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Minimal Convex Combinations of Three Sequential Laplace-Dirichlet Eigenvalues

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Abstract We study the shape optimization problem where the objective function is a convex combination of three sequential Laplace-Dirichlet eigenvalues. That is, for $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta \leq 1$, we consider $\inf\{\alpha\lambda_k(\Omega) + \beta\lambda_{k+1}(\Omega) + (1-\alpha-\beta) \times \lambda_{k+2}(\Omega) \colon \Omega$ open set in $\mathbb{R}^2$ and $|\Omega| \leq 1\}$. Here $\lambda_k(\Omega)$ denotes the k-th Laplace-Dirichlet eigenvalue and $|\cdot|$ denotes the Lebesgue measure. For $k = 1, 2$, the minimal values and minimizers are computed explicitly when the set of admissible domains is restricted to the disjoint union of balls. For star-shaped domains, we show that for $k = 1$ and $\alpha + 2\beta \leq 1$, the ball is a local minimum. For $k = 1, 2$, several properties of minimizers are studied computationally, including uniqueness, connectivity, symmetry, and eigenvalue multiplicity.

Keywords Shape optimization · Laplacian eigenvalues · Dirichlet boundary condition · Isoperimetric problems

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and $\{\lambda_k(\Omega), \psi_k(\mathbf{x}; \Omega)\}_{k=1}^{\infty}$ denote the eigenpairs of the Laplace-Dirichlet operator for the domain Ω (listed with multiplicity), satisfying

$$\begin{aligned} -\Delta\psi(\mathbf{x}) &= \lambda\psi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \psi(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega. \end{aligned} \tag{1.1}$$

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The eigenvalues $\lambda_k(\Omega)$ are characterized by the Courant-Fischer formulation

$$\lambda_k(\Omega) = \min_{\substack{E_k \subset H_0^1(\Omega) \\ \text{subspace of dim } k}} \max_{\psi \in E_k, \psi \neq 0} \frac{\int_\Omega |\nabla \psi|^2 d\Omega}{\int_\Omega \psi^2 d\Omega}, \tag{1.2}$$

where E_k is in general a k-dimensional subspace of $H_0^1(\Omega)$ and at the minimizer, $E_k = \text{span}(\{\psi_j(\mathbf{x};\Omega)\}_{j=1}^k)$. The ratio in (1.2) is referred to as the Rayleigh quotient. General references for Laplace-Dirichlet eigenvalues can be found in [5, 6, 8].

In this work, we consider the shape optimization problem where the objective function is a convex combination of three sequential Laplace-Dirichlet eigenvalues. That is, we consider the following (α,β)-parameterized optimization problem:

$$C_{\alpha,\beta}^{j*} = \inf_{\Omega\in\mathcal{A}} C_{\alpha,\beta}^j(\Omega) \quad \text{and} \quad \hat{\Omega}_{\alpha,\beta}^j = \left\{\Omega \in \mathcal{A} \colon C_{\alpha,\beta}^j(\Omega) = C_{\alpha,\beta}^{j*}\right\}, \tag{1.3}$$

where

$$T := \left\{(\alpha,\beta)\in\mathbb{R}^2 \colon \alpha \geq 0, \beta \geq 0, \alpha+\beta \leq 1\right\},$$

$$\mathcal{A} := \left\{\Omega \subset \mathbb{R}^2 \colon \Omega \text{ quasi-open and } |\Omega| \leq 1\right\},$$

$$C_{\alpha,\beta}^j(\Omega) := \alpha\lambda_j(\Omega) + \beta\lambda_{j+1}(\Omega) + (1-\alpha-\beta)\lambda_{j+2}(\Omega), \quad (\alpha,\beta)\in T \text{ and } \Omega \in \mathcal{A}.$$

We will also consider several reduced admissible classes. Let $\mathcal{B} \subset \mathcal{A}$, be the set of balls, i.e.,

$$\mathcal{B} := \{\Omega \in \mathcal{A} \colon \Omega \text{ is a ball}\}.$$

We'll use the notation $\mathcal{B} \sqcup \mathcal{B}$ to denote the class of domains consisting of the disjoint union of two balls. We say a domain Ω is F_N *representable* if

$$\Omega = \left\{(r,\theta) : r \leq R_N(\theta), \theta \in [0,2\pi]\right\}, \quad \text{where } R_N(\theta) = \sum_{k=-N}^{N} a_k e^{\imath k\theta} \text{ and } a_{-k} = \overline{a_k}. \tag{1.4}$$

Define the set of all F_N representable domains by

$$\mathcal{F}_N = \{\Omega \in \mathcal{A} \colon \Omega \text{ is } F_N \text{ representable}\}. \tag{1.5}$$

Note that $\mathcal{F}_\infty$ is the class of star-shaped, bounded domains. Finally, we use the notation $\mathcal{F}_N \sqcup \mathcal{F}_N$ to denote the class of domains consisting of the disjoint union of two F_N representable domains.

