Bounded Domain Problem for the Modified Buckley-Leverett Equation

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Abstract

The focus of the present study is the modified Buckley-Leverett (MBL) equation describing two-phase flow in porous media. The MBL equation differs from the classical Buckley-Leverett (BL) equation by including a balanced diffusive-dispersive combination. The dispersive term is a third order mixed derivatives term, which models the dynamic effects in the pressure difference between the two phases. The classical BL equation gives a monotone water saturation profile for any Riemann problem; on the contrast, when the dispersive parameter is large enough, the MBL equation delivers a non-monotone water saturation profile for certain Riemann problems as suggested by the experimental observations. In this paper, we first show that for the MBL equation, the solution of the finite interval $[0, L]$ boundary value problem converges to that of the half line $[0, +\infty)$ boundary value problem exponentially as $L \to +\infty$. This result provides a justification for the use of the finite interval in numerical studies for the half line problem [Y. Wang and C.-Y. Kao, *Central schemes for the modified Buckley-Leverett equation*, J. Comput. Sci. (2012), doi:10.1016/j.jocs.2012.02.001]. Furthermore, we numerically verify that the convergence rate is consistent with the theoretical derivation. Numerical results confirm the existence of non-monotone water saturation profiles consisting of constant states separated by shocks.

Keywords: conservation laws, dynamic capillarity, two-phase flows, porous media, shock waves, pseudo-parabolic equations
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1. Introduction

The classical Buckley-Leverett (BL) equation [3] is a simple model for two-phase fluid flow in a porous medium. One application is secondary recovery by water-drive in oil reservoir simulation. In one space dimension the equation has the standard conservation form

$$\begin{aligned} u_t + (f(u))_x &= 0 \qquad \text{in} \qquad && Q = \{(x,t) : x > 0, t > 0\} \\ u(x,0) &= 0 && x \in (0,\infty) \\ u(0,t) &= u_B && t \in [0,\infty). \end{aligned} \tag{1.1}$$

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The flux function $f(u)$ is defined as

$$f(u) = \begin{cases} 0 & u < 0, \\ \frac{u^2}{u^2+M(1-u)^2} & 0 \le u \le 1, \\ 1 & u > 1. \end{cases} \tag{1.2}$$

In this problem, $u : \bar{Q} \to [0,1]$ denotes the water saturation (e.g. $u = 1$ means pure water, and $u = 0$ means pure oil), u_B is a constant which indicates water saturation at $x = 0$, and $M > 0$ is the water/oil viscosity ratio. The classical BL equation (1.1) is a prototype for conservation laws with convex-concave flux functions. The graph of $f(u)$ and $f'(u)$ with $M = 2$ is given in Figure 1.1.

(a) $f(u) = \frac{u^2}{u^2+M(1-u)^2}$ (b) $f'(u) = \frac{2Mu(1-u)}{(u^2+M(1-u)^2)^2}$

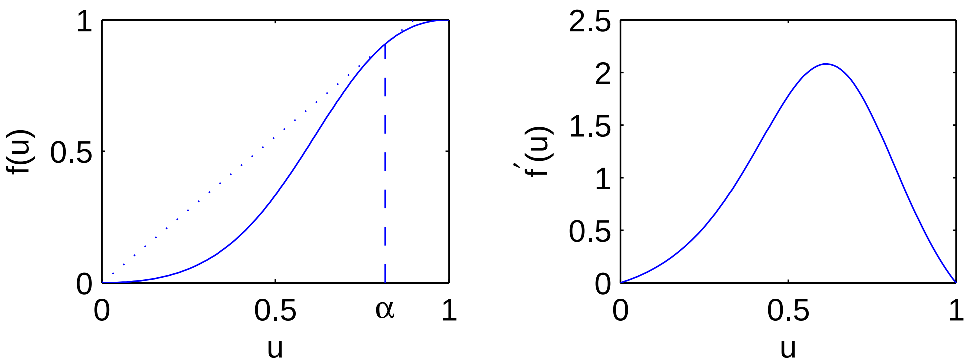

Figure 1.1: $f(u)$ and $f'(u)$ with $M = 2$.

Due to the possibility of the existence of shocks in the solution of the hyperbolic conservation laws (1.1), the weak solutions are sought. The function $u \in L^\infty(Q)$ is called a weak solution of the conservation laws (1.1) if

$$\int_Q \{u\phi_t + f(u)\phi_x\}\, dx\, dt = 0 \qquad \text{for all} \quad \phi \in C_0^\infty(Q).$$

Notice that the weak solution is not unique. Among the weak solutions, the entropy solution is physically relevant and unique. The weak solution that satisfies Oleinik entropy condition [11]

$$\frac{f(u) - f(u_l)}{u - u_l} \ge s \ge \frac{f(u) - f(u_r)}{u - u_r} \qquad \text{for all } u \text{ between } u_l \text{ and } u_r \tag{1.3}$$

is the entropy solution, where u_l, u_r are the function values to the left and right of the shock respectively, and the shock speed s satisfies Rankine-Hugoniot jump condition [10, 8]

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{1.4}$$

The classical BL equation (1.1) with flux function $f(u)$ as given in (1.2) has been well studied (see [9] for an introduction). Let α be the solution of $f'(u) = \frac{f(u)}{u}$, i.e.,

$$\alpha = \sqrt{\frac{M}{M+1}}. \tag{1.5}$$

The entropy solution of the BL equation (1.1) can be classified into two categories:

1. If $0 < u_B \le \alpha$, the entropy solution has a single shock at $\frac{x}{t} = \frac{f(u_B)}{u_B}$.
2. If $\alpha < u_B < 1$, the entropy solution contains a rarefaction between u_B and α for $f'(u_B) < \frac{x}{t} < f'(\alpha)$ and a shock at $\frac{x}{t} = \frac{f(\alpha)}{\alpha}$.

These two types of solutions are shown in Figure 1.2 for $M = 2$. In either case, the entropy

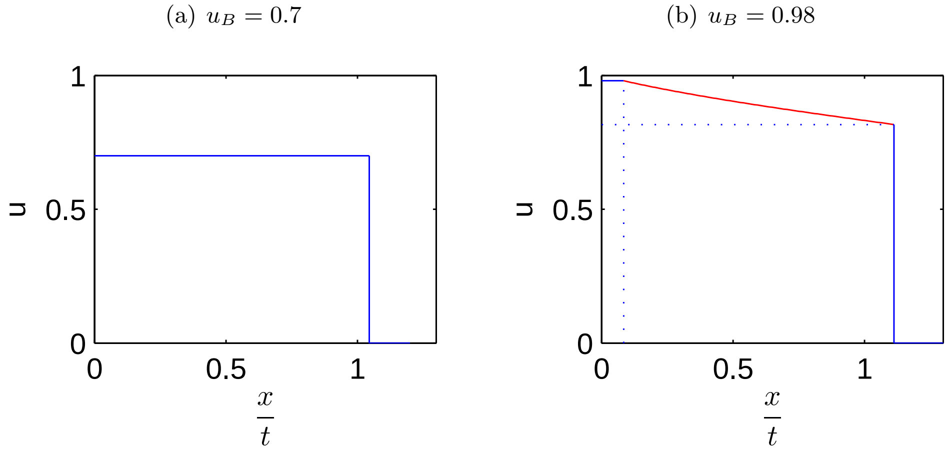

Figure 1.2: The entropy solution of the classical BL equation ($M = 2$, $\alpha = \sqrt{\frac{2}{3}} \approx 0.8165$). (a) $0 < u_B = 0.7 \le \alpha$, the solution consists of one shock at $\frac{x}{t} = \frac{f(u_B)}{u_B}$; (b) $\alpha < u_B = 0.98 < 1$, the solution consists of a rarefaction between u_B and α for $f'(u_B) < \frac{x}{t} < f'(\alpha)$ and a shock at $\frac{x}{t} = \frac{f(\alpha)}{\alpha}$.

solution of the classical BL equation (1.1) is a non-increasing function of x at any given time $t > 0$. However, the experiments of two-phase flow in porous medium reveal complex infiltration profiles, which may involve overshoot, i.e., profiles may not be monotone [5]. This suggests the need of modification to the classical BL equation.

To better describe the infiltration profiles, we go back to the origins of (1.1). Let S_i be the saturation of water/oil ($i = w, o$) and assume that the medium is completely saturated, i.e. $S_w + S_o = 1$. The conservation of mass gives

$$\phi \frac{\partial S_i}{\partial t} + \frac{\partial q_i}{\partial x} = 0 \tag{1.6}$$

where ϕ is the porosity of the medium (relative volume occupied by the pores) and q_i denotes the discharge of water/oil with $q_w + q_o = q$, which is assumed to be a constant in space due to the complete saturation assumption. Throughout of this work, we consider it constant in time as well. By Darcy's law

$$q_i = -k \frac{k_{ri}(S_i)}{\mu_i} \frac{\partial P_i}{\partial x}, \qquad i = w, o \tag{1.7}$$

where k denotes the absolute permeability, k_{ri} is the relative permeability and μ_i is the viscosity of water/oil. Instead of considering constant capillary pressure as adopted by the classical BL equation (1.1), Hassanizadeh and Gray [6, 7] have defined the dynamic capillary pressure as

$$P_c = P_o - P_w = p_c(S_w) - \phi\tau \frac{\partial S_w}{\partial t} \tag{1.8}$$

where $p_c(S_w)$ is the *static* capillary pressure and τ is a positive constant, and $\frac{\partial S_w}{\partial t}$ is the dynamic effects. Using Corey [4, 12] expressions with exponent 2, $k_{rw}(S_w) = S_w^2,\ k_{ro}(S_o) = S_o^2$, rescaling $x\frac{\phi}{q} \to x$ and combining (1.6)-(1.8), the single equation for the water saturation $u = S_w$ is

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{u^2}{u^2 + M(1-u)^2}\right] = -\frac{\partial}{\partial x}\left[\frac{\phi^2}{q^2}\frac{k(1-u)^2u^2}{\mu_w(1-u)^2 + \mu_o u^2}\frac{\partial}{\partial x}\left(\frac{p_c(u)}{\phi} - \tau\frac{\partial u}{\partial t}\right)\right] \tag{1.9}$$

where $M = \frac{\mu_w}{\mu_o}$ [14]. Linearizing the right hand side of (1.9) and rescaling the equation as in [13, 12], the modified Buckley-Leverett equation (MBL) is derived as

$$u_t + (f(u))_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt} \tag{1.10}$$

where the water fractional flow function $f(u)$ is given as in (1.2). Notice that, if P_c in (1.8) is taken to be constant, then (1.9) gives the classical BL equation; while if the dispersive parameter τ is taken to be zero, then (1.10) gives the viscous BL equation, which still displays monotone water saturation profile. The third order mixed derivative term $\epsilon^2\tau u_{xxt}$ in the MBL equation (1.10) plays an essential role. Van Duijn et al. [13] showed that the value τ is critical in determining the type of the solution profile. In particular, the solution profile of (1.10) is not monotone when τ is larger than the threshold value τ_*, which was numerically determined to be 0.61 [13]. The non-monotonicity of the solution profile is consistent with the experimental observations [5].

