Math 138 1/22/09.

Baire Category Theorem

Thm: X a complete ~~metric~~ space.

a) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of dense open sets in X then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

b) X is not a countable union of nowhere dense sets.

Pf: a) $\Rightarrow$ b).

recall: $A \subset X$ is dense iff $\bar{A} = X$.

$A \subset X$ is dense at $x \in X$ iff $\bar{A}$ contains a nbhd of X.

$A \subset X$ is nowhere dense iff A is not dense at any point $x \in X$ iff $(\bar{A})^{\circ} = \emptyset$.

if E_n is a sequence of nowhere dense subsets of X

then $(\overline{E_n})^\circ = \emptyset$

so $(\overline{E_n})^c$ is open and dense in X.

Part a) $\Rightarrow$ $\bigcap_{n=1}^{\infty} (\overline{E_n})^c$ is dense in X

$$= \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)^c$$

$\therefore$ $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \overline{E_n} \neq X$.

<u>Pf of a)</u>

Let W be a non-empty open set in X.

We will show that $W \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset$.

$U_1 \cap W$ is open and non-empty since U_1 is dense and open

Let $x_1 \in U_1 \cap W$ and choose $r_1 < 1$, s.t.

$B(r_1, x_1) \subset U_1 \cap W$

Then $\overline{B(\frac{r_1}{2}, x_1)} \subset U_1 \cap W$.

and $U_2 \cap B(\frac{r_1}{2}, x_1)$ is open and non empty. and we can choose $x_2 \in B(\frac{r_1}{2}, x_1)$ and r_2 s.t $B(\frac{r_2}{2}, x_2) \subset U_2 \cap B(\frac{r_1}{2}, x_1)$.

and s.t $\overline{B(\frac{r_2}{2}, x_2)} \subset U_2 \cap B(\frac{r_1}{2}, x_1)$

Having chosen $x_1, \dots, x_n$, $r_1, \dots, r_n$ in this same way, note that $U_{n+1} \cap B(\frac{r_n}{2}, x_n)$ is open and non empty. We can then choose $x_{n+1} \in B(\frac{r_n}{2}, x_n)$ and r_{n+1} s.t.

$$B(r_{n+1}, x_{n+1}) \subset U_{n+1} \cap B(\frac{r_n}{2}, x_n)$$

and then

$$\overline{B(\frac{r_{n+1}}{2}, x_{n+1})} \subset U_{n+1} \cap B(\frac{r_n}{2}, x_n)$$

We have (inductively) $r_n < 2^{-n} r_1$.
And if $n, m \geq N$ then

$x_n, x_m \in B(\frac{r_N}{2}, x_N)$.

Since $r_n \to 0$, the sequence

$\{x_n\}$ ~~has a limit~~ is Cauchy

and $\therefore$ has a limit (X is complete)

which we denote by x.

Since $x_n \in B(\frac{r_N}{2}, x_N)$ $\forall n \geq N$,

we have

$$x \in \overline{B(\frac{r_n}{2}, x_N)} \subset U_N \cap B(r_1, x_1)$$

$$\subset U_N \cap W.$$

$\forall N$.

so $x \in \left[\bigcap_{n=1}^{\infty} U_n \cap W\right]$. $\boxed{/\!/\!/}$

e.g. Let E_n be the set of all

$f \in C[0,1]$ for which $\exists\, x_0 \in [0,1]$ (depending on f) such that

$$|f(x)-f(x_0)| \le n\,|x-x_0| \qquad \forall x \in [0,1].$$

Recall that $C[0,1]$ is a complete metric space with the metric

$$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}.$$

Pf: (Completeness).