In what follows, we give previous results for this problem and state our own results with an outline of this paper.

1.1 Previous Results

The simplest problem of the general form (1.3) is the minimization of a single eigenvalue, i.e.,

$$\min_{\Omega} \lambda_j = \min_{\Omega\in\mathcal{A}} C_{1,0}^j(\Omega).$$

The existence of a minimizer for general j was recently shown to exist and have finite perimeter [4, 10]. It is well known that among all open, two-dimensional domains of equal area, the unique minimizer of $\lambda_1(\Omega)$ is a ball (Faber-Krahn inequality) and the unique minimizer of $\lambda_2(\Omega)$ is the disjoint union of two equal-area balls (Krahn-Szegö inequality); see [6]. Wolf and Keller [15] showed the minimizer of $\lambda_3(\Omega)$ is connected and that a ball is a local minimizer. It remains as a conjecture that a ball is a global minimizer of $\lambda_3(\Omega)$; see [6]. The minimizer of $\lambda_4(\Omega)$ is conjectured to be the disjoint union of two balls with radii which have ratio $j_{0,1}/j_{1,1}$ where $j_{m,n}$ is the $n-th$ zero of the $m-th$ order Bessel function J_m; see [6]. There are no theoretical results for the explicit optimal shapes for $j \geq 5$, however there are several computational studies in this area. In [13], minimizers for the first ten Laplace-Dirichlet eigenvalues were found numerically. Here, a level set approach was used to represent the domain and a relaxed formulation of the Laplace-Dirichlet problem was used to compute the eigenvalues. In [1], the same problem for $j \leq 15$ was studied using a high-accuracy meshless method for the eigenpair computation and the boundaries were parameterized using Fourier coefficients. The results were very similar to those in [13], except that an improved domain was found for the seventh eigenvalue. It was also observed that the minimizer for the thirteenth eigenvalue is not symmetric. The multiplicity of λ_j for the optimal domains was also investigated.

The shape optimization problem where the objective function is a convex combination of two sequential Laplace-Dirichlet eigenvalues, i.e., taking $\alpha + \beta = 1$ in $C^j_{\alpha,\beta}$:

$$\min_{\Omega \in \mathcal{A}} C^j_{\alpha,1-\alpha}(\Omega) \quad \text{for } \alpha \in [0,1], \tag{1.6}$$

has also been studied. In [15], the range of the first two Laplace-Dirichlet eigenvalues $(\lambda_1(\Omega), \lambda_2(\Omega))$ for a planar domain Ω of unit area was explored. The boundary of the range consists of the two rays $\{(\lambda_1,\lambda_2)\colon \lambda_2 = \lambda_1 \text{ and } \lambda_1 \geq \pi j^2_{0,1}\}$ and $\{(\lambda_1,\lambda_2)\colon \lambda_2 = \frac{j^2_{1,1}}{j^2_{0,1}}\lambda_1 \text{ and } \lambda_2 \geq 2\pi j^2_{0,1}\}$ and a curve connecting their endpoints which was determined numerically by studying (1.6) with $j = 1$. It was observed computationally that for $\alpha > 0$, the minimizer is connected, while it is known that for $\alpha = 0$, the minimizer is the disjoint union of two equal-area balls (Krahn-Szegö inequality). Thus, there is a topological change in the minimizer as $\alpha \downarrow 0$. In [2], the means of sequential eigenvalues are studied, i.e., (1.6) with $(\alpha,\beta) = (0.5, 0.5)$, and the connectivity of optimal domains is investigated. In particular, for $j = 1$ and 2, using an argument similar to that of Wolf and Keller [15], it is established that the minimizers are connected. In [12], (1.6) is studied and it is shown $C^{j*}_{\alpha,1-\alpha}$ is a Lipschitz continuous, non-increasing, concave function of α and the minimizer is upper hemicontinuous in α. Furthermore, for $j \leq 5$, properties of the minimizer (e.g., the number of connected components) are studied computationally as a function of α. For $j = 2$, it is shown that for $\alpha \in [0, \frac{1}{2}]$, the ball is a local minimizer.

