Math 138 1/27/09.

X is a non-empty set.

Def: <u>A topology on X</u>

is $\mathcal{T} \subset \mathcal{P}(X)$ s.t.

i) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

ii) $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T} \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

where A is any index set.

iii) $\{U_1, \dots U_n\} \subset \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

$\forall\ n \in \mathbb{Z}^+$.

$(X, \mathcal{T})$ is a <u>topological space</u>

Elements of $\mathcal{T}$ are called <u>open sets</u>

<u>e.g.</u> If $(X, \mathcal{T})$ is a top. space

and $Y \subset X$ then

$\mathcal{T}_Y \equiv \{ U \cap Y ; U \in \mathcal{T} \}$ is a topology on Y.

The <u>relative</u> topology induced by $\mathcal{T}$.

The open sets in this topology are called

<u>relatively</u> open.

Other vocabulary

- closed set - the complement of an open set.
- interior of $A \subset X$ - denoted by A°, this is the union of all open sets contained in A
- Closure of $A \subset X$ - denoted by $\bar{A}$, this is the intersection of all ~~comp~~ closed sets containing A.
- Boundary of $A \subset X$ - denoted by ∂A or bA

$$\partial A = \bar{A} \setminus A^\circ$$

- If $\bar{A} = X$, A is dense in X.
- If $(\bar{A})^\circ = \emptyset$, A is nowhere dense.

- A <u>nbhd</u> of $x \in X$ is a set $A \subset X$ s.t. $x \in A^\circ$.

- If $x \in X$ and $A \subset X$ then x is an accumulation pt. of A if $A \cap (U \setminus \{x\}) \neq \emptyset$ for every nbhd of x.

<u>Prop</u>: $A \subset X$

$C(A) = \{$accumulation pts of $A\}$.

- $\overline{A} = A \cup C(A)$
- A is closed iff $C(A) \subset A$.

<u>Pf</u>: Let F be any closed set containing A.

if $x \notin F$ then F^c is a nbhd of x

but $A \cap (F^c \setminus \{x\}) = \emptyset$

so $x \notin C(A)$

$\therefore \quad C(A) \subset F$

$\therefore \quad C(A) \subset \bar{A}$ (since F was arbitrary).

$\therefore \quad A \cup C(A) \subset \bar{A}$ (since $A \subset \bar{A}$).

For the opposite inclusion, it suffices to show that

$$\bar{A} / A \subset C(A)$$

which by de-Morgans laws is equivalent to.

$$C(A)^c \setminus A \subset \bar{A}^c$$

But if $x \in C(A)^c \setminus A$ then

$\exists \; U$, a nbhd of x s.t.

$$A \cap U = \emptyset \qquad \left(A \cap (U \setminus \{x\}) = \emptyset \text{ but } x \notin A \right).$$

Now $(U^\circ)^c$ is a closed set containing A which does not contain x

so $x \notin \bar{A}$.

For the last conclusion,

$$A \text{ is closed iff } A = \overline{A}$$
$$\text{iff } A = A \cup C(A)$$
$$\text{iff } C(A) \subset A.$$

$\blacksquare$

Neighborhood bases and base for a Topology:

$\mathcal{T}$ a topology on X

A nbhd base for $\mathcal{T}$ at $x \in X$

is a family $\mathcal{N} \subset \mathcal{T}$ s.t.

a) $x \in V \quad \forall V \in \mathcal{N}$

b) if $U \in \mathcal{T}$ and $x \in U$ then

$\exists\ V \in \mathcal{N}$ s.t. $V \subset U$.

A <u>base</u> for $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ which contains a nbhd base for each $x \in X$.

A: <u>Prop</u> $\mathcal{T}$ a topology on X
$\mathcal{E} \subset \mathcal{T}$

- $\mathcal{E}$ is a base for $\mathcal{T} \iff$

  every $U \in \mathcal{T}$ $(U \neq \emptyset)$ is a union of sets in $\mathcal{E}$.

<u>Pf:</u>

If every $U \in \mathcal{T}$ is a union of sets in $\mathcal{E}$ then $\mathcal{E}$ is a base by definition.

If $\mathcal{E}$ is a base and $x \in U \in \mathcal{T}$,

$\exists$ $V_x \in \mathcal{E}$ s.t $x \in V_x \subset U$.

So $U = \bigcup_{x \in U} V_x$

∎

B: <u>Prop:</u> $\mathcal{E} \subset P(X)$

- $\mathcal{E}$ is a base for a topology on X $\iff$

  The following two conditions hold:

  a) for each $x \in X$, $\exists$ $V \in \mathcal{E}$ s.t $x \in V$

  b) $U, V \in \mathcal{E}$ and $x \in U \cap V \Rightarrow \exists W \in \mathcal{E}$ s.t $x \in W \subset U \cap V$.

