Gateway to Exploring Mathematical Sciences (GEMS)

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The Monty Hall Problem

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1 The Monty Hall Problem

Problem created by Steve Selvin

- Letter to *American Statistician* in 1975
- Inspired by game show "Let's Make a Deal"
- Host of the show was Monty Hall.

The Problem

- A contestant is chosen from the studio audience.
- The contestant is shown one of three doors.
- Behind one door is a prize, the other two have nothing
- (Traditionally: prize is a car, other doors have a goat.)
- Monty then always opens one of the doors that does not contain the prize.
- Monty then always offers the contestant a chance to switch their door.
- Then the constestant opens the door.

Should the contestant switch when the offer is made?

- Problem became much more famous when asked to Marilyn Vos Savant's column as a brainteaser.
- Very easy problem to get wrong!

Probability is about information

- Today you'll be working with a six sided die
- Probability measures how much information you have
- Sides of the die symmetric
  (talk about Platonic solids, show other dice)

Principle of Insufficient Reason

- Unless you have a reason otherwise, all outcomes equally likely
- California weather: rain or not rain not equally likely
- Sides of a die: symmetric, no reason why one more likely than another

  $$\text{Prob}(1) = \text{Prob}(2) = \text{Prob}(3) = \text{Prob}(4) = \text{Prob}(5) = \text{Prob}(6) = x$$

- They must add up to 1

  $$x + x + x + x + x + x = 1 \Rightarrow 6x = 1 \Rightarrow x = 1/6.$$

- Probability of an outcome 1 over number of outcomes.
- Extra information can change the probabilities.

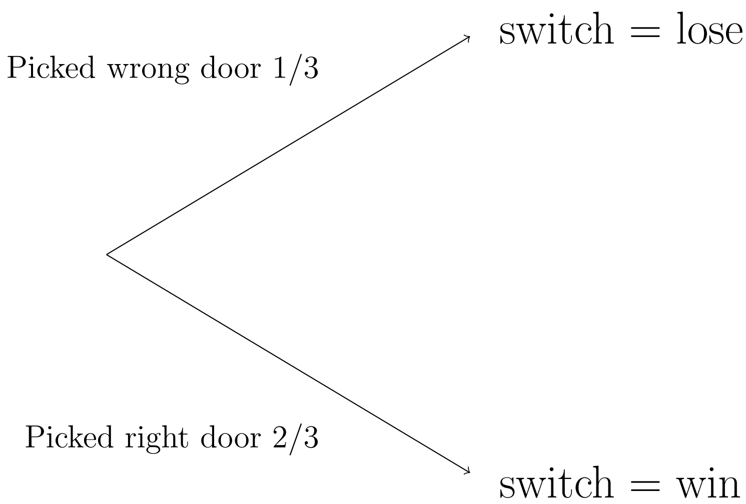
outcome	Prob	Info: not 6
1	1/6	1/5
2	1/6	1/5
3	1/6	1/5
4	1/6	1/5
5	1/6	1/5
6	1/6	0

- Learning the the die roll is not 6 changes the probabilities!

Monty Hall info changes the probabilities as well!

- Let's see if we can figure out how the probabilities change with the extra information.
- Build a *probability tree.*
- First branch tells if contestant first picked the right door or first picked the wrong door.

- Key insight: if the contestant picks the right door and switches they lose, but if they pick the wrong door and switch, they win!



- That means if you always switch,
  you have a 2/3 chance of winning!
- The extra information helps!

Does more information help more?

- Suppose you know that when Monty opens a door, he always opens the door with the lowest number
- Does this information allow the player to do better than 2/3?
- Let

$$P = \# \text{ of the door chosen initially by player}$$
$$L = \text{lower numbered door that remains}$$
$$H = \text{higher numbered door that remains}$$

- For example, if the player chooses door 3, $P = 3$, $L = 1$, and $H = 2$. If the player chooses door 2, $P = 2$, $L = 1$, and $H = 3$.
- So now let's look at the outcomes.

- Let p stand for prize and n for no prize. So the outcomes are

$$PLH \in \{pnn, npn, nnp\}.$$

(For instance, $PLH = npn$ means the prize was behind door L.)

- In case npn, Monty opens the higher door H.
- In cases pnn and nnp, Monty opens the lower door L. Hence

PLH	Probability	Opens door L	Opens door H
pnn	1/3	1/2	0
npn	1/3	0	1
nnp	1/3	1/2	0

- In other words, when Monty opens door H, the player knows with certainty that the prize is behind the lower number door L.
- So Monty opens H means player should switch.
- When Monty opens L, player has a 50/50 chance of having prize.
- Player can switch or not switch, won't change chances!
- Overall chance of player winning:

$$\underbrace{\left(\frac{1}{3}\right)}_{\text{chance of } H}(1) + \underbrace{\left(\frac{2}{3}\right)}_{\text{chance of } L}\left(\frac{1}{2}\right) = 1/3 + 1/3 = 2/3.$$

- So (perhaps surpringly), this extra information does not help!

