Math 138 1/29/09

Separation axioms. $(X, \mathcal{T})$

T_0: $x \neq y \Rightarrow \exists V \in \mathcal{T}$ s.t $x \in V$, $y \notin V$
or $\exists V \in \mathcal{T}$ s.t $x \notin V$, $y \in V$.

T_1: $x \neq y \Rightarrow \exists V \in \mathcal{T}$ s.t. $x \in V$, $y \notin V$.

T_2: $x \neq y \Rightarrow \exists U, V \in \mathcal{T}$ $U \cap V = \emptyset$
s.t $x \in U$, $y \in V$.
("Hausdorff Spaces").

T_3: X is T_1 and for any
closed $A \subset X$ and any $x \in X \setminus A$
there are $U, V \in \mathcal{T}$ $U \cap V = \emptyset$ s.t.
$x \in U$ $A \subset V$.

T_4: X is T_1 and for any disjoint
closed sets A, B $\exists U, V \in \mathcal{T}$, disjoint
s.t $A \subset U$, $B \subset V$.
("Normal Spaces").

Prop: $(X, \mathcal{T})$ is a T_1 space iff $\{x\}$ is closed for each $x \in X$.

Pf: If X is T_1 and $x \in X$ then for each $y \neq x$ $\exists\, U_y \in \mathcal{T}$ s.t. $y \in U_y$ $x \notin U_y$.

So $\{x\}^c = \bigcup_{y \neq x} U_y$ is open and $\{x\}$ is closed.

If $\{x\}$ is closed for each x and $y \neq x$, $\{x\}^c$ is an open set containing y and not x $\blacksquare$

$(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$ top spaces

<u>Def:</u> $f: X \to Y$ is continuous

iff $f^{-1}(V) \in \mathcal{T}_X$ $\forall\ V \in \mathcal{T}_Y$

iff $f^{-1}(K)$ is closed in X

$\forall\ K$ closed in Y.

<u>Def:</u> $f: X \to Y$ is continuous at $x \in X$

iff $f^{-1}(V)$ is a nbhd of x

for every $V \in \mathcal{T}_Y$ which is

a nbhd of $f(x)$.

<u>Prop</u> $f: X \to Y$ is cont. iff f is cont.
at each $x \in X$.

<u>Pf:</u> if f is cont. and $x \in X$
let V be a nbhd of $f(x)$.
$f^{-1}(V^\circ)$ is a nbhd of x so f is cont at x.

if f is cont at each $x \in X$, let $V \subset Y$ be an open set.

then V is a nbhd of $f(x)$ for each $x \in f^{-1}(V)$

$\therefore$ $f^{-1}(V)$ is a nbhd. of each $x \in f^{-1}(V)$.

$\therefore$ $f^{-1}(V) = f^{-1}(V)^{\circ}$ so $f^{-1}(V)$ is open.

$\therefore$ f is continuous.

<u>Prop</u> if $\mathcal{E}$ generates $\mathcal{T}_Y$ then $f: X \to Y$ is cont.

iff $f^{-1}(V) \in \mathcal{T}_X$ $\forall$ $V \in \mathcal{E}$.

<u>Pf:</u> if $U \in \mathcal{T}_Y$ is open then

$U = \bigcup_{y \in U} V_y$ where $V_y \in \mathcal{E}$. $y \in V_y$.

so $f^{-1}(U) = \bigcup_{y \in U} f^{-1}(V_y)$ is open.

The opposite implication is clear.

Weak topologies

X a set

$\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ a family of maps into top. spaces Y_α.

$\exists!$ topology $\mathcal{T}$ which is the weakest topology on X making all f_α continuous.

i.e. $\mathcal{T}$ is the smallest topology containing all sets of the form $f_\alpha^{-1}(U_\alpha)$ where $U_\alpha \in \mathcal{T}_{Y_\alpha}$, $\alpha \in A$.

$\mathcal{T}$ is called the weak topology generated by the $\{f_\alpha\}_{\alpha \in A}$.

e.g. Recall that

$X = \prod_{\alpha \in A} X_\alpha$ is the set of functions:

$f \in X$

$f : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ s.t $f(\alpha) \in X_\alpha \ \forall \alpha$

and the projection map $\pi_\alpha : X \to X_\alpha$.

is defined by

$\pi_\alpha(f) = f(\alpha)$.

The weak topology on X generated by the maps π_α $\alpha \in A$ is called the **product** topology on X.

