \cap W = \emptyset$

$A \subset V$, $B \subset W$ (normality).

Let $U_{1/2} = V$.

Then W^c is closed so

$$A \subset U_{1/2} \subset \overline{U_{1/2}} \subset W^c \subset B^c = U_1.$$

To get $U_{1/4}$, repeat the argument with $(A, U_{1/2}^c)$ replacing (A,B)

To get $U_{3/4}$, repeat replacing (A,B) by $(\overline{U_{1/2}}, B^c)$.

• Proceeding inductively, suppose we have chosen U_r $r = k \cdot 2^{-n}, 0 < k < 2^n$ $1 \le n \le N-1$. Then we find U_r for $r = (2j+1) 2^{-N}$

$$0 \le j < 2^{N-1}$$

by using $\overline{U_{j \cdot 2 \cdot 2^{-N}}}$, $(U_{(j+1) \cdot 2 \cdot 2^{-N}})^c$ in the above argument to replace (A, B)

i.e. we can choose an open set

$$U_{(2j+1)2^{-N}} = U_r \quad \text{s.t.}$$

$$\overline{U_{j \cdot 2 \cdot 2^{-N}}} \subset U_r \subset \overline{U_r} \subset U_{2(j+1) \cdot 2^{-N}}$$

The set $\{U_r\}$ have the desired properties

Lemma: (Urysohn) Let X be a normal space. If A and B are disjoint closed sets in X, there exists $f \in C(X, [0,1])$ s.t.

$$f = 0 \text{ on } A \quad \text{and} \quad f = 1 \text{ on } B.$$

Pf: Take U_r as in previous lemma.

and $U_r = X$ for $r > 1$.

Define $f(x) = \inf \{ r : x \in U_r \}$.

$$A \subset U_r \subset B^c \qquad 0 < r < 1.$$

$$\Rightarrow \quad f(x) = 0 \quad \forall x \in A.$$

$$f(x) = 1 \quad \forall x \in B.$$

$$0 \le f(x) \le 1 \quad \forall x \in X.$$

$$f(x) \in \cancel{[0}(-\infty, \alpha) \quad \text{iff} \quad \exists\, r < \alpha \text{ s.t. } x \in U_r.$$

$$\text{iff} \quad x \in \bigcup_{r<\alpha} U_r$$

so $f^{-1}(\cancel{[0}(-\infty, \alpha)) = \bigcup_{r<\alpha} U_r$ is open.

$f(x) \in (\alpha, 1]$

iff $\inf \{ r : x \in U_r \} > \alpha$.

iff $\exists \quad r > \alpha \quad s.t \quad x \notin U_r$

iff $\exists \quad s > \alpha \quad s.t \quad x \notin \overline{U_s}$ (recall $\overline{U_s} \subset U_r \quad s<r$)

iff $x \in \bigcup_{s>\alpha} (\overline{U_s})^c$

so $f^{-1}((\alpha, \infty))$ is open.

Since the sets $(-\infty, \alpha)$, $(\alpha, +\infty)$ generate the topology on $\mathbb{R}$, f is continuous

<u>Thm</u>: X a normal space

$A \subset X$, A closed

$f \in C(A, [a,b])$

$\Rightarrow \exists \ F \in C(X, [a,b])$ s.t

$F|_A = f$.

Pf: w.m.a. $f \in C[A, [0,1]]$

since $\frac{f-a}{b-a} \in C(A, [0,1])$.

We construct g_n continuous on X s.t.

$$0 \le g_n \le 2^{n-1}/3^n \quad \text{on } X$$

and $0 \le f - \sum_1^n g_j \le (2/3)^n$ on A.

as follows:

Let $B = f^{-1}[0, 1/3]$

$C = f^{-1}[2/3, 1]$

B and C are closed subsets of A
(i.e. $B = K \cap A$ for some closed subset K of X)

$\therefore$ B and C are closed in X (since A is closed in X).

By Urysohn, $\exists$ $g_1 : X \to [0, 1/3]$

s.t $g_1 = 0$ on B $\quad g_1 = 1/3$ on C

so $0 \le f - g_1 \le 2/3$ on A.

We can now repeat the reasoning with $\frac{3}{2}(f-g_1)$ in place of f.

(produce h_1 instead of g_1).

$$0 \le \frac{3}{2}(f-g_1) - h_1 \le 2/3. \quad \text{on } A$$

$$0 \le f-g_1-\frac{2}{3}h_1 \le (2/3)^2. \quad \text{on } A.$$

let $g_2 = \frac{2}{3}h_1$.

so $0 \le f-g_1-g_2 \le \left(\frac{2}{3}\right)^2$. on A.

and $0 \le g_2 \le \frac{2}{3}\cdot\frac{1}{3}$ on X

$0 \le g_1 \le 1/3$.