The classical BL equation (1.1) is hyperbolic, the MBL equation (1.10), however, is pseudo-parabolic. Unlike the finite domain of dependence for the classical BL equation (1.1), the domain of dependence for the MBL equation (1.10) is infinite. This naturally raises the question for the choice of domain size when numerical solutions are sought [15]. To answer this question, we study the MBL equation equipped with two types of domains. One is the half line boundary value problem

$$\begin{aligned} u_t + (f(u))_x &= \epsilon u_{xx} + \epsilon^2\tau u_{xxt} && \text{in} \quad Q = \{(x,t) : x > 0, t > 0\} \\ u(x,0) &= u_0(x) && x \in [0,\infty) \\ u(0,t) = g_u(t), \quad \lim_{x\to\infty} u(x,t) &= 0 && t \in [0,\infty) \\ u_0(0) &= g_u(0) && \text{compatibility condition} \end{aligned} \tag{1.11}$$

and the other one is finite interval boundary value problem

$$\begin{aligned} v_t + (f(v))_x &= \epsilon v_{xx} + \epsilon^2\tau v_{xxt} && \text{in} \quad \widetilde{Q} = \{(x,t) : x \in (0,L), t > 0\} \\ v(x,0) &= v_0(x) && x \in [0,L] \\ v(0,t) = g_v(t), \quad v(L,t) &= h(t) && t \in [0,\infty) \\ v_0(0) = g_v(0), \quad v_0(L) &= h(0) && \text{compatibility condition.} \end{aligned} \tag{1.12}$$

Considering

$$u_0(x) = \begin{cases} v_0(x) & \text{for} \quad x \in [0,L] \\ 0 & \text{for} \quad x \in [L,+\infty) \end{cases}, \qquad g_u(t) = g_v(t) \equiv g(t), \quad h(t) \equiv 0, \tag{1.13}$$

we will show the relation between the solutions of problems (1.11) and (1.12). To the best knowledge of the authors, there is no such study for MBL equation (1.10). Similar questions were answered for BBM equation [1, 2].

The water saturation during secondary recovery in oil reservoir is influenced by vast amount of factors. Given the current computing resources, it is not realistic to model the reservoir flow pattern using complete and concrete set of information, nor can one get such a detailed dataset. The MBL equation (1.10), from the practical point of view, is an extreme simplification of such a model, which does produce water saturation profiles qualitatively consistent with the experimental observations [5]. Problems (1.11) and (1.12) correspond to studying the water saturation through virtually the entire reservoir and a specific region of the reservoir respectively. For reservoir management it is usually sufficient to provide the general trends in the reservoir flow pattern for a specific region. By comparing the solutions of problems (1.11) and (1.12), it will help to determine the size of a specific region that should be taken into consideration.

The organization of this paper is as follows. Section 2 will bring forward the exact theory comparing the solutions of (1.11) and (1.12). The difference between the solutions of these two types of problems decays exponentially with respect to the length of the interval L for Riemann problem. This provides a theoretical justification for the choice of the computational domain. Section 3 provides the numerical comparison of the solutions of (1.11) and (1.12). The computational results show that the difference between the solutions of these two types of problems indeed decays exponentially with respect to L, and nearly exponentially with respect to $\frac{1}{\epsilon}$ as well. The numerical results also confirm that the water saturation profile strongly depends on the dispersive parameter τ value as studied in [13]. For $\tau > \tau_*$, the MBL equation (1.10) gives non-monotone water saturation profiles for certain Riemann problems as suggested by experimental observations [5]. Section 4 gives the conclusion of the paper and the possible future directions.

2. The half line problem versus the finite interval problem

Let $u(x,t)$ be the solution to the half line problem (1.11), and let $v(x,t)$ be the solution to the finite interval problem (1.12). We consider the natural assumptions (1.13). The goal of this section is to develop an estimate of the difference between u and v on the spatial interval $[0, L]$ at a given finite time t. The main result of this section is

Theorem 2.1 (The Main Theorem)**.** *If $u_0(x)$ satisfies*

$$u_0(x) = \begin{cases} C_u & x \in [0, L_0] \\ 0 & x > L_0 \end{cases} \tag{2.1}$$

where $L_0 < L$ and C_u are positive constants; $u(x,t)$ and $v(s,t)$ are solutions to (1.11) and (1.12) respectively; and assumption (1.13) holds, then

$$\| u(\cdot,t) - v(\cdot,t) \|_{H^1_{L,\epsilon,\tau}} \le D_{1;\epsilon,\tau}(t) e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + D_{2;\epsilon,\tau}(t) e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \tag{2.2}$$

for some $0 < \lambda < 1$, $D_{1;\epsilon,\tau}(t) > 0$ and $D_{2;\epsilon,\tau}(t) > 0$, where

$$\| Y(\cdot,t) \|_{H^1_{L,\epsilon,\tau}} := \sqrt{\int_0^L Y(x,t)^2 + (\epsilon\sqrt{\tau} Y_x(x,t))^2 \, dx}. \tag{2.3}$$

Notice that with the initial condition (2.1) we are considering a Riemann problem. Theorem 2.1 shows that the solution to the half line problem (1.11) can be approximated as accurately as one wants by the solution to the finite interval problem (1.12) in the sense that $D_{1;\epsilon,\tau}(t)$, $D_{2;\epsilon,\tau}(t)$, $\frac{\lambda L}{\epsilon\sqrt{\tau}}$ and $\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}$ can be controlled.

To prove theorem 2.1, we first derive the implicit solution formulae for the half line problem and the finite interval problem in section 2.1 and section 2.2 respectively. The implicit solution formulae are in integral form, which are derived by separating the x-derivative from the t-derivative, and formally solving a first order linear ODE in t and a second order non-homogeneous ODE in x. In section 2.3, we use Gronwall's inequality multiple times to obtain the desired result in Theorem 2.1.

2.1. Half line problem

In this section, we derive the implicit solution formula for the half line problem (1.11) (with $g_u(t) = g(t)$ as given in (1.13)). To solve (1.11), we first rewrite (1.11) by separating the x-derivative from the t-derivative,

$$\left(I - \epsilon^2\tau\partial_{xx}\right)\left(u_t + \frac{1}{\epsilon\tau}u\right) = \frac{1}{\epsilon\tau}u - (f(u))_x. \tag{2.4}$$

By using integrating factor method, we formally integrate (2.4) over $[0, t]$ to obtain

$$\left(I - \epsilon^2\tau\partial_{xx}\right)\left(u - e^{-\frac{t}{\epsilon\tau}}u_0\right) = \int_0^t \left(\frac{1}{\epsilon\tau}u - (f(u))_x\right) e^{-\frac{t-s}{\epsilon\tau}}\, ds. \tag{2.5}$$

Furthermore, we let

$$A = u - e^{-\frac{t}{\epsilon\tau}}u_0, \tag{2.6}$$

then (2.5) can be written as

$$A'' - \frac{1}{\epsilon^2\tau}A = \int_0^t \left(-\frac{1}{\epsilon^3\tau^2}u + \frac{1}{\epsilon^2\tau}(f(u))_x\right) e^{-\frac{t-s}{\epsilon\tau}}\, ds, \qquad \text{where } ' = \frac{d}{dx}. \tag{2.7}$$

Notice that (2.7) is a second-order non-homogeneous ODE in x-variable along with the boundary conditions

$$\begin{aligned} A(0,t) &= u(0,t) - e^{-\frac{t}{\epsilon\tau}}u_0(0) = g(t) - e^{-\frac{t}{\epsilon\tau}}g(0), \\ A(\infty,t) &= u(\infty,t) - e^{-\frac{t}{\epsilon\tau}}u_0(\infty) = 0. \end{aligned} \tag{2.8}$$

To solve (2.7), we first solve the corresponding linear homogeneous equation with the non-zero boundary conditions (2.8). We then find a particular solution for the non-homogeneous equation with zero boundary conditions by introducing a Green's function $G(x,\xi)$ and a kernel $K(x,\xi)$ for the non-homogeneous terms u and $(f(u))_x$ respectively. Combining the solutions for the two non-homogeneous terms and the homogeneous part with boundary conditions, we get the solution for equation (2.7) satisfying the boundary conditions (2.8):

$$\begin{aligned} A(x,t) &= -\frac{1}{\epsilon^3\tau^2}\int_0^t\int_0^{+\infty} G(x,\xi)u(\xi,s)e^{-\frac{t-s}{\epsilon\tau}}\, d\xi\, ds \\ &\quad + \frac{1}{\epsilon^2\tau}\int_0^t\int_0^{+\infty} K(x,\xi)f(u)e^{-\frac{t-s}{\epsilon\tau}}\, d\xi\, ds \\ &\quad + \left(g(t) - e^{-\frac{t}{\epsilon\tau}}g(0)\right)e^{-\frac{x}{\epsilon\sqrt{\tau}}} \end{aligned} \tag{2.9}$$

where the Green's function $G(x,\xi)$ and the kernel $K(x,\xi)$ are

$$G(x,\xi) = \frac{\epsilon\sqrt{\tau}}{2}\left(e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right), \tag{2.10}$$

$$K(x,\xi) = -\frac{\partial G(x,\xi)}{\partial \xi} = \frac{1}{2}\left(e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right). \tag{2.11}$$

To recover the solution for the half line problem (1.11), we refer to the definition of A in (2.6). Thus, the implicit solution formula for the half line problem (1.11) is

$$\begin{aligned} u(x,t) = & -\frac{1}{2\epsilon^2\tau\sqrt{\tau}}\int_0^t\int_0^{+\infty}\left(e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right)u(\xi,s)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & +\frac{1}{2\epsilon^2\tau}\int_0^t\int_0^{+\infty}\left(e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right)f(u)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & +\left(g(t) - e^{-\frac{t}{\epsilon\tau}}g(0)\right)e^{-\frac{x}{\epsilon\sqrt{\tau}}} + e^{-\frac{t}{\epsilon\tau}}u_0(x). \end{aligned} \tag{2.12}$$