Let $\{f_n\}$ be a Cauchy sequence and choose n_j s.t. $j=0,1,2,\ldots$

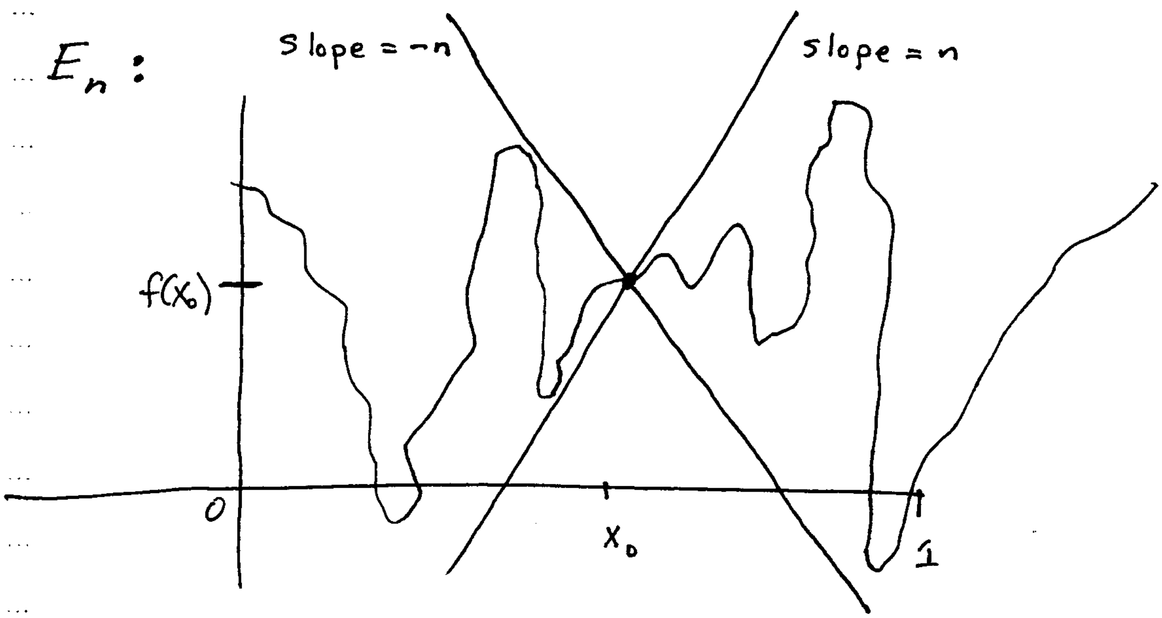
$$\sup_{[0,1]}\left(|f_{n_j} - f_{n_{j-1}}|\right) < 2^{-j}$$

then $g_N \equiv \sum_{j=0}^{N} (f_{n_{j+1}} - f_{n_j}) = f_{n_N} - f_{n_0}$

converges absolutely and uniformly to a continuous limit g.

So, $f_{n_N} \xrightarrow[N \to \infty]{} f_{n_1} + g \in C[0,1]$.
(uniformly).

E_n:



<u>Claim</u>: E_n is nowhere dense in $C[0,1]$.

To see this, we first approximate an arbitrary $f \in C[0,1]$ by a piecewise linear function.

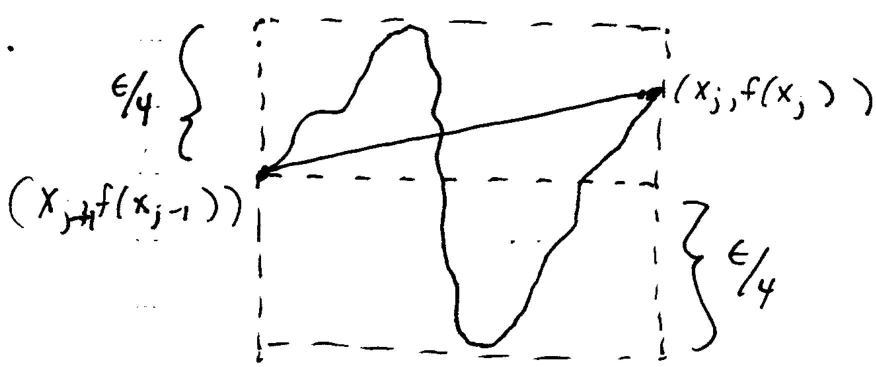
Given $f \in C[0,1]$ and $\epsilon > 0$, we may choose (by continuity and compactness).