Having found $g_1, \dots, g_{n-1}$

s.t $0 \le g_k \le 2^{k-1}/3^k \quad k \le n-1$

and $0 \le f - \sum_{k=1}^{n-1} g_k \le \left(\frac{2}{3}\right)^{n-1}$ on A

we repeat step 1 for

$$\left(\frac{3}{2}\right)^{n-1}\left(f - \sum_{k=1}^{n-1} g_k\right)$$

to produce h_n s.t. $0 \le h_n \le 1/3$

$$0 \le \left(\frac{3}{2}\right)^{n-1}\left(f - \sum_{u=1}^{n-1} g_u\right) - h_n \le \frac{2}{3}.$$

so that

$$0 \le f - \sum_{u=1}^{n-1} g_u - \left(\frac{3}{2}\right)^{n-1} h_n \le \left(\frac{2}{3}\right)^n$$

and let $g_n = \left(\frac{2}{3}\right)^{n-1} h_n$

so that $0 \le g_n \le \left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{3}$.

Let $F = \sum_{n=1}^{\infty} g_n$.

Since $\|g_n\|_\infty \le 2^{n-1}/3^n$

the series is uniformly convergent

and $\therefore$ F is continuous.

We also have on A.

$$0 \le f - F \le (2/3)^n \quad \text{for all } n.$$

So $f = F$ on A.

__Corollary:__

If X is normal, $A \subset X$ is closed and $f \in C(A)$

$\exists \; F \in C(X)$ s.t $F|A = f$.

__Pf:__ $f = u + iv$ where u, v are real valued

so we assume that f is real vld.

Put $g = \dfrac{f}{1+|f|}$

then $|g| < 1$.

so $g \in C(A, (-1,1))$

and by the Tietze ext thm

$\exists \; G \in C(X, [-1,1])$ with

$G|A = g$.

on A we have.

$$G = \frac{f}{1+|f|}$$

so on $[f \geq 0] \cap A$

we have.

$$G = \frac{f}{1+f}$$

$$G + Gf = f \quad , \quad f = \frac{G}{1-G}$$

and on $[f<0] \cap A$.

$$G = \frac{f}{1-f} \qquad f = \frac{G}{1+G}.$$

In any case.

$$f = \frac{G}{1-|G|}. \quad \text{on } A.$$

The problem is that

G may take the values 1 or -1 on X.

Let $B = G^{-1}(\{-1, 1\})$.

$\boxed{B \text{ is closed.}}$

By Urysohn's lemma. $\exists$

$h \in C(X, [0,1])$ s.t.

$h = 1$ on A $h = 0$ on B.

Then $hG = G$ on A.

$|hG| < 1$ on X and.

$$\therefore \quad F = \frac{hG}{1 - |hG|}$$

gives an example as desired.

e.g. $\mathbb{C}^{\mathbb{R}} = \{ f : f : \mathbb{R} \to \mathbb{C} \}$.

with the product topology.

$C(\mathbb{R}) \subset \mathbb{C}^{\mathbb{R}}$ is a subspace.

Since all continuous fcns are Borel measurable and pointwise limits of Borel msble fcns are Borel msble,

The sequential limits of $C(\mathbb{R})$ is a subset of the Borel msble fcns. and $\therefore$ a proper subset of $\mathbb{C}^{\mathbb{R}}$.

But $C(\mathbb{R})$ is dense in $\mathbb{C}^{\mathbb{R}}$

Given $f \in \mathbb{C}^{\mathbb{R}}$, the sets

$$\{ g \in \mathbb{C}^{\mathbb{R}} : |g(x_j) - f(x_j)| < \epsilon,\ j = 1, \dots, n \}$$
$$x_1, \dots, x_n \in \mathbb{R},\ n \in \mathbb{N},\ \epsilon > 0 \}$$

form a nbhd base at f

and each such set contains continuous functions.

If we want to describe topological notions (closure, interior, denseness etc) compactness)

~~int~~ in terms of convergence

then we will need a notion more general than sequential convergence.

Some suggested study problems from Folland!

4.1 2, 5, 6, 8, 13. , 9

4.2 15, 20., 22, 24, 25.

4.3. 31, 33, 37, ~~41~~

4.4. 39, 41, 43

4.5. 49, 51, 53, 55

4.6. 61, 63, 64.

4.7 66, 67, 70

4.8 72, 74