2.2. Finite interval problem

The implicit solution for the finite interval problem (1.12) (with $g_v(t) = g(t)$ as given in (1.13)) can be solved in a similar way. The only difference is that the additional boundary condition $h(t)$ at $x = L$ in (1.12) gives different boundary conditions for the non-homogeneous ODE in x-variable. Denote

$$A^L = v - e^{-\frac{t}{\epsilon\tau}}v_0, \tag{2.13}$$

then it satisfies

$$(A^L)'' - \frac{1}{\epsilon^2\tau}A^L = \int_0^t\left(-\frac{1}{\epsilon^3\tau^2}v + \frac{1}{\epsilon^2\tau}(f(v))_x\right)e^{-\frac{t-s}{\epsilon\tau}}\,ds \quad \text{where} \quad ' = \frac{d}{dx} \tag{2.14}$$

with the boundary conditions

$$\begin{aligned} A^L(0,t) &= v(0,t) - e^{-\frac{t}{\epsilon\tau}}v_0(0) = g(t) - e^{-\frac{t}{\epsilon\tau}}g(0), \\ A^L(L,t) &= v(L,t) - e^{-\frac{t}{\epsilon\tau}}v_0(L) = h(t) - e^{-\frac{t}{\epsilon\tau}}h(0). \end{aligned}$$

These boundary conditions affect both the homogeneous solution and the particular solution of (2.14) as follows

$$\begin{aligned} A^L(x,t) = & -\frac{1}{\epsilon^3\tau^2}\int_0^t\int_0^L G^L(x,\xi)v(\xi,s)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & +\frac{1}{\epsilon^2\tau}\int_0^t\int_0^L K^L(x,\xi)f(v)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & +c_1(t)\phi_1(x) + c_2(t)\phi_2(x) \end{aligned} \tag{2.15}$$

where the Green's function $G^L(x,\xi)$, the kernel $K^L(x,\xi)$ and the bases for the homogeneous solutions are

$$G^L(x,\xi) = \frac{\epsilon\sqrt{\tau}}{2(e^{\frac{2L}{\epsilon\sqrt{\tau}}}-1)}\left(e^{\frac{x+\xi}{\epsilon\sqrt{\tau}}}+e^{\frac{2L-(x+\xi)}{\epsilon\sqrt{\tau}}}-e^{\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}-e^{\frac{2L-|x-\xi|}{\epsilon\sqrt{\tau}}}\right), \tag{2.16}$$

$$\begin{aligned} K^L(x,\xi) = & -\frac{1}{2(e^{\frac{2L}{\epsilon\sqrt{\tau}}}-1)}\left(e^{\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{\frac{2L-(x+\xi)}{\epsilon\sqrt{\tau}}}\right. \\ & \left.+\mathrm{sgn}(x-\xi)e^{\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}-\mathrm{sgn}(x-\xi)e^{\frac{2L-|x-\xi|}{\epsilon\sqrt{\tau}}}\right), \end{aligned} \tag{2.17}$$

$$c_1(t) = g(t)-e^{-\frac{t}{\epsilon\tau}}g(0), \qquad c_2(t)=h(t)-e^{-\frac{t}{\epsilon\tau}}h(0), \tag{2.18}$$

$$\phi_1(x) = \frac{e^{\frac{L-x}{\epsilon\sqrt{\tau}}}-e^{\frac{-L+x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}}-e^{-\frac{L}{\epsilon\sqrt{\tau}}}}, \qquad \phi_2(x)=\frac{e^{\frac{x}{\epsilon\sqrt{\tau}}}-e^{-\frac{x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}}-e^{-\frac{L}{\epsilon\sqrt{\tau}}}}. \tag{2.19}$$

Thus, the implicit solution formula for the finite interval problem (1.12) is

$$\begin{aligned} v(x,t) = & -\frac{1}{2\epsilon^2\tau\sqrt{\tau}(e^{\frac{2L}{\epsilon\sqrt{\tau}}}-1)}\int_0^t\int_0^L\left(e^{\frac{x+\xi}{\epsilon\sqrt{\tau}}}+e^{\frac{2L-(x+\xi)}{\epsilon\sqrt{\tau}}}-e^{\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right. \\ & \left.-e^{\frac{2L-|x-\xi|}{\epsilon\sqrt{\tau}}}\right)v(\xi,s)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & -\frac{1}{2\epsilon^2\tau(e^{\frac{2L}{\epsilon\sqrt{\tau}}}-1)}\int_0^t\int_0^L\left(e^{\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{\frac{2L-(x+\xi)}{\epsilon\sqrt{\tau}}}+\mathrm{sgn}(x-\xi)e^{\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right. \\ & \left.-\mathrm{sgn}(x-\xi)e^{\frac{2L-|x-\xi|}{\epsilon\sqrt{\tau}}}\right)f(v)e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds \\ & +c_1(t)\phi_1(x)+c_2(t)\phi_2(x)+e^{-\frac{t}{\epsilon\tau}}v_0(x). \end{aligned} \tag{2.20}$$

2.3. Comparisons

In this section, we will prove that the solution $u(x,t)$ to the half line problem can be approximated as accurately as one wants by the solution $v(x,t)$ to the finite interval problem as stated in Theorem 2.1.

Due to the difference in the integration domains, we do not use (2.12) and (2.20) directly for the comparison. Instead, we decompose $u(x,t)$ ($v(x,t)$ respectively) into two parts: $U(x,t)$ and $u_L(x,t)$ ($V(x,t)$ and $v_L(x,t)$ respectively), such that $U(x,t)$ ($V(x,t)$ respectively) has zero initial condition and boundary conditions at $x=0$ and $x=L$. We estimate the difference between $u(\cdot,t)$ and $v(\cdot,t)$ by estimating the differences between $u_L(\cdot,t)$ and $v_L(\cdot,t)$, $U(\cdot,t)$ and $V(\cdot,t)$, then applying the triangle inequality.

2.3.1. Definitions and lemmas

To assist the proof of Theorem 2.1 in section 2.3.3, we introduce some new notations in this section. We first decompose $u(x,t)$ as sum of two terms $U(x,t)$ and $u_L(x,t)$, such that

$$u(x,t)=U(x,t)+u_L(x,t) \qquad x\in[0,+\infty)$$

where

$$u_L = e^{-\frac{t}{\epsilon\tau}} u_0(x) + c_1(t) e^{-\frac{x}{\epsilon\sqrt{\tau}}} + \left(u(L,t) - c_1(t) e^{-\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{t}{\epsilon\tau}} u_0(L) \right) \phi_2(x) \tag{2.21}$$

and $c_1(t)$ and $\phi_2(x)$ are given in (2.18) and (2.19) respectively. With this definition, u_L takes care of the initial condition $u_0(x)$ and boundary conditions $g(t)$ at both $x = 0$ and $x = L$ for $u(x,t)$. Then U satisfies an equation slightly different from the equation u satisfies in (1.11):

$$\begin{aligned} U_t - \epsilon U_{xx} - \epsilon^2 \tau U_{xxt} &= \left(u_t - \epsilon u_{xx} - \epsilon^2 \tau u_{xxt} \right) - \left((u_L)_t - \epsilon (u_L)_{xx} - \epsilon^2 \tau (u_L)_{xxt} \right) \\ &= -\left(f(u) \right)_x + \frac{1}{\epsilon\tau} u_L(x,t) \end{aligned} \tag{2.22}$$

In addition, $U(x,t)$ has zero initial and boundary conditions on $[0, L]$, i.e.,

$$U(x,0) = 0, \qquad U(0,t) = 0, \qquad U(L,t) = 0. \tag{2.23}$$

Similarly, for $v(x,t)$, let

$$v(x,t) = V(x,t) + v_L(x,t) \qquad x \in [0, L]$$

where

$$v_L = e^{-\frac{t}{\epsilon\tau}} v_0(x) + c_1(t)\phi_1(x) + c_2(t)\phi_2(x) \tag{2.24}$$

and $c_1(t)$, $c_2(t)$ and $\phi_1(x)$, $\phi_2(x)$ are given in (2.18) and (2.19) respectively. With this definition, v_L takes care of the initial condition $v_0(x)$ and boundary conditions $g(t)$ and $h(t)$ at $x = 0$ and $x = L$ for $v(x,t)$. Then V satisfies an equation slightly different from the equation v satisfies in (1.12):

$$V_t - \epsilon V_{xx} - \epsilon^2 \tau V_{xxt} = -\left(f(v) \right)_x + \frac{1}{\epsilon\tau} v_L(x,t) \tag{2.25}$$

with

$$V(x,0) = 0, \qquad V(0,t) = 0, \qquad V(L,t) = 0. \tag{2.26}$$

Since, in the end, we want to study the difference between $U(x,t)$ and $V(x,t)$, we define

$$W(x,t) = V(x,t) - U(x,t) \qquad \text{for} \qquad x \in [0, L].$$

Because of (2.22) and (2.25), we have

$$W_t - \epsilon W_{xx} - \epsilon^2 \tau W_{xxt} = -\left(f(v) - f(u) \right)_x + \frac{1}{\epsilon\tau} (v_L - u_L). \tag{2.27}$$

In lieu of (2.23) and (2.26), $W(x,t)$ also satisfies

$$W(x,0) = 0, \qquad W(0,t) = 0, \qquad W(L,t) = 0. \tag{2.28}$$

Now, to estimate $\| u - v \|$, we can estimate $\| W \| = \| V - U \|$ and estimate $\| u_L - v_L \|$ separately. These estimates are done in section 2.3.3.

Next, we state the lemmas needed in the proof of Theorem 2.1. The proof of the lemmas can be found in the appendix Appendix A and [14]. In all the lemmas, we assume $0 < \lambda < 1$, $\epsilon > 0$, $\tau > 0$ and $u_0(x)$ satisfies (2.1).

Lemma 2.2. $f(u) = \frac{u^2}{u^2+M(1-u)^2} \leq Du$ *where* $D = \frac{f(\alpha)}{\alpha}$ *and* $\alpha = \sqrt{\frac{M}{M+1}}$.

Lemma 2.3. *(i)* $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \lambda\xi}{\epsilon\sqrt{\tau}}} \, d\xi \leq \frac{2\epsilon\sqrt{\tau}}{1-\lambda^2}$.

(ii) $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \xi}{\epsilon\sqrt{\tau}}} \, d\xi \leq \frac{\epsilon\sqrt{\tau}}{e(1-\lambda)}$.

(iii) $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} |u_0(\xi)| \, d\xi \leq 2C_u \epsilon\sqrt{\tau} e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}$.

Lemma 2.4. *(i)* $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \mathrm{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \lambda\xi}{\epsilon\sqrt{\tau}}} \, d\xi \leq \frac{2\epsilon\sqrt{\tau}}{1-\lambda^2}$.

(ii) $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \mathrm{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \xi}{\epsilon\sqrt{\tau}}} \, d\xi \leq \epsilon\sqrt{\tau} + \frac{\epsilon\sqrt{\tau}}{e(1-\lambda)}$.

(iii) $\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \mathrm{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} |u_0(\xi)| \, d\xi \leq 2C_u \epsilon\sqrt{\tau} e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}$.