$$x_0 = 0 < x_1 < x_2 < \dots < x_M = 1.$$

s.t $\quad |f(x) - f(x_{j-1})| < \epsilon/4$

$\forall \quad x_{j-1} < x \le x_j$ and for each j

For each $j = 1, \dots, M$ let $l_j(x)$ be the line (Affine linear) function which passes through $(x_{j-1}, f(x_{j-1}))$ and $(x_j, f(x_j))$.. Then for all x s.t $x_{j-1} \le x \le x_j$ we have

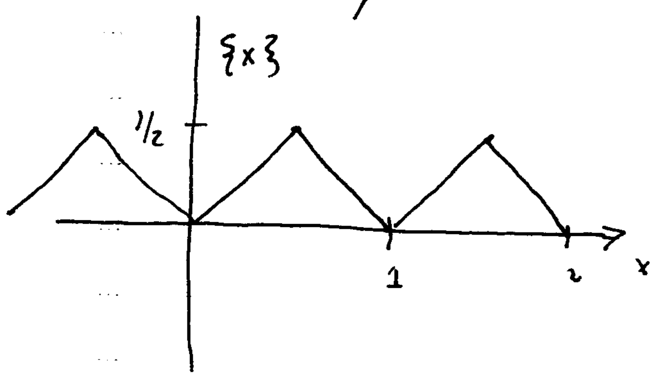


$$|l_j(x) - f(x)| \le |f(x) - f(x_j)| + |l_j(x) - f(x_j)|.$$
$$\le \epsilon/4 + \epsilon/4$$

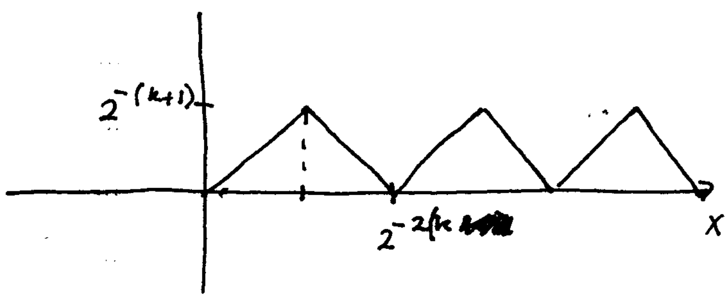
Let f_0 be the piecewise linear function which is equal to $l_j(x)$ for $x_{j-1} \le x \le x_j$, $j = 1, \dots, M$.

then $|f - f_0| < \epsilon/2$ on $[0,1]$.

For $x \in \mathbb{R}$, let $\{x\}$ denote the distance from x to the nearest integer. and let



$$g_k(x) = 2^{-k}\{2^{2k}x\}.$$



The slope of each linear segment in the graph of g_k

is
$$\frac{2^{-(k+1)}}{2^{-2k-1}} = \frac{2^{-k}}{2^{-2k}} = 2^k.$$

Also, $|g_k| \le 2^{-(k+1)}$

Let n be fixed.

Given $f \in C[0,1]$ and $\epsilon > 0$.