The present work is motivated by [7], where (1.3) is considered for $j = 1$. They show that a minimizer of $C^1_{\alpha,\beta}$ has no more than 2 connected components and prove that for a subset of $(\alpha,\beta) \in T$, the minimizer is connected. They also conjecture that the minimizer of (1.3) for $j = 1$ is connected unless $\beta = 1$.

For a more extensive discussion of related work in this area, please consult [12].

1.2 Results and Outline

In Sect. 2 we give some continuity results for the minimum values and minimizers of $C^j_{\alpha,\beta}(\Omega)$ over the admissible class $\mathcal{A}$. We also show (in Proposition 2) that a ball is a local minimizer of $C^j_{\alpha,\beta}(\Omega)$ for $\alpha+2\beta \leq 1$ in the admissible class $\mathcal{F}_\infty$. In Sect. 3, we discuss the minimizers of $C^j_{\alpha,\beta}(\Omega)$ over the admissible class, $\mathcal{B}\sqcup\mathcal{B}$, consisting of the disjoint union of balls. Here, the solution can be written explicitly in terms of zeros of Bessel functions. In Sect. 4, we describe a computational method for minimizing $C^j_{\alpha,\beta}(\Omega)$ over the admissible class, $\mathcal{F}_N \sqcup \mathcal{F}_N$, consisting of the disjoint union of two F_N representable domains. The method is used to numerically investigate several properties of the minimizers for (1.3). In particular, for $j=1$ and 2, we answer the following questions (1) For what values of $(\alpha,\beta)\in T$ is the minimizer unique, have symmetry, or is connected? (2) Are there values $(\alpha,\beta)\in T$ for which the minimizer does not vary continuously? (3) For varying values of $(\alpha,\beta)\in T$, what are the multiplicities of the first few Laplace-Dirichlet eigenvalues for the minimizing domains? (4) For what values of $(\alpha,\beta)\in T$ does the optimal solution agree with the optimizer over the admissible class $\mathcal{B}\sqcup\mathcal{B}$? In Sect. 5, we conclude with a brief discussion.

2 Results for the Minimum of $C^j_{\alpha,\beta}(\Omega)$ over the Admissible Sets $\mathcal{A}$ and $\mathcal{F}_\infty$

In this section we give some analytical results for the shape optimization problem of minimizing $C^j_{\alpha,\beta}(\Omega)$ over the admissible classes $\mathcal{A}$ and $\mathcal{F}_\infty$.

Since $C^j_{\alpha,\beta}(\Omega)$ is a non-decreasing and Lipschitz continuous function of the Laplace-Dirichlet eigenvalues, the recent results of [4, 10] show that the infimum in (1.3) exists and that every minimizer has finite perimeter. For a parameterized optimization function, such as in (1.3), the optimal value and minimizing set, when viewed as a function of the parameter, inherit some continuity properties from the objective function. We make these statements precise for (1.3) in the following proposition, which is a direct generalization of [12, Proposition 1] and we state without proof. Recall that a set valued function $\Gamma\colon A\to B$ is *upper hemicontinuous* at a point $a\in A$ if for all sequences $\{a_n\}$ such that $a_n\to a$ and all sequences $\{b_n\}$ such that $b_n\in\Gamma(a_n)$, there exist a $b\in\Gamma(a)$ such that $b_n\to b$.

Proposition 1 *Consider the* (α,β)*-parameterized shape optimization problem* (1.3). *For each* $j\in\mathbb{N}$ *the following statements hold*:

1. *For each* $(\alpha,\beta)\in T$, $C^{j,*}_{\alpha,\beta}$ *exists and* $\hat{\Omega}^j_{\alpha,\beta}$ *is a non-empty and closed set. Furthermore, every* $\Omega\in\hat{\Omega}^j_{\alpha,\beta}$ *has finite perimeter.*
2. *The optimal value*, $C^{j,*}_{\alpha,\beta}$, *is a non-increasing*, *Lipschitz continuous*, *and concave function in both* α *and* β.
3. *As a set-valued function of* (α,β), $\hat{\Omega}^j_{\alpha,\beta}$ *is upper hemicontinuous.*

We now restrict our attention to $\mathcal{F}_\infty\subset\mathcal{A}$, the class of domains which are star-shaped and bounded. The following proposition shows that for $j=1$ and a large subset of (α, β)-values in T, the ball is a local minimizer. Our computational results, presented in Sect. 4, suggest it is a global minimizer.