Lemma 2.5. *(i)* $\left| \phi_1(x) - e^{-\frac{x}{\epsilon\sqrt{\tau}}} \right| = e^{-\frac{L}{\epsilon\sqrt{\tau}}} |\phi_2(x)|$.

(ii) $|\phi_2(x)| \leq 1$ *for* $x \in [0, L]$.

(iii) $|\phi_2'(x)| \leq \frac{2}{\epsilon\sqrt{\tau}}$ *if* $\epsilon \ll 1$ *for* $x \in [0, L]$.

Notice that the constraint $\lambda \in (0,1)$ is crucial in Lemmas 2.3, 2.4. Last but not least, the norm that is used in Theorem 2.1 and its proof is given in (2.3).

2.3.2. A proposition

In this section, we will give a critical estimate, which is essential in the calculation of maximum difference $\| u_L(\cdot,t) - v_L(\cdot,t) \|_\infty$ in section 2.3.3. By comparing $u_L(x,t)$ and $v_L(x,t)$ given in (2.21) and (2.24) respectively, it is clear that the coefficient $u(L,t) - c_1(t)e^{-\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{t}{\epsilon\tau}} u_0(L)$ for $\phi_2(x)$ appeared in (2.21) needs to be compared with the corresponding coefficient $c_2(t)$ for $\phi_2(x)$ appeared in (2.24). We thus define a space-dependent function

$$U_{c_2}(x,t) = u(x,t) - c_1(t)e^{-\frac{x}{\epsilon\sqrt{\tau}}} - e^{-\frac{t}{\epsilon\tau}} u_0(x) \tag{2.29}$$

and establish the following proposition

Proposition 2.6.

$$|U_{c_2}(L,t)| \leq a_\tau(t) e^{\frac{b_\tau t}{\epsilon\tau}} e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + c_\tau \frac{t}{\epsilon\tau} e^{\frac{(b_\tau - 1)t}{\epsilon\tau}} e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \tag{2.30}$$

for some positive τ-dependent constants a_τ, b_τ and c_τ.

Proof. Based on the implicit solution formula (2.12), Lemma 2.2 and the relationship between U_{c_2} and u (2.29), we can get an inequality in terms of U_{c_2}

$$\begin{aligned}
|U_{c_2}(x,t)| \leq\ & \frac{1}{2\epsilon^2\tau\sqrt{\tau}}\left[\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||U_{c_2}(\xi,s)|\,e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds\right.\\
&+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||c_1(s)|\,e^{-\frac{\xi}{\epsilon\sqrt{\tau}}}e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds\\
&\left.+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||u_0(\xi)|\,e^{-\frac{t}{\epsilon\tau}}\,d\xi\,ds\right]\\
&+\frac{D}{2\epsilon^2\tau}\left[\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||U_{c_2}(\xi,s)|\,e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds\right.\\
&+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||c_1(s)|\,e^{-\frac{\xi}{\epsilon\sqrt{\tau}}}e^{-\frac{t-s}{\epsilon\tau}}\,d\xi\,ds\\
&\left.+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||u_0(\xi)|\,e^{-\frac{t}{\epsilon\tau}}\,d\xi\,ds\right].
\end{aligned} \tag{2.31}$$

To show that $U_{c_2}(x,t)$ decays exponentially with respect to x, we pull out an exponential term by writing $U_{c_2}(x,t) = e^{-\frac{\lambda x}{\epsilon\sqrt{\tau}}}e^{-\frac{t}{\epsilon\tau}}\tilde{U}(x,t)$, where $0<\lambda<1$, such that

$$\tilde{U}(x,t) = e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}e^{\frac{t}{\epsilon\tau}}U_{c_2}(x,t), \tag{2.32}$$

then (2.31) can be rewritten in terms of $\tilde{U}(x,t)$ as follows

$$\begin{aligned}
\left|\tilde{U}(x,t)\right| \leq\ & \frac{1}{2\epsilon^2\tau\sqrt{\tau}}\left[\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right|e^{\frac{\lambda x-\lambda\xi}{\epsilon\sqrt{\tau}}}\left|\tilde{U}(\xi,s)\right|\,d\xi\,ds\right.\\
&+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||c_1(s)|\,e^{\frac{\lambda x-\xi}{\epsilon\sqrt{\tau}}}e^{\frac{s}{\epsilon\tau}}\,d\xi\,ds\\
&\left.+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}-e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right|e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}\,|u_0(\xi)|\,d\xi\,ds\right]\\
&+\frac{D}{2\epsilon^2\tau}\left[\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right|e^{\frac{\lambda x-\lambda\xi}{\epsilon\sqrt{\tau}}}\left|\tilde{U}_(\xi,s)\right|\,d\xi\,ds\right.\\
&+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right||c_1(s)|\,e^{\frac{\lambda x-\xi}{\epsilon\sqrt{\tau}}}e^{\frac{s}{\epsilon\tau}}\,d\xi\,ds\\
&\left.+\int_0^t\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}}+\operatorname{sgn}(x-\xi)e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right|e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}\,|u_0(\xi)|\,d\xi\,ds\right].
\end{aligned} \tag{2.33}$$

Because of Lemmas 2.3–2.4, we can get the following estimate for $\left|\tilde{U}(\cdot,t)\right|_\infty$

$$
\begin{aligned}
\left|\tilde{U}(\cdot,t)\right|_\infty \leq\ & \frac{1}{2\epsilon^2\tau\sqrt{\tau}}\left[\frac{2\epsilon\sqrt{\tau}}{1-\lambda^2}\int_0^t |\tilde{U}(\cdot,s)|_\infty\, ds + \frac{\epsilon\sqrt{\tau}}{e(1-\lambda)}\int_0^t |c_1(s)|e^{\frac{s}{\epsilon\tau}}\, ds\right.\\
& \left. +2C_u\epsilon\sqrt{\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}\int_0^t 1\, ds\right]\\
& +\frac{D}{2\epsilon^2\tau}\left[\frac{2\epsilon\sqrt{\tau}}{1-\lambda^2}\int_0^t |\tilde{U}(\cdot,s)|_\infty\, ds + \epsilon\sqrt{\tau}\left(1+\frac{1}{e(1-\lambda)}\right)\int_0^t |c_1(s)|e^{\frac{s}{\epsilon\tau}}\, ds\right.\\
& \left. +2C_u\epsilon\sqrt{\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}\int_0^t 1\, ds\right]\\
\leq\ & \int_0^t \frac{b_\tau}{\epsilon\tau}|\tilde{U}(\cdot,s)|_\infty\, ds + \int_0^t \frac{\tilde{a}_\tau(s)}{\epsilon\tau}\, ds
\end{aligned}
\tag{2.34}
$$

where

$$
\begin{aligned}
b_\tau &= \frac{1+D\sqrt{\tau}}{1-\lambda^2}, \qquad \tilde{a}_\tau(t) = a_\tau e^{\frac{t}{\epsilon\tau}} + c_\tau e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}},\\
a_\tau &= \frac{|c_1(\cdot)|_\infty(1+D\sqrt{\tau}(e(1-\lambda)+1))}{2e(1-\lambda)}, \quad c_\tau = C_u(1+D\sqrt{\tau}).
\end{aligned}
$$

By Gronwall's inequality, inequality (2.34) gives that

$$
\left|\tilde{U}(\cdot,t)\right|_\infty \leq \int_0^t \frac{\tilde{a}_\tau(t-s)}{\epsilon\tau}e^{\frac{b_\tau(t-s)}{\epsilon\tau}}\, ds \ \leq\ \left(a_\tau e^{\frac{t}{\epsilon\tau}} + c_\tau\frac{t}{\epsilon\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}\right)e^{\frac{b_\tau t}{\epsilon\tau}}.
$$

Hence $|U_{c_2}(x,t)| \leq \left|\tilde{U}(\cdot,t)\right|_\infty e^{\frac{-\lambda x}{\epsilon\sqrt{\tau}}}e^{-\frac{t}{\epsilon\tau}} \leq \left(a_\tau e^{\frac{t}{\epsilon\tau}} + c_\tau\frac{t}{\epsilon\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}\right)e^{\frac{b_\tau t}{\epsilon\tau}}e^{\frac{-\lambda x}{\epsilon\sqrt{\tau}}}e^{-\frac{t}{\epsilon\tau}}$ i.e., $U_{c_2}(x,t)$ decays exponentially with respect to x. In particular, when $x=L$, we have

$$
|U_{c_2}(L,t)| \leq a_\tau e^{\frac{b_\tau t}{\epsilon\tau}}e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + c_\tau\frac{t}{\epsilon\tau}e^{\frac{(b_\tau-1)t}{\epsilon\tau}}e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \tag{2.35}
$$

as given in (2.30). □

2.3.3. Proof of Theorem 2.1

In this section, we will first find the maximum difference of $\| u_L(\cdot,t)-v_L(\cdot,t)\|_\infty$, then we will derive $\| u_L(\cdot,t)-v_L(\cdot,t)\|_{H^1_{L,\epsilon,\tau}}$ and $\| W(\cdot,t)\|_{H^1_{L,\epsilon,\tau}} = \| U(\cdot,t)-V(\cdot,t)\|_{H^1_{L,\epsilon,\tau}}$. Combining these two, we will get the estimate for $\| u(\cdot,t)-v(\cdot,t)\|_{H^1_{L,\epsilon,\tau}}$ as given in (2.2).

Proposition 2.7. *If $u_0(x)$ satisfies (2.1), then*

$$
\| u_L - v_L \|_\infty \leq E_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}
$$

where $E_{1;\epsilon,\tau}(t) = |c_1(\cdot)|_\infty + a_\tau e^{\frac{b_\tau t}{\epsilon\tau}}$ and $E_{2;\epsilon,\tau}(t) = c_\tau\frac{t}{\epsilon\tau}e^{\frac{(b_\tau-1)t}{\epsilon\tau}}$.

