Perfect simulation with PRAR

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When at first you don’t succeed...

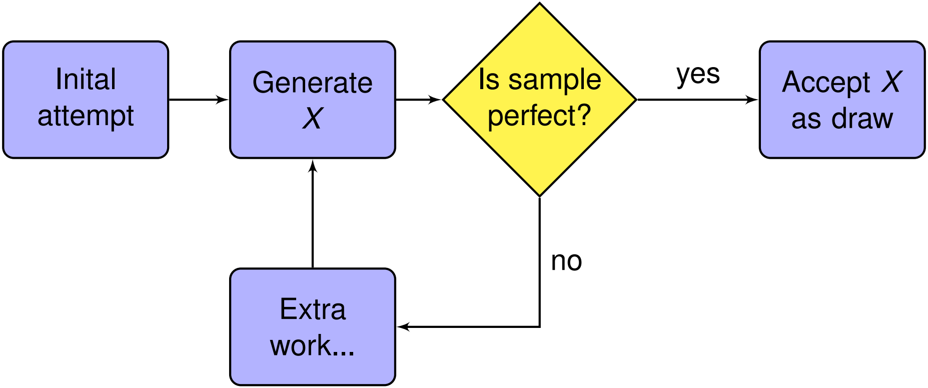
Why do we fail...so we might learn to pick ourselves up.

-Batman Begins

Perfect simulation

Definition (Perfect simulation)

A perfect simulation algorithm draws variates exactly from a target distribution in a random number of steps.



Types of perfect simulation methods

Markov chain based

- Coupling from the past [Propp-Wilson 1996]
- FMMR [Fill-Machida-Murdoch-Rosenthal 2000]
- Randomness Recycler [Fill-H. 2001]

Acceptance Rejection based

- Sequential Acceptance Rejection [H. 2006]
- Retrospective sampling [Papaspiliopoulos-Roberts 2006]
- Partially Recursive Acceptance Rejection (PRAR)

Applications

	Markov chain type	Acceptance rejection type
SDE		Retrospective
Scalable (ex: permanent)	Can't do	Sequential AR
Weakly interacting dimensions	CFTP, RR	?

Applications

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Weakly interacting dimensions	CFTP, RR	PRAR

Basic Acceptance/Rejection

Ingredients

- A set $B \subset A$ that we want to sample uniformly from
- The ability to sample uniformly from A

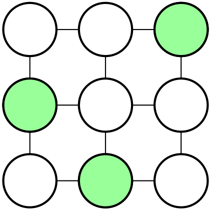
The Algorithm

- Draw X uniformly from A
- If X in B, output X as sample and quit
- Else, startover (so go back to first step)

A representative problem

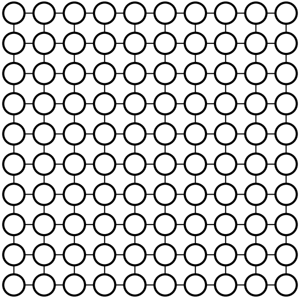
Definition

An *independent set* of a graph is a subset of nodes where no edge has both endpoints in the subset.



Basic AR for independent sets

- Draw X uniformly from subsets of nodes
- If X is not an independent set, go to first step



Chance of accepting for 10 by 10 grid $< 10^{-10}$.

Idea: start by determining single node

Split set of nodes into $\{v\}$ and $V \setminus \{v\}$

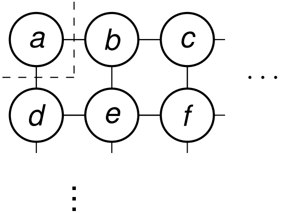
- Include v in independent set with probability $1/2$
- Draw independent set for $V \setminus \{v\}$
- Accept if together they make an independent set

Key observation:

- Don't need entire independent set on $V \setminus \{v\}$
- Just need for the immediate neighbors of v

One at a time

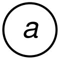
Build up the sample one node at a time



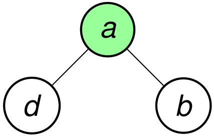
- First make *a* in set or out each with probability $1/2$;
- If *a* out, always accept
- Else find out if *b* and *d* are in set, that determines acceptance for *a*