Proposition 2 *The ball is a local minimizer of $C^1_{\alpha,\beta}(\Omega)$ over the admissible class $\mathcal{F}_\infty$ for the set* $\{(\alpha, \beta) \in T : \alpha + 2\beta \leq 1\}$.

Proof Our proof is a generalization of the proof that the ball is a local minimum of $\lambda_3(\Omega)$ given in [15, Thm. 8.3], to which we refer the reader for details. Consider the nearly circular domain $\Omega_\epsilon = \{(r, \theta) : \; r < R(\theta, \epsilon), \; \theta \in [0, 2\pi]\}$, where

$$R(\theta, \epsilon) := 1 + \epsilon \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} + \epsilon^2 \sum_{k=-\infty}^{\infty} b_k e^{ik\theta} + O\big(\epsilon^3\big), \quad a_n = \overline{a_{-n}} \text{ and } b_n = \overline{b_{-n}}.$$

Using the asymptotic formulas for $|\Omega_\epsilon|\lambda_k(\Omega_\epsilon)$ given in [15, App. A], the following holds. If $a_2 \neq 0$,

$$C^1_{\alpha,\beta}(\Omega_\epsilon) = \pi\big[\alpha j^2_{0,1} + (1-\alpha) j^2_{1,1}\big] + 2\epsilon\pi j^2_{1,1}(1 - \alpha - 2\beta)|a_2| + O\big(\epsilon^2\big) \tag{2.1}$$

and if $a_2 = 0$,

$$\begin{aligned} C^1_{\alpha,\beta}(\Omega_\epsilon) = \pi\big[\alpha j^2_{0,1} + (1-\alpha) j^2_{1,1}\big] + A\alpha\epsilon^2 + B(1-\alpha)\epsilon^2 \\ + (1 - \alpha - 2\beta)C\epsilon^2 + O\big(\epsilon^3\big) \end{aligned} \tag{2.2}$$

where

$$A = 4\pi j^2_{0,1} \sum_{n=1}^{\infty} \left(1 + j_{0,1} \frac{J'_n(j_{0,1})}{J_n(j_{0,1})}\right) |a_n|^2$$

$$B = 2\pi j^2_{1,1} \sum_{\ell} \left(1 + j_{1,1} \frac{J'_{\ell-1}(j_{1,1})}{J_{\ell-1}(j_{1,1})}\right) |a_\ell|^2$$

$$C = 2\pi j^2_{1,1} \left| b_2 - \sum_{\ell} \left(\frac{1}{2} + j_{1,1} \frac{J'_\ell(j_{1,1})}{J'_\ell(j_{1,1})}\right) a_{1+\ell} a_{1-\ell} \right|.$$

Here, A and B are both non-negative constants, dependent on $\{a_n\}$, which vanish only if $a_n = 0$ for all n. C is a non-negative constant, dependent on both $\{a_n\}$ and b_2. In both (2.1) and (2.2), if $\alpha + 2\beta \leq 1$ and $0 \leq \alpha \leq 1$, any perturbation of the ball increases $C^1_{\alpha,\beta}$, showing that the ball is a local minimizer. □

3 Minimum of $C^1_{\alpha,\beta}(\Omega)$ and $C^2_{\alpha,\beta}(\Omega)$ over the Union of Two Disjoint Balls, $\mathcal{B} \sqcup \mathcal{B}$

Consider the disjoint union of two balls, $D_r \in \mathcal{B} \sqcup \mathcal{B}$, with radii given by $r_1 := r$ and $r_2 := \sqrt{\pi^{-1} - r^2}$ where $r^2 \in [0, (2\pi)^{-1}]$. Note that the measure of D_r is exactly one and that the second ball is larger than the first. The first eigenfunction is supported in the larger ball, so

$$\lambda_1(D_r) = \pi\left(1-\pi r^2\right)^{-1} j_{0,1}^2.$$

The second, third, and fourth eigenvalues depend on the ratio of the ball sizes. We compute

$$\lambda_2(D_r) = \lambda_3(D_r) = \frac{\pi j_{1,1}^2}{1-\pi r^2} \quad \text{and} \quad \lambda_4(D_r) = \frac{\pi j_{2,1}^2}{1-\pi r^2}$$

$$\text{for } r^2 \in I_1 := \left[0, \frac{1}{\pi}\frac{j_{0,1}^2}{j_{2,1}^2 + j_{0,1}^2}\right]$$

$$\lambda_2(D_r) = \lambda_3(D_r) = \frac{\pi j_{1,1}^2}{1-\pi r^2} \quad \text{and} \quad \lambda_4(D_r) = \frac{j_{0,1}^2}{r^2}$$