Proof. By the definition of u_L and v_L given in (2.21) and (2.24) and the assumption that $u_0(x) = v_0(x)$ for $x \in [0, L]$, we can get their difference

$$u_L(x,t) - v_L(x,t) = c_1(t)\left(e^{-\frac{x}{\epsilon\sqrt{\tau}}} - \phi_1(x)\right) + \left(U_{c_2}(L,t) - h(t) + e^{-\frac{t}{\epsilon\tau}}h(0)\right)\phi_2(x)$$

Combining Lemmas 2.5(i), 2.5(ii), inequality (2.35), and $h(t) \equiv 0$, we have

$$\| u_L(\cdot,t) - v_L(\cdot,t) \|_\infty \le E_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \tag{2.36}$$

where

$$E_{1;\epsilon,\tau}(t) = |c_1(\cdot)|_\infty + a_\tau e^{\frac{b_\tau t}{\epsilon\tau}} \quad \text{and} \quad E_{2;\epsilon,\tau}(t) = c_\tau \frac{t}{\epsilon\tau} e^{\frac{(b_\tau - 1)t}{\epsilon\tau}}. \tag{2.37}$$

□

Proposition 2.8. *If $u_0(x)$ satisfies (2.1), and $E_{1;\epsilon,\tau}(t), E_{2;\epsilon,\tau}(t)$ are as in proposition 2.7, then*

$$\| u_L(\cdot,t) - v_L(\cdot,t) \|_{H^1_{L,\epsilon,\tau}} \le \sqrt{5L}\left(E_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}\right).$$

Proof. Because of the definition of u_L and v_L given in (2.21) and (2.24), Lemma 2.5(iii) and inequality (2.35), we have that

$$\begin{aligned} \| (u_L(\cdot,t) - v_L(\cdot,t))_x \|_\infty &\le |c_1(t)|\, e^{-\frac{L}{\epsilon\sqrt{\tau}}} \left|\phi_2'(x)\right| + |U_{c_2}(L,t)| \left|\phi_2'(x)\right| \\ &\le \frac{2}{\epsilon\sqrt{\tau}}\left(E_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}\right). \end{aligned} \tag{2.38}$$

Now, combining (2.36) and (2.38), we obtain that

$$\begin{aligned} \| u_L(\cdot,t) - v_L(\cdot,t) \|_{H^1_{L,\epsilon,\tau}} &= \sqrt{\int_0^L |u_L - v_L|^2 + \left|\epsilon\sqrt{\tau}\,(u_L - v_L)_x\right|^2 dx} \\ &\le \sqrt{5L}\left(E_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}\right). \end{aligned} \tag{2.39}$$

□

Proposition 2.9. *If $u_0(x)$ satisfies (2.1), then*

$$\| W(\cdot,t) \|_{H^1_{L,\epsilon,\tau}} \le \gamma_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + \gamma_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}$$

where the coefficients are given by

$$\begin{aligned} \gamma_{1;\epsilon,\tau}(t) &= e^{\frac{(M+1)^2 t}{2M\epsilon\sqrt{\tau}}}\left(\frac{(M+1)^2\sqrt{\tau}}{2M} + 1\right)\sqrt{L}\left(\frac{t}{\epsilon\tau}|c_1(\cdot)|_\infty + \frac{a_\tau}{b_\tau}(e^{\frac{b_\tau t}{\epsilon\tau}} - 1)\right) \\ \gamma_{2;\epsilon,\tau}(t) &= e^{\frac{(M+1)^2 t}{2M\epsilon\sqrt{\tau}}}\left(\frac{(M+1)^2\sqrt{\tau}}{2M} + 1\right)\sqrt{L}c_\tau \\ &\quad \cdot\left(\frac{t}{\epsilon\tau(b_\tau - 1)}e^{\frac{(b_\tau - 1)t}{\epsilon\tau}} - \frac{1}{(b_\tau - 1)^2}(e^{\frac{(b_\tau - 1)t}{\epsilon\tau}} - 1)\right). \end{aligned} \tag{2.40}$$

Proof. Multiplying the governing equation of W (2.27) by $2W$, integrating over $[0, L]$, and using integration by parts, we get

$$\begin{aligned} & \frac{d}{dt}\int_0^L W^2 + (\epsilon\sqrt{\tau}W_x)^2\,dx \\ = & -\epsilon\int_0^L 2W_x^2\,dx + \int_0^L 2W_x\,(f(v)-f(u))\,dx + \frac{2}{\epsilon\tau}\int_0^L W(v_L-u_L)\,dx. \end{aligned}$$

Therefore, using the norm (2.3), and $f'(u) \le \frac{(M+1)^2}{2M} := C$, we have

$$\begin{aligned} & \frac{d}{dt}\parallel W(\cdot,t)\parallel^2_{H^1_{L,\epsilon,\tau}} \\ \le & \; 2\int_0^L |W_x||f'(\eta)||v-u|\,dx + \frac{2\sqrt{L}}{\epsilon\tau}\parallel v_L-u_L\parallel_\infty \parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \\ \le & \; 2C\int_0^L |W_x|\left(|W| + \parallel v_L-u_L\parallel_\infty\right)\,dx + \frac{2\sqrt{L}}{\epsilon\tau}\parallel v_L-u_L\parallel_\infty \parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \\ \le & \; \frac{2C}{\epsilon\sqrt{\tau}}\left(\parallel W(\cdot,t)\parallel^2_{H^1_{L,\epsilon,\tau}} + \parallel v_L-u_L\parallel_\infty \sqrt{L}\parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}}\right) \\ & \quad + \frac{2\sqrt{L}}{\epsilon\tau}\parallel v_L-u_L\parallel_\infty \parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \\ = & \; \frac{2C}{\epsilon\sqrt{\tau}}\parallel W(\cdot,t)\parallel^2_{H^1_{L,\epsilon,\tau}} + \left(\frac{2C}{\epsilon\sqrt{\tau}} + \frac{2}{\epsilon\tau}\right)\sqrt{L}\parallel v_L-u_L\parallel_\infty \parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}}. \end{aligned}$$

Hence,

$$\frac{d}{dt}\parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \le \frac{C}{\epsilon\sqrt{\tau}}\parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} + \left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\parallel v_L-u_L\parallel_\infty.$$

By Gronwall's inequality and (2.36)

$$\begin{aligned} & \parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \\ \le & \int_0^t \left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\parallel v_L-u_L\parallel_\infty e^{\frac{C(t-s)}{\epsilon\sqrt{\tau}}}\,ds \\ \le & \; e^{\frac{Ct}{\epsilon\sqrt{\tau}}}\left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\int_0^t E_{1;\epsilon,\tau}(s)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + E_{2;\epsilon,\tau}(s)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}\,ds \\ \le & \left(e^{\frac{Ct}{\epsilon\sqrt{\tau}}}\left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\int_0^t E_{1;\epsilon,\tau}(s)\,ds\right)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} \\ & + \left(e^{\frac{Ct}{\epsilon\sqrt{\tau}}}\left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\int_0^t E_{2;\epsilon,\tau}(s)\,ds\right)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \\ \le & \; e^{\frac{Ct}{\epsilon\sqrt{\tau}}}\left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\left(t|c_1(\cdot)|_\infty + \frac{a_\tau\epsilon\tau}{b_\tau}(e^{\frac{b_\tau t}{\epsilon\tau}}-1)\right)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} \\ & + e^{\frac{Ct}{\epsilon\sqrt{\tau}}}\left(\frac{C}{\epsilon\sqrt{\tau}} + \frac{1}{\epsilon\tau}\right)\sqrt{L}\frac{c_\tau}{\epsilon\tau}\left(\frac{\epsilon\tau}{b_\tau-1}te^{\frac{(b_\tau-1)t}{\epsilon\tau}} - (\frac{\epsilon\tau}{b_\tau-1})^2(e^{\frac{(b_\tau-1)t}{\epsilon\tau}}-1)\right)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}. \end{aligned}$$

Hence

$$\parallel W(\cdot,t)\parallel_{H^1_{L,\epsilon,\tau}} \le \gamma_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + \gamma_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}}$$

where $\gamma_{1;\epsilon,\tau}(t)$ and $\gamma_{2;\epsilon,\tau}(t)$ are given in (2.40). □

Now we prove Theorem 2.1.

Proof to Theorem 2.1. Combine Propositions 2.9 and 2.8

$$\begin{aligned} \| u(\cdot,t)-v(\cdot,t)\|_{H^1_{L,\epsilon,\tau}} &\leq \| W(\cdot,t)\|_{H^1_{L,\epsilon,\tau}} + \| v_L(\cdot,t)-u_L(\cdot,t)\|_{H^1_{L,\epsilon,\tau}} \\ &= D_{1;\epsilon,\tau}(t)e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}} + D_{2;\epsilon,\tau}(t)e^{-\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}} \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} & D_{1;\epsilon,\tau}(t) = \gamma_{1;\epsilon,\tau}(t) + \sqrt{5L}E_{1;\epsilon,\tau}(t) \\ = \;& e^{\frac{(M+1)^2 t}{2M\epsilon\sqrt{\tau}}}\left(\frac{(M+1)^2\sqrt{\tau}}{2M}+1\right)\sqrt{L}\left(\frac{t}{\epsilon\tau}|c_1(\cdot)|_\infty + \frac{a_\tau}{b_\tau}(e^{\frac{b_\tau t}{\epsilon\tau}}-1)\right) \\ & +\sqrt{5L}(|c(\cdot)|_\infty + a_\tau e^{\frac{b_\tau t}{\epsilon\tau}}), \\ & D_{2;\epsilon,\tau}(t) = \gamma_{2;\epsilon,\tau}(t) + \sqrt{5L}E_{2;\epsilon,\tau}(t) \\ = \;& e^{\frac{(M+1)^2 t}{2M\epsilon\sqrt{\tau}}}\left(\frac{(M+1)^2\sqrt{\tau}}{2M}+1\right)\sqrt{L}c_\tau\cdot\left(\frac{t}{\epsilon\tau(b_\tau-1)}e^{\frac{(b_\tau-1)t}{\epsilon\tau}} - \frac{1}{(b_\tau-1)^2}(e^{\frac{(b_\tau-1)t}{\epsilon\tau}}-1)\right) \\ & +\sqrt{5L}c_\tau\frac{t}{\epsilon\tau}e^{\frac{(b_\tau-1)t}{\epsilon\tau}}. \end{aligned}$$

□

This result shows that $\| u(\cdot,t)-v(\cdot,t)\|_{H^1_{L,\epsilon,\tau}}$ exponentially decays in L, i.e., if $\frac{\lambda L}{\epsilon\sqrt{\tau}}$ and $\frac{\lambda(L-L_0)}{\epsilon\sqrt{\tau}}$ converge to infinity, then the solution $v(x,t)$ of the finite interval problem converges exponentially to the solution $u(x,t)$ of the half line problem in the sense of $\|\cdot\|_{H^1_{L,\epsilon,\tau}}$. This can be achieved by either letting $\epsilon\to 0$ or $L\to\infty$. In the first case, the extreme situation is $\epsilon = 0$, the half line problem (1.11) becomes hyperbolic and the domain of dependence is finite, one only need to consider the finite interval problem. In the second case, for a fixed final time t, so long as $L > \max\left\{\left(\frac{(M+1)^2}{2M}+\frac{b_\tau}{\sqrt{\tau}}\right)\frac{t}{\lambda}, \left(\frac{(M+1)^2}{2M}+\frac{b_\tau-1}{\sqrt{\tau}}\right)\frac{t}{\lambda}+L_0\right\}$, the exponent of all the exponential functions in the upper bound of (2.41) becomes negative, and hence $\| u(\cdot,t)-v(\cdot,t)\|_{H^1_{L,\epsilon,\tau}}\to 0$ as $L\to\infty$. Theorem 2.1 gives a theoretical justification for using the solution of the finite interval problem to approximate the solution of the half line problem with appropriate choice of L and ϵ [15].