<u>Pf:</u> If $\mathcal{E}$ is a base for a topology $\mathcal{T}$ then $\mathcal{E} \subset \mathcal{T}$, a) holds by definition and $U \cap V$ is open so if $x \in U \cap V$ then $\exists\, W \in \mathcal{E}$ s.t $x \in W \subset U \cap V$.

If a) and b) hold for $\mathcal{E}$

let $\mathcal{T} = \{ U \subset X :$ for every $x \in U$ $\exists\, V \in \mathcal{E}$ s.t. $x \in V \subset U \}$

then • $\emptyset \in \mathcal{T}$ and by a), $X \in \mathcal{T}$.

• It is clear that any union of sets in $\mathcal{T}$ is still in $\mathcal{T}$.

• If U, V are in $\mathcal{T}$ and $x \in U \cap V$

then $\exists\, U_1 \in \mathcal{E}$ s.t $x \in U_1 \subset U$

and $\exists\, V_1 \in \mathcal{E}$ s.t $x \in V_1 \subset V$

~~With $W = U_1 \cap V_1$ we have~~

~~$x \in W \subset$~~ By condition b) $\exists\, W \in \mathcal{E}$ s.t $x \in W \subset U_1 \cap V_1 \subset U \cap V$.

So $U \cap V \in \mathcal{T}$ and by induction, any finite intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

$\therefore$ $\mathcal{T}$ is a topology which has $\mathcal{E}$ as a base.

$\boxtimes$

Prop: $\mathcal{E} \subset \mathcal{P}(X)$

(The topology generated by $\mathcal{E}$ is the smallest topology on X which contains $\mathcal{E}$).

The topology $\mathcal{T}$ generated by $\mathcal{E}$ consists of $\emptyset, X$ and all unions of finite intersections of members of $\mathcal{E}$.

Pf: The collection of sets in question is clearly contained in $\mathcal{T}$, so it suffices to show that this collection (denote it by $\mathcal{C}$) is a topology.

By Prop B

the set of finite intersections of sets in $\mathcal{E}$ together with X

is a base $\mathcal{E}^*$ for a topology. $\mathcal{T}^*$

By proposition A, the nonempty sets in $\mathcal{T}^*$ are unions of sets in $\mathcal{E}^*$.
So $\mathcal{T}^* = \mathcal{C}$. ▨

<u>Vocabulary:</u>

$(X, \mathcal{T})$ is <u>1st countable</u>

if each $x \in X$ has a <u>countable</u> nbhd base for $\mathcal{T}$.

<u>remark:</u> if X is 1st countable then at each $x \in X$ there is a nbhd base $\{U_j\}_{j=1}^{\infty}$ s.t. $U_j \supset U_{j+1}$ $\forall j$.

$(X, \mathcal{T})$ is <u>second</u> <u>countable</u> if $\mathcal{T}$ has a countable base.

$(X, \mathcal{T})$ is <u>seperable</u> if it has a countable dense subset.

<u>Prop</u>: Every second countable space is separable.

<u>Pf</u>: Let $\mathcal{E}$ be a countable base for X.

$\mathcal{E} = \{U_j\}_{j=1}^{\infty}$ For each j, let $x_j \in U_j$

Then $\left(\overline{\bigcup_{j=1}^{\infty} \{x_j\}}\right)^c \equiv V$ is open.

If $V \neq \emptyset$ then $\exists\, j$ s.t. $U_j \subset V$

but $x_j \in U_j$ and V contains no x_j.

So $V = \emptyset$. and $\{x_j\}_{j=1}^{\infty}$ is dense.

$\blacksquare$

$(X, \mathcal{T})$

A sequence $\{x_j\}_{j=1}^{\infty} \subset X$ converges to $x \in X$ if for any nbhd V of x $\exists$ $N \in \mathbb{Z}^+$ s.t $x_j \in V$ $\forall$ $j > N$.

In 1st countable spaces, topological concepts such as closure and continuity can be defined in terms of sequential convergence. This is not the case in general (as we will see later).

Prop: $(X, \mathcal{T})$ is 1st countable. $A \subset X$.

- $x \in \overline{A}$ iff $\exists$ $\{x_j\}_{j=1}^{\infty} \subset A$ s.t $x_j \to x$.

Pf: if $x \notin \overline{A}$ and $\{x_j\}_{j=1}^{\infty} \subset A$ then $(\overline{A})^c$ is a nbhd of x containing no x_j.

If $x \in \overline{A}$, let $\{U_j\}_{j=1}^{\infty}$ be a countable nbhd base of x s.t $U_j \supset U_{j+1}$ $\forall j$.

For each j, $A \cap (\mathcal{U}_j \setminus \{x\}) \neq \emptyset$.

so we can choose $x_j \in A \cap (\mathcal{U}_j \setminus \{x\})$.

If V is any nbhd of x then $\exists\, j$

s.t $\mathcal{U}_j \subset V$, $\therefore$ $x_k \in V$ $\forall\, k \geq j$.

so $x_j \to x$. $\blacksquare$