$$\text{for } r^2 \in I_2 := \left[\frac{1}{\pi}\frac{j_{0,1}^2}{j_{2,1}^2 + j_{0,1}^2}, \frac{1}{\pi}\frac{j_{0,1}^2}{j_{1,1}^2 + j_{0,1}^2}\right]$$

$$\lambda_2(D_r) = \frac{j_{0,1}^2}{r^2} \quad \text{and} \quad \lambda_3(D_r) = \lambda_4(D_r) = \frac{\pi j_{1,1}^2}{1-\pi r^2}$$

$$\text{for } r^2 \in I_3 := \left[\frac{1}{\pi}\frac{j_{0,1}^2}{j_{1,1}^2 + j_{0,1}^2}, \frac{1}{2\pi}\right].$$

For the domain D_r, the convex combination of eigenvalues of the first three eigenvalues, $C_{\alpha,\beta}^1$, is

$$C_{\alpha,\beta}^1(D_r) = \begin{cases} [\alpha j_{0,1}^2 + (1-\alpha) j_{1,1}^2]\frac{\pi}{1-\pi r^2} & \text{for } r^2 \in I_1 \cup I_2 \\ \beta\frac{j_{0,1}^2}{r^2} + [\alpha j_{0,1}^2 + (1-\alpha-\beta) j_{1,1}^2]\frac{\pi}{1-\pi r^2} & \text{for } r^2 \in I_3 \end{cases} \tag{3.1}$$

and the convex combination of eigenvalues of the second through fourth eigenvalues, $C_{\alpha,\beta}^2$, is

$$C_{\alpha,\beta}^2(D_r) = \begin{cases} (\alpha+\beta)\frac{\pi j_{1,1}^2}{1-\pi r^2} + (1-\alpha-\beta)\frac{\pi j_{2,1}^2}{1-\pi r^2} & \text{for } r^2 \in I_1 \\ (\alpha+\beta)\frac{\pi j_{1,1}^2}{1-\pi r^2} + (1-\alpha-\beta)\frac{j_{0,1}^2}{r^2} & \text{for } r^2 \in I_2 \\ \alpha\frac{j_{0,1}^2}{r^2} + (1-\alpha)\frac{\pi j_{1,1}^2}{1-\pi r^2} & \text{for } r^2 \in I_3. \end{cases} \tag{3.2}$$

Proposition 3 *Define the partition*, $T = T_1^1 \cup T_2^1$, *by*

$$T_1^1 := \left\{(\alpha,\beta) \in T : \beta \le \frac{j_{1,1}^2}{2(j_{1,1}^2 - j_{0,1}^2)} - \frac{\alpha}{2}\right\}$$

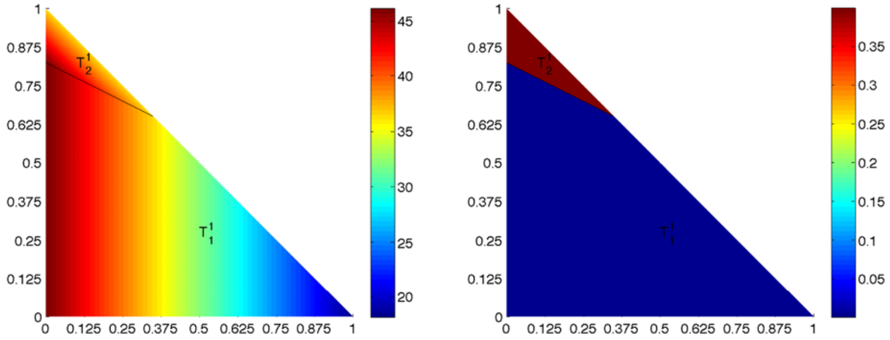

Fig. 1 (*left*) The value of $C^{1\circ}_{\alpha,\beta} := \inf\{C^1_{\alpha,\beta}(D_r) : r^2 \in [0, (2\pi)^{-1}]\}$ and (*right*) the corresponding optimal parameter r for $(\alpha,\beta) \in T$. See Proposition 3 and Sect. 3

$$T_2^1 := \left\{(\alpha,\beta) \in T : \beta > \frac{j_{1,1}^2}{2(j_{1,1}^2 - j_{0,1}^2)} - \frac{\alpha}{2}\right\}.$$