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Break into small groups so that each group has an even number of people. Each person should have a six-sided die.

The warmup

1. Try rolling your die 10 times, and record the number of times the die showed a 1 or 2. ____________

   What percentage of the time did a 1 or 2 come up in your individual rolls? ____________

2. Compare your number to the others in your group. Add up over your whole group the number of times you rolled a 1 or 2. What percentage of the time did a 1 or 2 come up in your group? ____________

The Monty Hall problem

1. Now break up your group into pairs of two people. One of each pair will play the host "Monty Hall" while the other person will be the player.

   - Have the host roll a die to determine which door gets the prize: on a 1 or 2 it is door number 1, for 3 or 4 it is door number 2, and for 5 or 6 it is door number 3. (Do not show the player which door has the prize!)
   - Next have player choose a door, either 1, 2, or 3.
   - The host then rolls the die. If the player chose the door with the prize, then if the number is 1,2,3, the host tells the player the lower numbered door that does not have a prize. If the number is 4,5,6, the host tells the player the higher numbered door that does not have a prize. For example: the prize is behind door number 2 and the player chose door number 2. The host rolls the die and if it is 3, tells the player the prize is not behind door number 1.

     Note: it is important that the host roll the die every time, even if the host does not have a choice as to which door to open. Otherwise the player could figure out if they had the right door by seeing if the host needed to roll a die or not!
   - The player then has the option to switch your door or not. (Remember, for this version of the problem, you should always switch!)
   - Finally, the host tells you whether or not you won the prize.

2. Repeat this game 10 times, and record how many times you got the prize.

3. Now switch roles: the player is now Monty Hall, and Monty Hall is now the player. Again play a total of 10 games and see how many are won.

4. When you are done, compare with the results from the other pairs in your group. What was the total winning percentage when you switched every time? ____________.

The Monty Hall problem variation By using the "always switch" strategy with the original rules, the chance of the player getting the prize is 2/3, or 66.66…%. If the rules Monty follows changes, then the player should change their strategy as well!

1. Now repeat the simulated game that you tried before, except now the rule for Monty is a little different.

   Instead of rolling a die to determine which door to reveal to the player, now Monty will always select the lower numbered door. For example, if the player picked 2, and the prize was behind door number 2, then Monty would always reveal the lower numbered door to the player.

   It turns out that in this case, if

If you have time... Now it's your turn! Can you come up with a new set of rules for Monty to follow? With your rules, what should the player do in order to maximize their chance of winning the prize?

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   - Have the host roll a die to determine which door gets the prize: on a 1 or 2 it is door number 1, for 3 or 4 it is door number 2, and for 5 or 6 it is door number 3. (Do not show the player which door has the prize!)
   - Next have player choose a door, either 1, 2, or 3.
   - The host then "opens a door" for the player by telling the player the lowest numbered door that does not contain a prize.
   - The player then has the option to switch your door or not. (Remember, for this version of the problem, you should always switch!)
   - Finally, the host tells you whether or not you won the prize.
2. Repeat this game 10 times, and record how many times you got the prize.
3. Now switch roles: the player is now Monty Hall, and Monty Hall is now the player. Again play a total of 10 games and see how many are won.
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The Monty Hall problem variation By using the "always switch" strategy with the original rules, the chance of the player getting the prize is $2/3$, or $66.66\ldots\%$. If the rules Monty follows changes, then the player should change their strategy as well!

1. Now repeat the simulated game that you tried before, except now the rule for Monty is a little different.

Now after the player chooses a door, Monty rolls a die and does the following

If the die is 1,2,3, or 4 and the player has chosen the correct door, Monty reveals the lowest numbered door without a prize.

If the die is 5 or 6 and the player has chosen an incorrect door, Monty also reveals the lowest numbered door without a prize.

Otherwise Monty does not reveal any door.

2. The player then gets a chance to switch doors if they wish. Have the player always switch doors. If there is more than one door to switch to, have them choose randomly (die roll: 1,2,3 choose lower number door, with die roll 4,5,6 choose higher number door.)

Try playing this game 10 times as the host and the player, with the always switching strategy.

Combine your data with the others from your group. From your data, what is your guess for the chance that the player wins with the always switch strategy?

Can you calculate the chance exactly?

Is there a better strategy that you could use?