3. Numerical Results

In this section, we numerically verify the convergence rate provided in Theorem 2.1. For the seek of this sole purpose, we employ a simple low order scheme proposed by Duijn et al. Developing effective high order numerical schemes to solve the MBL equation is not the focus of this paper, please refer to [15] for higher order central schemes.

We adopt the finite difference scheme given in [13] to solve the following finite interval initial boundary value problem

$$\begin{aligned} & v_t + (f(v))_x = \epsilon v_{xx} + \epsilon^2\tau v_{xxt} \\ & v(x,0) = u_B\chi_{\{x=0\}} + 0\chi_{\{0<x\leq L\}} \\ & v(0,t) = u_B, \qquad v(L,t) = 0. \end{aligned} \tag{3.1}$$

3.1. Verification of Theorem 2.1

We have verified the theoretical approximation given by Theorem 2.1 for various choices of parameters, and similar results have been obtained. In this section, we demonstrate the results with $\tau = 1$ and $u_B = 0.9$. Let $v(x,t)$ be the numerical solution to the finite interval boundary value problem (3.1), and $u(x,t)$ be the numerical solution to the corresponding half line initial boundary value problem

$$\begin{aligned} &u_t + (f(u))_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt} \\ &u(x,0) = u_B \chi_{\{x=0\}} + 0\chi_{\{x>0\}} \\ &u(0,t) = u_B. \end{aligned} \tag{3.2}$$

For the seek of numerical verification to Theorem 2.1, we compare $u(x,t)$ and $v(x,t)$ in the sense of $\| \cdot \|_\infty$ norm. Proposition 2.7 and Theorem 2.1, with $L_0 = 0$, give that

$$\| u(\cdot,t) - v(\cdot,t) \|_\infty \leq G_{\epsilon,\tau}(t) e^{-\frac{\lambda L}{\epsilon\sqrt{\tau}}}. \tag{3.3}$$

We numerically verify that $\| u - v \|_\infty$ decays exponentially with respect to L as given in (3.3). For a fixed time $t = 0.1$, we solve (3.1) for two choices of diffusion coefficient, one smaller $\epsilon = 0.1$ and one larger $\epsilon = 1$.

(a) $\epsilon = 0.1$
We numerically solved (3.1) for $L = 0.5, 1, 2, 4, 8, 16, 32, 64, 128$, and the obtained numerical solutions differ by only machine-ϵ when $L \geq 64$. Hence, we use the solution of (3.1) corresponding to $L = 128$ as the numerical approximation for the half line problem (3.2) for numerical comparison purpose, i.e., we take $u = v_{\{L=128\}}$. Figure 3.1(a) shows that $\| u - v \|_\infty$ decays

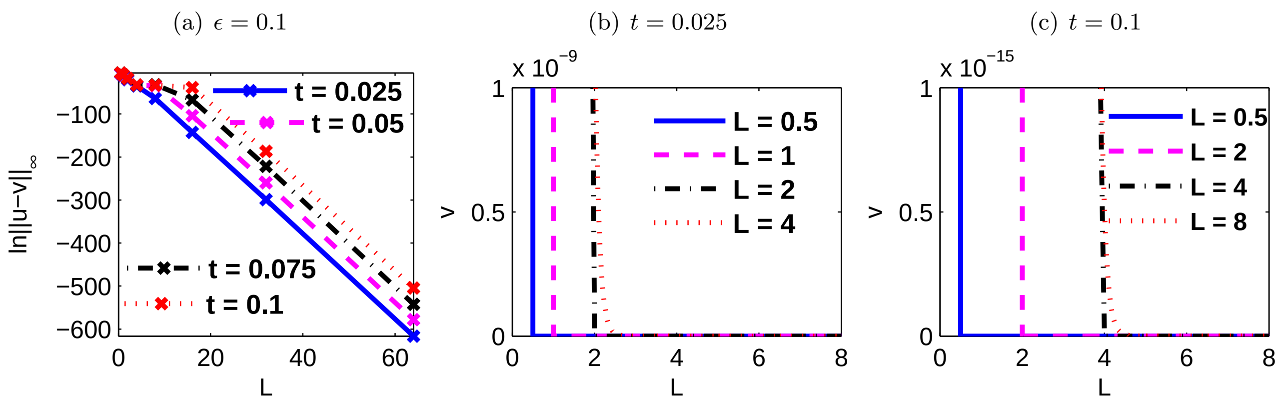

Figure 3.1: $\epsilon = 0.1$, $\tau = 1$, $u_B = 0.9$, $M = 2$ are fixed and $\Delta x = 0.005$, $\Delta t = 0.001$ are used. (a): $\| u - v \|_\infty$ decays exponentially with respect to L; (b), (c): solution profiles with different domain size L's at $t = 0.025$ and $t = 0.1$ respectively.

exponentially with respect to the domain size L at time $t = 0.025$, $t = 0.05$, $t = 0.075$ and $t = 0.1$. Furthermore, it displays that $\| u - v \|_\infty$ increases as time progresses. This numerically shows that $G_{\epsilon,\tau}(t)$ in (3.3) is increasing with respect to t. Notice that in Figure 3.1(a), when $L \leq 8$, the $\| u - v \|_\infty$ decay rate is smaller than that for $L \geq 16$. This is especially significant when $t > 0.05$. This is because when L is too small, at a later time, the domain size is not sufficiently large for the wave propagation, which in turn makes $\| u - v \|_\infty$ unproportionally large. This can be seen from Figures 3.1(b) and 3.1(c), which give the zoom-in view of the solution profiles gotten with different domain size L's at $t = 0.025$ (Figure 3.1(b)) and t = 0.1 (Figure 3.1(c)). Based on Figure 3.1(b), it is clear that at $t = 0.025$, $L = 2$ is too small to capture the leading shock; while Figure 3.1(c) shows that at a later time $t = 0.1$, even $L = 4$ is too small to capture the leading shock, in fact, even $L = 8$ is too small to capture the correct solution, which corresponds to a significant drop of decay rate for $L \leq 8$ in Figure 3.1(a) at $t = 0.1$. In addition, the λ in (3.3) is estimated based on

$$\lambda = \texttt{slope} \times \left(-\epsilon\sqrt{\tau}\right), \tag{3.4}$$

where $\texttt{slope}$ is the slope of the line $\ln \| u - v \|_\infty$ versus L. Based on Figure 3.1(a), λ approximately equals 0.9866, 0.9883, 0.9902, 0.9709 for t equals 0.025, 0.05, 0.075, 0.1 respectively.

(b) $\epsilon = 1$
The increment of ϵ enhances the diffusion effect in the solution profiles. It is natural to enlarge the domain size to achieve the correct solution. Therefore for the larger $\epsilon = 1$, we numerically solved (3.1) for $L = 0.5$, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 and the obtained numerical solutions differ by only machine-ϵ when $L \geq 256$. Hence in this case, we use the solution of (3.1) corresponding to $L = 512$ as the numerical approximation for the half line problem (3.2), i.e., we take $u = v_{\{L=512\}}$. Figure 3.2(a) again shows that $\| u - v \|_\infty$ decays exponentially with

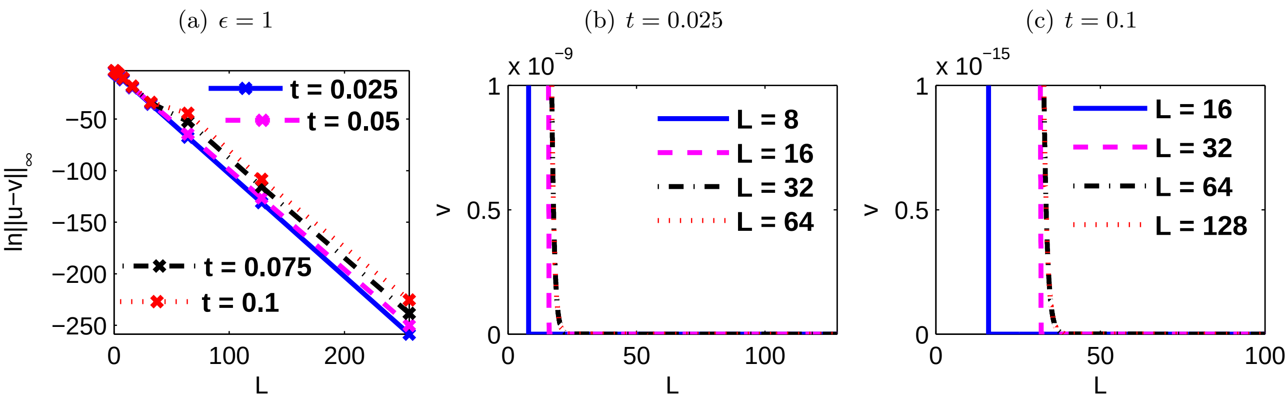

Figure 3.2: $\epsilon = 1$, $\tau = 1$, $u_B = 0.9$ and $M = 2$ are fixed and $\Delta x = 0.005$, $\Delta t = 0.001$ are used. (a): $\| u - v \|_\infty$ decays exponentially with respect to L; (b), (c): solution profiles with different domain size L's at $t = 0.025$ and $t = 0.1$ respectively.

respect to the domain size L, and for fixed L, $\| u - v \|_\infty$ increases as time progresses. Figure 3.2(b) shows that when $t = 0.025$, $L = 16$ is too small to capture the leading shock; whereas Figure 3.2(c) shows that when $t = 0.1$, even $L = 32$ is not sufficiently large to ensure the accuracy of the solution, which corresponds to the significant drop of decay rate for $L \leq 32$ in Figure 3.2(a) when $t = 0.1$. In addition, (3.4) is used to estimate the λ value, and based on Figure 3.2(a), λ approximately equals 0.9949, 0.9669, 0.9696, 0.9429 for t equals 0.025, 0.05, 0.075, 0.1 respectively.

For both smaller and larger diffusion effects, $\epsilon = 0.1$ and $\epsilon = 1$ respectively, we have shown numerically that when ϵ, τ and u_B are fixed, $\| u - v \|_\infty$ decays exponentially with respect to L as given in (3.3). We also numerically showed that $G_{\epsilon,\tau}(t)$ in (3.3) is increasing with respect to t. In addition, Figures 3.1(b) and 3.2(b) demonstrate that at a fixed time ($t = 0.025$), bigger $\epsilon = 1$ requires larger domain size to ensure the accuracy of the solution. This can be seen in Figures 3.1(c) and 3.2(c) ($t = 0.1$) too. Furthermore, We numerically estimated that λ value in (3.3)

$$\lambda \in (0.94, 0.99).$$

This estimate gives the convergence rate for using the solution to the finite interval boundary value problem to approximate that to the half line problem. It provides a way to estimate the domain size needed to achieve a desired accuracy.