If $A = \{1, \ldots, n\}$, ~~inters~~

unions of finite intersection of sets of the form $\pi_\alpha^{-1}(U_\alpha)$ $U_\alpha \in \mathcal{T}_{X_\alpha}$

all look like

$\bigcap_{i=1}^{n} \pi^{-1}(V_i)$ for some V_i open in X_i.

(since $\pi_{\alpha_i}^{-1}(X_i) = X$)

and $\bigcap_{i=1}^{n} \pi^{-1}(V_i) = \prod_{i=1}^{n} V_i \subset \prod_{i=1}^{n} X_i$.

Prop: X_α Hausdorff for each $\alpha \in A$

$\Rightarrow$ $X = \prod_{\alpha \in A} X_\alpha$ is Hausdorff.

if $x \in X$, $y \in X$ and $x \neq y$.

$\exists\ \alpha$ s.t $\pi_\alpha(x) \neq \pi_\alpha(y)$.

Since X_α is Hausdorff $\exists\ V_x, V_y \in \mathcal{T}_\alpha$

$V_x \cap V_y = \emptyset$ s.t $\pi_\alpha(x) \in V_x$, $\pi_\alpha(y) \in V_y$

then $\pi_\alpha^{-1}(V_x)$ is a nbhd of x

and $\pi_\alpha^{-1}(V_y)$ is a nbhd of y.

and $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \emptyset$.

Prop: $X = \prod_{\alpha \in A} X_\alpha$ $(X_\alpha, \mathcal{T}_\alpha)$

$(Y, \mathcal{T}_Y$

$f: Y \to X$ is cont iff $\pi_\alpha \circ f$ is cont. for each α.

Pf: If $f: Y \to X$ is cont.

then $\pi_\alpha \circ f$ is cont. $\forall \alpha$.

(let $U_\alpha \subset X_\alpha$ be open. then

$$(\pi_\alpha \circ f)^{-1}(U_\alpha) = f^{-1}(\pi_\alpha^{-1}(U_\alpha)).$$

is open in Y).

Suppose $\pi_\alpha \circ f$ is cont. for each $\alpha \in A$.

then for each open $U_\alpha \subset X_\alpha$.

$f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open.

Since the product top. on X is generated by the sets $\pi_\alpha^{-1}(U_\alpha)$, f is cont.

Prop (topology of pointwise convergence).

If $X_\alpha = X \quad \forall \alpha$ then $\prod_{\alpha \in A} X_\alpha$

is $X^A = \{$ functions from A to $X\}$.)

~~if $(X, \mathcal{T})$ is top sp.~~

if $(X, \mathcal{T})$ is a top. space

and $\{f_n\}$ is a sequence in X^A

then $f_n \to f$ in the product topology

iff $f_n \to f$ pointwise.

Pf: Suppose that $f_n \to f$ in the product topology. Then for each

$a \in A$ $\quad \pi_a(f) = f(a)$. and if

$V \subset X$ is a nbhd of $f(a)$ in X

then $\pi_a^{-1}(V)$ is a nbhd of f in X^A.

Since $f_n \to f$, $\exists N$ s.t $\forall n \geq N$

$f_n \in \pi_a^{-1}(V)$ so $\pi_a(f_n) = f_n(a) \in V \ \forall n \geq N$.

$\therefore f_n(\alpha) \to f(\alpha) \quad \forall \alpha \in A$.

Now Suppose that $f_n \to f$ pointwise.

The open sets in the product topology are precisely the arbitrary unions of finite intersections of sets of the form $\pi_a^{-1}(U)$ where $a \in A$ and $U \subset X$ is open.

So the sets

$$N(\alpha_1, \dots, \alpha_k, U_1, \dots, U_k) = \{ g \in X^A : g(\alpha_j) \in U_j \quad 1 \le j \le k \}.$$

~~form a subbasis~~

where $\alpha_j \in A$ and U_j is a nbhd of $f(\alpha_j)$

form a nbhd base at f.

Fix such a set. $W = N(\alpha_1, \dots, \alpha_k, U_1, \dots U_k)$.

Since $f_n \to f$ p.w.

$$f_n(\alpha_j) \to f(\alpha_j) \qquad 1 \le j \le k.$$

so $\exists$ N s.t. $\forall n \ge N$ $f_n \in W$

$\therefore$ $f_n \to f$ in the product topology.