Then for $C^1_{\alpha,\beta}(D_r)$ as defined in (3.1),

$$\min_{r^2 \in [0,(2\pi)^{-1}]} C^1_{\alpha,\beta}(D_r) \equiv C^{1\circ}_{\alpha,\beta} = \begin{cases} \pi(\alpha j_{0,1}^2 + (1-\alpha) j_{1,1}^2) & \text{if } (\alpha,\beta) \in T_1^1 \\ 2\pi((\alpha+\beta) j_{0,1}^2 + (1-\alpha-\beta) j_{1,1}^2) & \text{if } (\alpha,\beta) \in T_2^1 \end{cases}$$

with minimizer given by

$$r^2 = \begin{cases} 0 & \text{if } (\alpha,\beta) \in T_1^1 \\ \frac{1}{2\pi} & \text{if } (\alpha,\beta) \in T_2^1. \end{cases}$$

Remark 4 The optimal objective function values, $C^{1\circ}_{\alpha,\beta}$, and corresponding optimal parameters, r, are plotted in Fig. 1 for $(\alpha,\beta) \in T$. The optimal objective function values for particular values $(\alpha,\beta) \in T$ are given in Table 1 (top).

Proof We first note that the value of $C^1_{\alpha,\beta}(D_r)$ for $r^2 \in I_1 \cup I_2$ is a monotone increasing function in r. Thus the minimum occurs for $r = 0$, which implies the minimizer over this interval is a single ball, D_0. The optimal value is $C^1_{\alpha,\beta}(D_0) = \pi(\alpha j_{0,1}^2 + (1-\alpha) j_{1,1}^2)$. For fixed $(\alpha,\beta) \in T$, we now find the radius r which minimizes $C^1_{\alpha,\beta}(D_r)$ for $r^2 \in I_3$. Since $C^1_{\alpha,\beta}$ is a continuous function, this occurs at critical values, r, where either $\frac{dC^1_{\alpha,\beta}(D_r)}{dr} = 0$ or values where $C^1_{\alpha,\beta}(D_r)$ is not differentiable. For the interval I_3, the critical radius r_1^*, satisfying $\frac{dC_{\alpha,\beta}(D_r)}{dr} = 0$, is

$$r_1^{*2} = \frac{1}{\pi} \frac{\sqrt{\beta} j_{0,1}}{\sqrt{\beta} j_{0,1} + \sqrt{\alpha j_{0,1}^2 + (1-\alpha-\beta) j_{1,1}^2}}.$$

Thus we consider the following three critical values

$$C^1_{\alpha,\beta}(D_r) = \begin{cases} C^1_{\alpha,\beta}(D_0) = \pi(\alpha j^2_{0,1} + (1-\alpha) j^2_{1,1}) & \text{if } r^2 = 0 \\ \pi(\sqrt{\beta} j_{0,1} + \sqrt{\alpha j^2_{0,1} + (1-\alpha-\beta) j^2_{1,1}})^2 & \text{if } r^2 = r^{*2}_1 \\ 2\pi((\alpha+\beta) j^2_{0,1} + (1-\alpha-\beta) j^2_{1,1}) & \text{if } r^2 = \frac{1}{2\pi}. \end{cases} \tag{3.3}$$

The result now follows from a comparison of the values in (3.3). □

A consequence of the following proposition is that the minimizers in Proposition 3 are also minimizers over $\mathcal{U}$, the class of domains consisting of the disjoint union of an arbitrary number of balls.

Proposition 5 *Any disconnected minimizer Ω of $C^1_{\alpha,\beta}(\Omega)$ over $\mathcal{U}$ has exactly two connected components*, *i.e.*, $\Omega \in \mathcal{B} \sqcup \mathcal{B}$.

Proof If is clear that the minimizer has at most three components. Suppose Ω is a minimizer of $C^1_{\alpha,\beta}$ with three components. Each component must support exactly one eigenvalue and consequently the radii of the three balls, r_1, r_2, and r_3 satisfy

$$\begin{aligned} \min_{r_1,r_2,r_3} \quad & j^2_{0,1}\left(\frac{\alpha}{r_1^2} + \frac{\beta}{r_2^2} + \frac{1-\alpha-\beta}{r_3^2}\right) \\ \text{s.t.} \quad & \pi\left(r_1^2 + r_2^2 + r_3^2\right) = 1 \\ & r_j \geq 0 \quad j = 1,2,3. \end{aligned}$$