3.2. Effect of ϵ

The theoretical approximation given by Theorem 2.1 does not only provide a way to estimate the computational domain size needed to capture the solution to the half line problem, it also shows how the diffusion coefficient ϵ affects $\| u(\cdot,t) - v(\cdot,t) \|$. In this section, we verify this numerically based on (3.3).

For a fixed time $t = 0.05$, we numerically solve for u and $v_{L=0.1}$ for seven representative $\epsilon = 1/1000, 1/800, 1/500, 1/400, 1/250, 1/200, 1/125$. The numerical solution $u(x,t)$ of the half line problem was obtained by doubling the domain size starting with $L = 0.1$, until the solutions do not differ more than the `tolerance` $= 10^{-10}$.

Figure 3.3(a) shows the plot $\ln \| u - v \|_\infty$ versus $\frac{1}{\epsilon}$ at various time $t = 0.0125$, $t = 0.025$, $t = 0.0375$, $t = 0.05$. It is shown that $\ln \| u - v \|_\infty$ and $\frac{1}{\epsilon}$ are not linearly related, this is because

(a) $L = 0.1$ (b) $t = 0.0125$ (c) $t = 0.05$

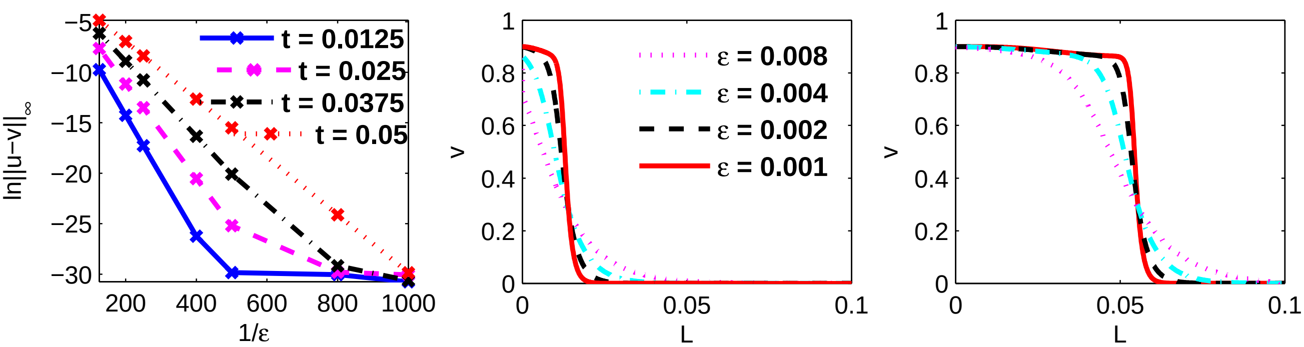

Figure 3.3: $L = 0.1$, $\tau = 1$, $u_B = 0.9$, $M = 2$ are fixed, and $\Delta x = 10^{-4}$, $\Delta t = 10^{-5}$ are used. (a): $\| u - v \|_\infty$ decays with respect to $1/\epsilon$; (b), (c): solution profiles with different ϵ's at $t = 0.0125$ and $t = 0.05$ respectively.

if we take logarithm on (3.3),

$$\ln \| u(\cdot,t) - v(\cdot,t) \|_\infty \le \ln G_{\epsilon,\tau}(t) - \frac{\lambda L}{\sqrt{\tau}} \cdot \frac{1}{\epsilon}, \tag{3.5}$$

the $G_{\epsilon,\tau}(t)$ appeared in the upper bound in (3.5) also depends on ϵ. Hence the upper bound in (3.5) is not a linear function of $\frac{1}{\epsilon}$. Notice that for a fixed $\frac{1}{\epsilon}$, $\| u - v \|_\infty$ increases as time progresses, this confirms again that $G_{\epsilon,\tau}(t)$ is increasing with respect to t. For any fixed time, as $\frac{1}{\epsilon}$ increases, the rate $\ln \| u - v \|_\infty$ decreases slows down. Hence, $G_{\epsilon,\tau}(t)$ decays slower than exponential decay with respect to $\frac{1}{\epsilon}$. As a matter of fact, $G_{\epsilon,\tau}(t)$ could be even increasing with respect to $\frac{1}{\epsilon}$. Figures 3.3(b), 3.3(c) give the solution profiles for all the four different ϵ's at $t = 0.0125$ and $t = 0.05$ respectively. They both show that smaller ϵ smears out the solution less. Comparing Figures 3.3(b) and 3.3(c), it shows that the diffusion effects do not change qualitatively as time progresses.

3.3. Monotonicity of solution profiles

In this section, we numerically verify that the MBL equation (3.1) does deliver non-monotone water saturation profiles as observed in experiments [5]. Van Duijn et al [13] provided a numerical u_B–τ bifurcation diagram. In particular, it explicitly spelled out the conditions for the existence of non-monotone solutions. Van Duijn et al [13] show that when the dispersive coefficient τ is larger than the threshold value $\tau_* \approx 0.61$, it corresponds to an interval $[\underline{u}, \bar{u}]$, and if the post-shock value u_B lies inside this interval, then the solution will contain two shocks, one from u_B to $\bar{u}$, and another one from $\bar{u}$ to 0, with the shock speed $\frac{f(u_B)-f(\bar{u})}{u_B-\bar{u}}$ and $\frac{f(\bar{u})}{\bar{u}}$ respectively.

Taking the convergence estimate given in section 3.1 into consideration, we have chosen large enough computational domains to seek the solution of (3.1). The numerical experiments in Figure 3.4 are carried out for three representative τ values 0.2, 1, 5. Since $\tau = 0.2$ is less that $\tau_* \approx 0.61$, Figures 3.4(a) 3.4(d) 3.4(g) all display monotone solution profiles. However, $\tau = 1$ is larger than $\tau_* \approx 0.61$, Figures 3.4(b) 3.4(e) 3.4(h) show the solutions for three representative u_B values 0.9, α, 0.75 respectively. Notice that Figure 3.4(e) is the only one that displays non-monotone behavior. This is because among all the three chosen u_B values, $u_B = \alpha$ is the only one that lies inside $[\underline{u}_{\tau=1}, \bar{u}_{\tau=1}]$. Another choice of $\tau = 5$ corresponds to $\underline{u}_{\tau=5} \approx 0.68$ and $\bar{u}_{\tau=5} \approx 0.98$. 3.4(c) 3.4(f) 3.4(i) show that all the three representative u_B values 0.9, α, 0.75 lie inside $[\underline{u}_{\tau=5}, \bar{u}_{\tau=5}]$, hence all display non-monotone solution profiles.

4. Discussion

The MBL equation (1.10) describes the flow scenario during the secondary recovery in oil reservoir. For this simplified model, we studied the relationship between two types of problem (1.11) and (1.12). Theorem 2.1 shows that the solution to the half line problem can be approximated by that of the finite interval problem. At a fixed final time t, given an error tolerance, Theorem 2.1 provides a way to determine the finite domain size L, such that the difference between the solutions of the half line and finite domain problems does not exceed the prescribed error tolerance. In the physical application, this will help oil companies to determine the size of a specific region to consider, so that the water saturation obtained by studying this specific region versus investigating the entire reservoir does not differ more than a prescribed error tolerance.

Theorem 2.1 provides a theoretical justification for using finite domain to seek the numerical solution of the MBL equation (1.10). The convergence rates are numerically verified to be consistent with the theoretical estimates. In particular, the difference between solutions of the half line and finite domain problems decay exponentially with respect to the domain size L, and nearly exponentially with respect to $\frac{1}{\epsilon}$.

In addition to its simple form, MBL equation (1.10) delivers water saturation profiles qualitatively consistent with the experimental results [5]. The numerical solutions for qualitatively different parameter values τ and initial conditions u_B show that the jump locations are consistent with the theoretical calculation and the plateau heights are consistent with the numerically

