Math 138 2/3/09

notations.

X, a set

$B(X,\mathbb{R})$ = bdd. real fcns on X.
$B(X,\mathbb{C})$ = " complex " " X.

$(X,\mathcal{T})$, a top. space

$C(X,\mathbb{R})$ = cont. real fcns on X
$C(X,\mathbb{C})$ = ...

$BC(X,\mathbb{R})$ = bdd, cont, real ...
$BC(X,\mathbb{C})$ = " " , complex ...

$f \in B(X) = B(X,\mathbb{C})$

$\|f\|_u = \sup \{ |f(x)| : x \in X \}$.

makes $B(X)$ a metric space.

If $\{f_n\}$ is a Cauchy sequence in $B(X)$
(given $\epsilon > 0$ $\exists N$ s.t. $\|f_n - f_m\|_u < \epsilon$ $\forall n, m \geq N$).

$\{f_n(x)\} \subset \mathbb{C}$ is Cauchy for each $x \in X$

so $f(x) \equiv \lim_{n\to\infty} f_n(x)$ exists.

and $f \in B(X)$.

For each $x \in X$ ~~[illegible]~~ $|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$

given $\epsilon > 0$ $\exists N$ s.t.

$$\|f_n - f_m\|_u < \epsilon/2.$$

and given x, $\exists M_x$ s.t $\quad |f_m(x) - f(x)| < \epsilon/2$

$\forall m \geq M$.

so $\forall n \geq N$ and any $x \in X$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{2M_x}(x)| + |f_{2M_x}(x) - f(x)|$$

$$\leq \epsilon.$$

and $f_n \xrightarrow{u} f$.

<u>Prop</u> $(X, \mathcal{T})$ a top. space.

$BC(X)$ is closed in $B(X)$ (in $\|\cdot\|_u$)

($\therefore$ $BC(X)$ is complete).

<u>Pf:</u> if $\|f_n - f\|_u \to 0$ with $\{f_n\} \subset BC(X)$

note that for any $x, y \in X$ and any n

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)|$$
$$+ |f_n(y) - f(y)|.$$

Given $\epsilon > 0$ $\exists N$ s.t $|f(x) - f_n(x)| < \epsilon/3$ $\forall n \geq N$

and $|f(y) - f_n(y)| < \epsilon/3$ $\forall n \geq N$

Since f_N is continuous. $\exists$ a nbhd V of x

s.t $\forall\, y \in V$. $|f_N(x) - f_N(y)| < \epsilon/3$.

$\left(V = f_N^{-1}\left(\{ z \in \mathbb{C} : |z - f_N(x)| < \epsilon/3 \} \right) \right)$.

$\therefore\ |f(x) - f(y)| < \epsilon \quad \forall\, y \in V$.

So f is continuous at x.

As x is arbitrary in X, f is cont. /.

5.1 exercise 10 (Connectedness).

$(X, \mathcal{T})$ is disconnected iff $\exists$ non-empty $U \in \mathcal{T}$, $V \in \mathcal{T}$ s.t $U \wedge V = \phi$ and $X = U \cup V$.

$(X, \mathcal{T})$ is connected if it is not disconnected

a) X is connected iff ϕ and X are the only subsets of X which are both open and closed.

pf: If X is disconnected then $\exists$ nonempty ~~sets~~ open sets U and V s.t

$U \cap V = \phi$ and $U \cup V = X$.

So $U^c = V$ and $V^c = U$

and U and V are both open and closed.

If $\exists$ $V \neq \phi$ and $V \neq X$ s.t.

V is both open and closed.

then V^c is both open and closed and non-empty.

and $V \cup V^c = X$ while $V \cap V^c = \phi$

so X is disconnected.

b). If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected subsets of X such that $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in A} E_\alpha$ is connected (in the relative topology).

Pf: If $\bigcup_{\alpha \in A} E_\alpha$ is disconnected, there are U, V open in X s.t.

$$U \cap \bigcup_{\alpha \in A} E_\alpha \quad , \quad V \cap \bigcup_{\alpha \in A} E_\alpha$$

are disjoint and non-empty

and $\left(U \cap \bigcup_{\alpha \in A} E_\alpha\right) \cup \left(V \cap \bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} E_\alpha$.

If $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$ then $\exists\, x \in \bigcap_{\alpha \in A} E_\alpha$ and

we may assume that $x \in U \cap \bigcup_{\alpha \in A} E_\alpha$.

Since $V \cap \bigcup_{\alpha \in A} E_\alpha$ is non-empty, $\exists\, \alpha'$ s. $E_{\alpha'} \cap V \neq \emptyset$

but $x \in E_\alpha'$ so $E_{\alpha'} \cap U$ is non-empty

$\therefore$ $E_\alpha' = (U \cap E_{\alpha'}) \cup (V \cap E_\alpha')$ is disconnected.

c) If $A \subset X$ is connected, then $\bar{A}$ is connected.

Pf: Suppose that $\bar{A}$ is disconnected.

Then there are $U, V \subset X$ non empty and open s.t. $U \cap V = \emptyset$ and.

$$\bar{A} = (\bar{A} \cap U) \cup (\bar{A} \cap V)$$

if $A \cap U = \emptyset$ then U^c is a closed set containing A
so $\bar{A} \subset U^c$

$\bar{A} \cap U \neq \emptyset \Rightarrow A \cap U \neq \emptyset$
and similarly $A \cap V \neq \emptyset$

so $A = (A \cap U) \cup (A \cap V)$. is disconnected.

d) Every point of X is contained in a maximal connected subset of X, and this subset is closed.
(the connected component of x).

Pf: The union of all connected subsets of X containing $\{x\}$ is connected by part b).
($\{x\}$ is connected). By part c) this union is closed.

4.2.22. If X is an infinite set with the co-finite topology, then every $f \in C(X)$ is constant.

<u>Pf:</u> The open sets are

$\{U \subset X : U = \emptyset \text{ or } U^c \text{ is finite}\}$.

If $f(x) = z$ for some $x \in X$ and $z \in \mathbb{C}$ then for any $\epsilon > 0$.

$f^{-1}(\{w : |w - z| < \epsilon\})$ is an ~~open~~ nbhd of x in X. So all but finitely many points of X are mapped into $\{w : |w-z| < \epsilon\}$ by f. Clearly, f cannot take another value z_1 outside $\{w : |w-z|<\epsilon\}$. Since $\epsilon > 0$ was arbitrary, f must be constant.