This is a convex objective function in the squared variables over a compact subset of an affine subspace of $\mathbb{R}^3$. The minimizer is given by

$$r_1^2 = \frac{1}{\pi}\frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}+\sqrt{1-\alpha-\beta}}, \qquad r_2^2 = \frac{1}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}+\sqrt{1-\alpha-\beta}},$$

$$r_3^2 = \frac{1}{\pi}\frac{\sqrt{1-\alpha-\beta}}{\sqrt{\alpha}+\sqrt{\beta}+\sqrt{1-\alpha-\beta}}$$

with optimal value

$$\begin{aligned} C_{\alpha,\beta}(3\text{ balls}) &= \pi j^2_{0,1}(\sqrt{\alpha}+\sqrt{\beta}+\sqrt{1-\alpha-\beta})\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{1-\alpha-\beta}}\right) \\ &\geq 9\pi j^2_{0,1}. \end{aligned}$$

The result now follows from a direct comparison with the values obtained in Proposition 3. □

Proposition 5 is analogous to [7, Thm. 2(b)] for the restricted class, $\mathcal{U}$. We next consider $C^j_{\alpha,\beta}$ for $j = 2$ over $\mathcal{B} \sqcup \mathcal{B}$.

Proposition 6 *Denote* $Q = [Q_L, Q_R] = [\frac{j_{0,1}^2}{j_{1,1}^2 + j_{0,1}^2}, \frac{j_{1,1}^2}{j_{1,1}^2 + j_{0,1}^2}]$. *We define the partition,* $T = T_1^2 \cup T_2^2 \cup T_3^2 \cup T_4^2$, *by*

$$T_1^2 := \left\{ (\alpha, \beta) \in T : \beta \le \frac{\alpha j_{1,1}^2 + (1-\alpha) j_{2,1}^2 - (\sqrt{\alpha} j_{0,1} + \sqrt{1-\alpha} j_{1,1})^2}{(j_{2,1}^2 - j_{1,1}^2)} \text{ and } \alpha \in Q \right\}$$

$$T_2^2 := \left\{ (\alpha, \beta) \in T : \beta \le \frac{(3 j_{1,1}^2 - 2 j_{0,1}^2 - j_{2,1}^2)\alpha - (2 j_{1,1}^2 - j_{2,1}^2)}{(j_{2,1}^2 - j_{1,1}^2)} \text{ and } \alpha \ge Q_R \right\}$$

$$\begin{aligned} T_3^2 := \Bigg\{ (\alpha, \beta) \in T : \ & \beta \ge \frac{j_{2,1}^2 - j_{1,1}^2 - j_{0,1}^2}{j_{2,1}^2 - j_{1,1}^2} - \alpha \text{ and } \alpha \le Q_L, \\ & \beta \ge \frac{\alpha j_{1,1}^2 + (1-\alpha) j_{2,1}^2 - (\sqrt{\alpha} j_{0,1} + \sqrt{1-\alpha} j_{1,1})^2}{(j_{2,1}^2 - j_{1,1}^2)} \text{ and } \alpha \in Q, \\ & \beta \ge \frac{(3 j_{1,1}^2 - 2 j_{0,1}^2 - j_{2,1}^2)\alpha - (2 j_{1,1}^2 - j_{2,1}^2)}{(j_{2,1}^2 - j_{1,1}^2)} \text{ and } \alpha \ge Q_R \Bigg\} \end{aligned}$$

$$T_4^2 := \left\{ (\alpha, \beta) \in T : \beta \le \frac{j_{2,1}^2 - j_{1,1}^2 - j_{0,1}^2}{j_{2,1}^2 - j_{1,1}^2} - \alpha \text{ and } \alpha \le Q_L \right\}$$

Then for $C_{\alpha,\beta}^2(D_r)$ *as defined in* (3.2),

$$\min_{r^2 \in [0, (2\pi)^{-1}]} C_{\alpha,\beta}^2(D_r) \equiv C_{\alpha,\beta}^{2\circ} = \begin{cases} \pi(\sqrt{\alpha} j_{0,1} + \sqrt{1-\alpha} j_{1,1})^2 & (\alpha, \beta) \in T_1^2 \\ 2\pi(\alpha j_{0,1}^2 + (1-\alpha) j_{1,1}^2) & (\alpha, \beta) \in T_2^2 \\ \pi((\alpha+\beta) j_{1,1}^2 + (1-\alpha-\beta) j_{2,1}^2) & (\alpha, \beta) \in T_3^2 \\ \pi(j_{1,1}^2 + j_{0,1}^2) & (\alpha, \beta) \in T_4^2 \end{cases}$$

with minimizer given by

$$r^2 = \begin{cases} \pi(\sqrt{\alpha} j_{0,1} + \sqrt{1-\alpha} j_{1,1})^2 & (\alpha, \beta) \in T_1^2 \\ \frac{1}{2\pi} & (\alpha, \beta) \in T_2^2 \\ 0 & (\alpha, \beta) \in T_3^2 \\ \frac{1}{\pi} \frac{j_{0,1}^2}{j_{1,1}^2 + j_{0,1}^2} & (\alpha, \beta) \in T_4^2. \end{cases}$$