Choose f_0 as constructed above and let
$$S_{max} = \max_{j \in [1,M]} \left| \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \right|$$

Choose k s.t. $2^{-k} < \frac{\epsilon}{2}$

$$2^k > 2n + S_{max}.$$

so that for each j,

(if m_j is the slope of the j^{th} segment in the graph of f_0)

$$2n + m_j < 2^k \qquad (m_j - 2^k < -2n)$$

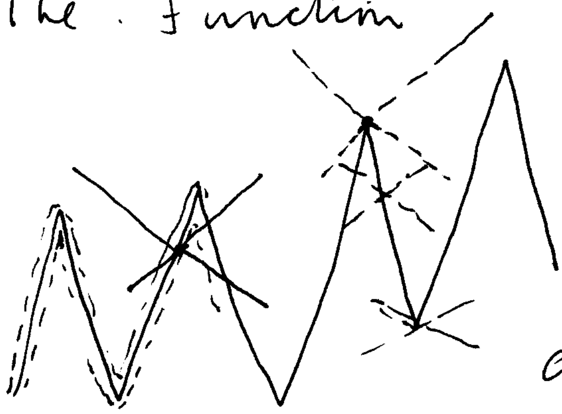
and
$$2n - m_j < 2^k \qquad (m_j + 2^k > 2n).$$

Then $|f_0 + g_k - f| < \epsilon$

and $f_0 + g_k$ is piecewise linear with the magnitude of the slope of each linear segment larger than $2n$.

Verification of these facts is an exercise with the triangle inequality.

The function $f_0 + g_n$ is not in E_n



and if $\delta > 0$ is suff small then any cont fcn. $h \in C[0,1]$ s.t. $\| f_0 + g_n - h \| < \delta$ has $h \notin \cancel{C[0,1]}\ E_n$.

$\boxed{\text{It follows that } (\overline{E_n})^\circ = \emptyset \ \ (E_n \text{ is nowhere dense}).}$

If $f \in C[0,1]$ is differentiable at $x_0 \in [0,1]$ then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \quad \text{exists.}$$

and $\exists\ \delta > 0$ s.t. if $|x - x_0| < \delta$ then

$$\cancel{\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right|}$$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq 2L.$$

i.e. $|f(x) - f(x_0)| \leq 2L\,|x - x_0| \quad \forall |x - x_0| < \delta.$

By cont. on $[0,1]$ $\exists$ M s.t.
$\forall$. $|x-x_0| > \delta$ $x \in [0,1]$.

$$|f(x) - f(x_0)| \leq \cancel{M} \delta \leq M|x-x_0|.$$

~~s.t.~~
so $\forall$ $x \in [0,1]$

$$|f(x)-f(x_0)| \leq \max(M,L)\,|x-x_0|.$$

let $n \in \mathbb{Z}^+$ s.t $n \geq \max(M,L)$
then
$$|f(x)-f(x_0)| \leq n\,|x-x_0|$$
and $f \in E_n$.

So the fcns $f \in C[0,1]$ which are diff. at least at one point in $[0,1]$ are contained in
$$\bigcup_{n=1}^{\infty} E_n.$$
(a countable union of nowhere dense sets)

The Baire category theorem says that $C[0,1]$ cannot be $\bigcup_{n=1}^{\infty} E_n$. so there must be nowhere differentiable fcns.

~~In fact~~, A countable union of nowhere dense sets is called "of first (Baire) category"

or "meager".

The complement of such a set is "2nd category"

or "residual".

The nowhere diff. fcns are residual in $C[0,1]$.

The 1st example of such a fcn was produced by Weierstress.
who proved that

$$f(x) \equiv \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

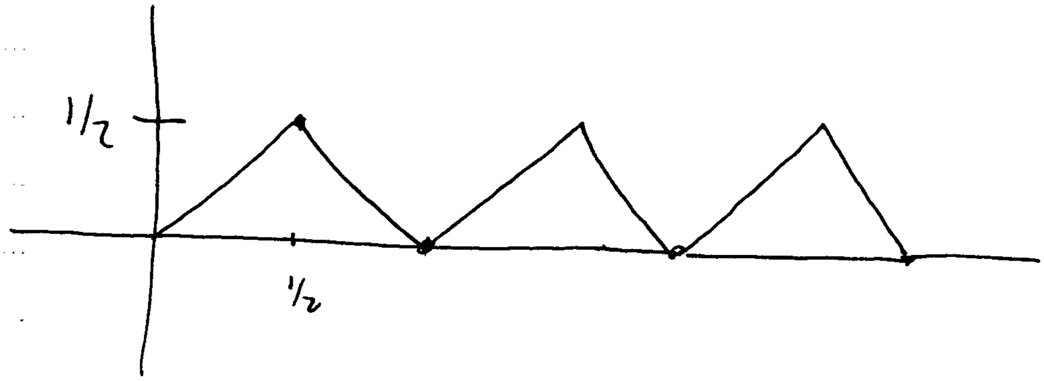
with $0 < a < 1$, $b \in \mathbb{Z}^+$, b odd.