Examples

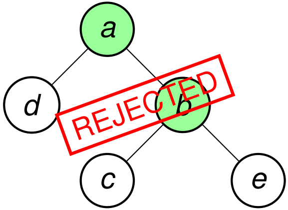
a out
Result: *a* is out of set



a in: put *b* and *d* on queue
b out
d out
Result: *a* is in, *b* is out, *d* is out



a in: put *b* and *d* on queue
b in: put *c* and *e* on queue
c out
e out
Result: with *c* and *e* out, *b* is also in, so reject *a* and everything removed, start over with *a*



The general algorithm

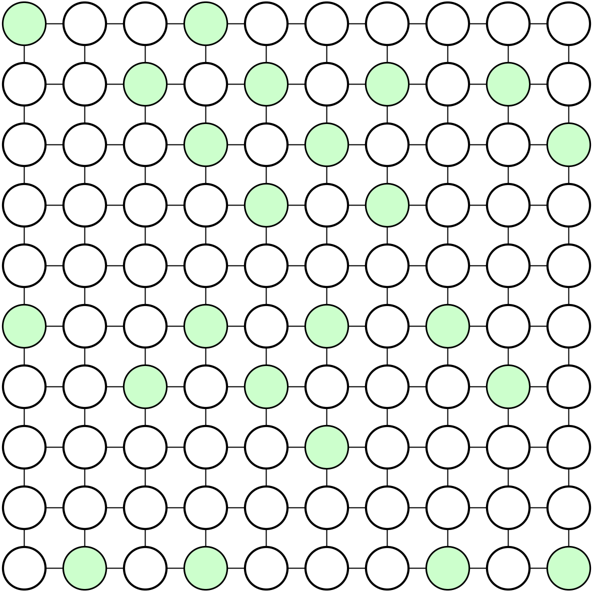
Begin with all nodes in list

- Put a node v off of list
- form its recursion tree
- Fix the values of all nodes in tree
- Continue until list empty

To form recursion tree

- With probability 1/2, node is out
- Otherwise node could be in: add neighbors to tree as children
- If all neighbors/children are out, then node is in...
- ...so reject and remove subtree formed by parent of node

A sample run on 10 by 10 grid



Analysis

Nodes that try to be “in” form node percolation process

- Since node in with probability 1/2, looks like at critical temperature
- But nodes that are “in” remove their siblings...
- ...which leads to subcritical behavior

More generally

- If interactions weak, tree shrinks in size on average
- Artificial phase transition

Method works for Markov random fields

Not just for independent sets...

$$\Omega = C^V, \quad f(x) = \left[\prod_{v \in V} a(x(v))\right] \left[\prod_{\{v,w\} \in E} b(x(v), x(w))\right]$$

This includes

- The Ising model
- Spin glasses
- Strauss model
- Kelly-Ripley interaction models
- Dimensions only interact locally

Strauss process

Like independent set, configuration is subset of nodes

- $x(v) = 1$ means in subset, $x(v) = 0$ means out
- $a(x(v)) = \lambda^{x(v)}$
- $b(x(v), x(w))$ is $\gamma < 1$ if $x(v) = x(w) = 1$
- Strauss density:

$$\lambda^{\text{\# of occupied nodes}} \gamma^{\text{\# of edges with both endpoints occupied}}.$$

Using PRAR with Strauss

Changes to PRAR

- Initial value for v is 1 with probability $\lambda/(1+\lambda)$
- If $x(v) = 1$ proposed...
- ...only check neighbor with probability $1 - \gamma$

Running time polynomial

- Let Δ be the maximum degree of the graph
- Expected linear time algorithm if

$$\frac{\Delta(1-\gamma)\lambda}{1+\lambda} < 1$$

Conclusion

Partially Recursive Acceptance/Rejection

- Basic acceptance/rejection exponentially slow in high dimensions
- PRAR: by doing acceptance/rejection in right order...
- ...can turn an exponential algorithm into a polynomial one
- Applies to models where dimensions interact weakly
- Gives linear expected running time on those models