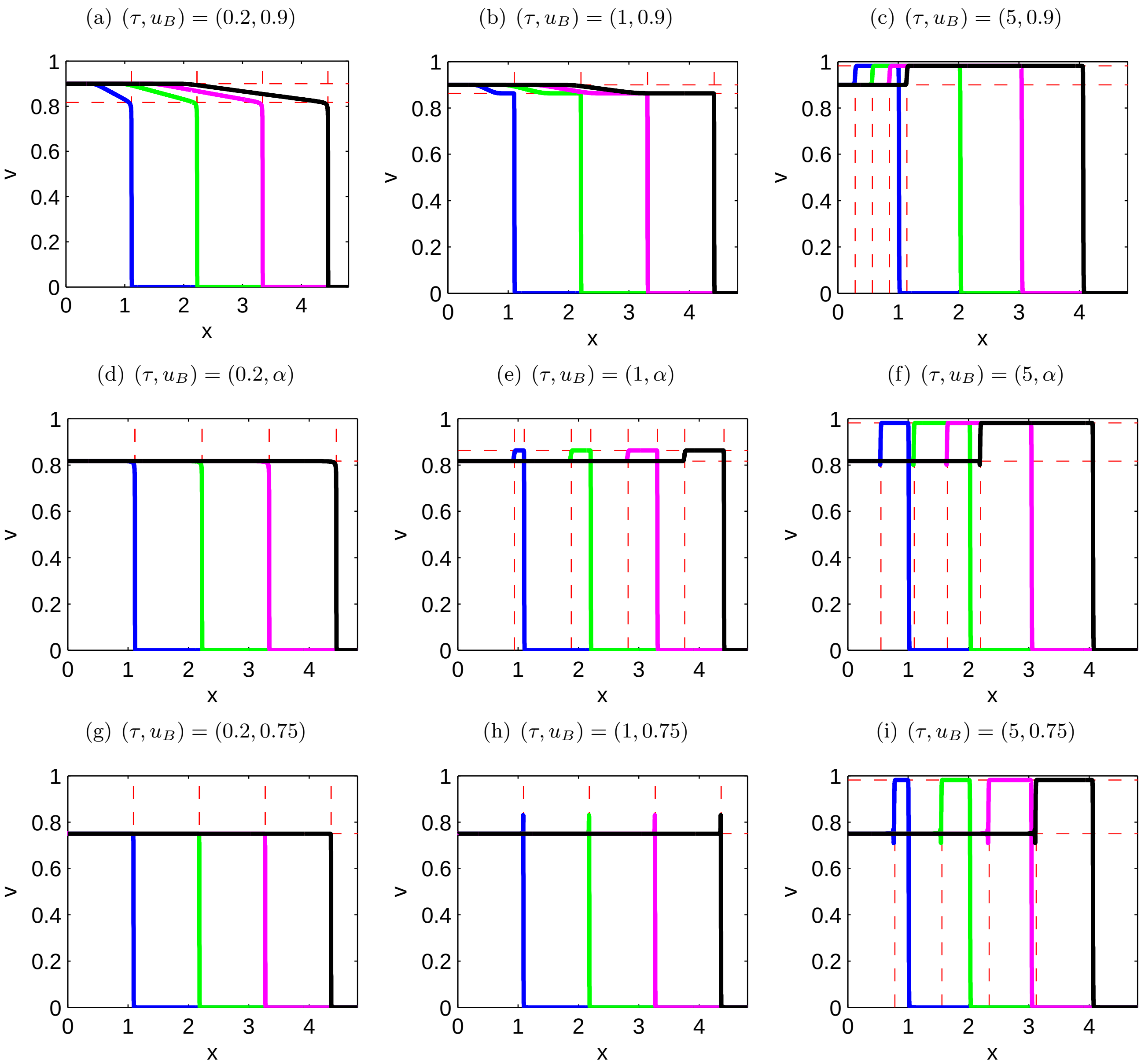

Figure 3.4: Numerical solutions to MBL equation (3.1) with different (τ, u_B)'s. $M = 2$, $\epsilon = 0.001$, $T = 4000 \times \epsilon$, $\Delta x = \frac{\epsilon}{10}$, $\frac{\Delta t}{\Delta x} = 0.1$, $L = 6$. The solution profies at $\frac{1}{4}T$ (blue), $\frac{2}{4}T$ (green), $\frac{3}{4}T$ (magenta) and T (black) are displayed to demonstrate the time evolution. In figures (d) – (f), $\alpha = \sqrt{\frac{M}{M+1}} = \sqrt{\frac{2}{3}}$ for $M = 2$. The red dashed lines indicate the theoretical shock locations and plateau values.

obtained values given in [13]. In particular, the numerical solutions give non-monotone water saturation profiles, for certain τ and u_B values, which is consistent with the experimental observations.

In [14, 12], the two-dimensional space extension of the modified Buckley-Leverett equation has been derived. One of the future directions is to study the difference between the solution to the quarter-plane problem and finite domain problem to 2D MBL equation, and develop numerical schemes to solve it efficiently.

Appendix A. Proof of the lemmas

Proof to lemma 2.2. Let $g(u) = \frac{f(u)}{u} = \frac{u}{u^2+M(1-u)^2}$, then

$$g'(u) = \frac{M-(1+M)u^2}{(u^2+M(1-u)^2)^2}\begin{cases} >0 & \text{if} \quad 0<u<\sqrt{\frac{M}{M+1}} \\ =0 & \text{if} \quad u=\sqrt{\frac{M}{M+1}} \\ <0 & \text{if} \quad u>\sqrt{\frac{M}{M+1}} \end{cases}$$

and hence $g(u)$ achieves its maximum at $u = \sqrt{\frac{M}{M+1}}$. Therefore, $\frac{f(u)}{u} = g(u) \leq D$, where $D = \frac{f(\alpha)}{\alpha}$ and $\alpha = \sqrt{\frac{M}{M+1}}$, and in turn, we have that $f(u) \leq Du$ for all $0 \leq u \leq 1$. □

Proof to lemma 2.3 (i).

$$\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right| e^{\frac{\lambda x-\lambda\xi}{\epsilon\sqrt{\tau}}}\, d\xi = \epsilon\sqrt{\tau}\frac{-2+2e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}}}{\lambda^2-1} \leq \frac{2\epsilon\sqrt{\tau}}{1-\lambda^2} \quad \text{if } \lambda\in(0,1).$$

□

Proof to lemma 2.3 (ii).

$$\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right| e^{\frac{\lambda x-\xi}{\epsilon\sqrt{\tau}}}\, d\xi = xe^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}} \leq \frac{\epsilon\sqrt{\tau}}{e(1-\lambda)} \quad \text{if } \lambda\in(0,1).$$

□

Proof to lemma 2.3 (iii). Based on the assumption on u_0 in (2.1)

$$\begin{aligned}\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}|u_0(\xi)|\, d\xi &\leq \int_0^{+\infty} e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}|u_0(\xi)|\, d\xi \\ &\leq C_u e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}\int_0^{L_0} e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\, d\xi := C_u y_1(x)\end{aligned}$$

Calculating $y_1(x)$ with the assumption that $\lambda \in (0,1)$, we get

$$y_1(x) = \begin{cases} e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}\int_0^{L_0} e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\, d\xi \leq 2\epsilon\sqrt{\tau}e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} \leq 2\epsilon\sqrt{\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}} & \text{for} \quad x\in[0,L_0] \\ e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}}\int_0^{L_0} e^{\frac{\xi}{\epsilon\sqrt{\tau}}}\, d\xi \leq \epsilon\sqrt{\tau}e^{\frac{(\lambda-1)x+L_0}{\epsilon\sqrt{\tau}}} \leq \epsilon\sqrt{\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}} & \text{for} \quad x\in[L_0,+\infty) \end{cases}$$

Therefore, we get the desired inequality

$$\int_0^{+\infty}\left|e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} - e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}}\right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}}|u_0(\xi)|\, d\xi \leq 2C_u\epsilon\sqrt{\tau}e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}.$$

□

Proof to lemma 2.4 (i).

$$\begin{aligned}
&\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \lambda \xi}{\epsilon\sqrt{\tau}}}\, d\xi \\
= &\frac{\epsilon\sqrt{\tau}}{\lambda^2 - 1} \left(-2 + 2\lambda e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}} - 2(\lambda-1) e^{-\frac{2x}{\epsilon\sqrt{\tau}}} \right) \le \frac{2\epsilon\sqrt{\tau}}{1-\lambda^2} \qquad \text{if } \lambda \in (0,1).
\end{aligned}$$

□

Proof to lemma 2.4 (ii).

$$\begin{aligned}
&\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x - \xi}{\epsilon\sqrt{\tau}}}\, d\xi \\
= &\frac{2e^{\frac{(\lambda-3)x}{\epsilon\sqrt{\tau}}} - 2e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}}}{\frac{-2}{\epsilon\sqrt{\tau}}} + x e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}} \le \epsilon\sqrt{\tau} + \frac{\epsilon\sqrt{\tau}}{e(1-\lambda)} \qquad \text{if } \lambda \in (0,1).
\end{aligned}$$

□

Proof to lemma 2.4 (iii). Based on the assumption on u_0 in (2.1)

$$\begin{aligned}
&\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} |u_0(\xi)|\, d\xi \\
\le \;& C_u e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} \int_0^{L_0} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| d\xi := \; C_u y_3(x)
\end{aligned}$$

Calculating $y_3(x)$ with the assumption that $\lambda \in (0,1)$, we get

$$y_3(x) \le \begin{cases} e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}} \int_0^x (e^{-\frac{\xi}{\epsilon\sqrt{\tau}}} + e^{\frac{\xi}{\epsilon\sqrt{\tau}}})\, d\xi + e^{\frac{(\lambda+1)x}{\epsilon\sqrt{\tau}}} \int_x^{L_0} e^{-\frac{\xi}{\epsilon\sqrt{\tau}}}\, d\xi \le 2\epsilon\sqrt{\tau} e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}, & x \in [0, L_0] \\ e^{\frac{(\lambda-1)x}{\epsilon\sqrt{\tau}}} \int_0^{L_0} (e^{-\frac{\xi}{\epsilon\sqrt{\tau}}} + e^{\frac{\xi}{\epsilon\sqrt{\tau}}})\, d\xi \le \epsilon\sqrt{\tau} e^{\frac{(\lambda-1)x + L_0}{\epsilon\sqrt{\tau}}} \le \epsilon\sqrt{\tau} e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}, & x \in [L_0, +\infty) \end{cases}$$

Therefore, we get the desired inequality

$$\int_0^{+\infty} \left| e^{-\frac{x+\xi}{\epsilon\sqrt{\tau}}} + \operatorname{sgn}(x-\xi) e^{-\frac{|x-\xi|}{\epsilon\sqrt{\tau}}} \right| e^{\frac{\lambda x}{\epsilon\sqrt{\tau}}} |u_0(\xi)|\, d\xi \le 2C_u \epsilon\sqrt{\tau} e^{\frac{\lambda L_0}{\epsilon\sqrt{\tau}}}.$$

□

Proof to lemma 2.5 (i).

$$\left| \phi_1(x) - e^{-\frac{x}{\epsilon\sqrt{\tau}}} \right| = e^{-\frac{L}{\epsilon\sqrt{\tau}}} \left| \frac{e^{-\frac{x}{\epsilon\sqrt{\tau}}} - e^{\frac{x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{L}{\epsilon\sqrt{\tau}}}} \right| = e^{-\frac{L}{\epsilon\sqrt{\tau}}} \left| \phi_2(x) \right|.$$

□

Proof to lemma 2.5 (ii). Since $\phi_2(x) = \frac{e^{\frac{x}{\epsilon\sqrt{\tau}}} - e^{-\frac{x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{L}{\epsilon\sqrt{\tau}}}}$, we see that $\phi_2'(x) = \frac{1}{\epsilon\sqrt{\tau}} \frac{e^{\frac{x}{\epsilon\sqrt{\tau}}} + e^{-\frac{x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{L}{\epsilon\sqrt{\tau}}}} > 0$ and hence $\phi_2(x) \le \phi_2(L) = 1$ for $x \in [0, L]$. □

Proof to lemma 2.5 (iii). $\phi_2'(x) = \frac{1}{\epsilon\sqrt{\tau}} \frac{e^{\frac{x}{\epsilon\sqrt{\tau}}} + e^{-\frac{x}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{L}{\epsilon\sqrt{\tau}}}}$ gives that $\phi_2''(x) = \frac{1}{\epsilon^2\tau}\phi_2(x) > 0$, and hence $\phi_2'(x) \le \phi_2'(L) = \frac{1}{\epsilon\sqrt{\tau}} \frac{e^{\frac{L}{\epsilon\sqrt{\tau}}} + e^{-\frac{L}{\epsilon\sqrt{\tau}}}}{e^{\frac{L}{\epsilon\sqrt{\tau}}} - e^{-\frac{L}{\epsilon\sqrt{\tau}}}} = \frac{1}{\epsilon\sqrt{\tau}} \frac{e^{\frac{2L}{\epsilon\sqrt{\tau}}} + 1}{e^{\frac{2L}{\epsilon\sqrt{\tau}}} - 1} \le \frac{2}{\epsilon\sqrt{\tau}}$ if $\epsilon \ll 1$ for $x \in [0, L]$. □

Acknowledgments

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