$$ab > 1 + \frac{3\pi}{2}$$

is continuous and nowhere differentiable.

An example more along the lines of our proof was given by Van der Waerden.

For $x \in \mathbb{R}$, let $\{x\}$ be the distance from x to the nearest integer



<u>Claim</u>: $$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}$$

is continuous and nowhere differentiable

<u>Pf</u>: Since $\left| \frac{\{10^n x\}}{10^n} \right| \leq 10^{-n}$

the series converges uniformly on $[0,1]$, and thus represents a cont. fcn with period 1 on $\mathbb{R}$.

Fix $x \in [0,1)$.
We will show that the derivative $f'(x)$ cannot exist.

let $x = 0.a_1 a_2 a_3 \ldots$ (base 10).

where we always choose an expansion ending in 0's if possible.

<u>Case I</u> $0.a_{n+1} a_{n+2}, \ldots \leq 1/2$

then

$$\{10^n x\} = 0.a_{n+1} a_{n+2} \ldots$$

<u>Case II</u> $0.a_{n+1} a_{n+2} \ldots > 1/2$

then

$$\{10^n x\} = 1 - 0.a_{n+1} a_{n+2} \ldots$$

Let

$$h_m = -10^{-m} \quad \text{if } a_m = 4 \text{ or } 9$$
$$h_m = 10^{-m} \quad \text{else}$$

and consider

$$\frac{f(x+h_m) - f(x)}{h_m}$$

$$= \sum_{n=0}^{\infty} \frac{\{10^n(x+h_m)\} - \{10^n x\}}{10^n \cdot h^m}$$

$$= 10^m \sum_{n=0}^{\infty} \frac{\{10^n(x \pm 10^{-m})\} - \{10^n x\}}{10^n(\pm 10^{-m})}.$$

$$= 10^{+m} \sum_{n=0}^{\infty} \pm \left(\frac{\{10^n(x \pm 10^{-m})\} - \{10^n x\}}{10^n} \right)$$

$$= 10^{+m} \sum_{n=0}^{m-1} \pm \left(\frac{\{10^n(x \pm 10^{-m})\} - \{10^n x\}}{10^n} \right)$$

$$\cancel{= \left(10^m \sum_{n=0}^{m-1} \pm \left(\frac{\{10^n x \pm 10^{n-m}\} - \{10^n x\}}{10^n} \right)\right)}$$

if n^{th} digit of x is 4 or 9,
subtract 1 from m^{th} digit.

$(m > n)$. (new n^{th} digit is 3 or 4 or 8 or 9.)

if n^{th} digit of x is 0, 1, 2, 3, 5, 6, 7, 8
add 1 to m^{th} digit
new n^{th} digit is same as old

In any event,
we stay in the same case.

and for ($n < m$)

$$\{10^n(x \pm 10^{-m})\} - \{10^n x\} = \pm 10^{n-m}.$$

So our sum is.

$$= 10^m \sum_{n=0}^{m-1} \frac{\pm 10^{n-m}}{10^n}$$

$$= \sum_{n=0}^{m-1} (\pm 1)$$

which is even or odd according to the parity of $m-1$.

So the limit

$$\lim_{h_m \to 0} \frac{f(x+h_m) - f(x)}{h_m}$$

can't exist.