Remark 7 The optimal objective function values, $C_{\alpha,\beta}^{2\circ}$, and corresponding optimal parameters, r, are plotted in Fig. 2 for $(\alpha, \beta) \in T$. The optimal objective function values for particular values $(\alpha, \beta) \in T$ are given in Table 2 (top).

Proof We first note that the value of $C_{\alpha,\beta}^2(D_r)$, as defined in (3.2), for $r^2 \in I_1$ is a monotone increasing function in r. Thus, the minimum occurs for $r = 0$, indi-

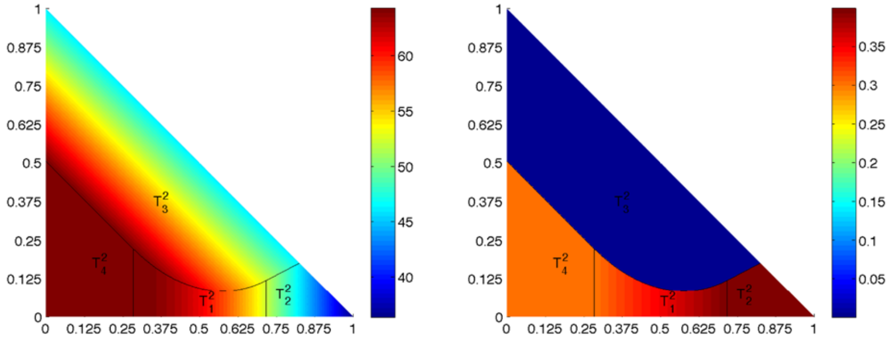

Fig. 2 (*left*) The value of $C^{2\circ}_{\alpha,\beta} := \inf\{C^2_{\alpha,\beta}(D_r) \colon r^2 \in [0, (2\pi)^{-1}]\}$ and (*right*) the corresponding optimal parameter r for $(\alpha,\beta) \in T$. See Proposition 6 and Sect. 3

cating the optimal domain is a single ball D_0. The optimal value is $C^2_{\alpha,\beta}(D_0) = (\alpha+\beta)\pi j^2_{1,1} + (1-\alpha-\beta)\pi j^2_{2,1}$. For fixed $(\alpha,\beta) \in T$, we now find the radius r which minimizes $C^2_{\alpha,\beta}(D_r)$ for $r^2 \in I_2$ and I_3. For the interval I_2, the critical radius r_2^* satisfying $\frac{dC^2_{\alpha,\beta}(D_r)}{dr} = 0$ is

$$r_2^{*2} = \frac{1}{\pi} \frac{\sqrt{(1-\alpha-\beta)}\, j_{0,1}}{\sqrt{(1-\alpha-\beta)}\, j_{0,1} + \sqrt{(\alpha+\beta)}\, j_{1,1}}.$$

For the interval I_3, the critical radius r_3^* satisfying $\frac{dC^2_{\alpha,\beta}(D_r)}{dr} = 0$ is

$$r_3^{*2} = \frac{1}{\pi} \frac{\sqrt{\alpha}\, j_{0,1}}{\sqrt{\alpha}\, j_{0,1} + \sqrt{1-\alpha}\, j_{1,1}}.$$

Thus, we consider the following five critical values

$$C^2_{\alpha,\beta}(D_r) = \begin{cases} \pi((\alpha+\beta) j^2_{1,1} + (1-\alpha-\beta) j^2_{2,1}) & \text{if } r^2 = 0 \\ \pi(\sqrt{(1-\alpha-\beta)}\, j_{0,1} + \sqrt{(\alpha+\beta)}\, j_{1,1})^2 & \text{if } r^2 = r_2^{*2} \\ \pi(j^2_{1,1} + j^2_{0,1}) & \text{if } r^2 = \frac{1}{\pi} \frac{j^2_{0,1}}{j^2_{1,1} + j^2_{0,1}} \\ \pi(\sqrt{\alpha}\, j_{0,1} + \sqrt{1-\alpha}\, j_{1,1})^2 & \text{if } r^2 = r_3^{*2} \\ 2\pi(\alpha j^2_{0,1} + (1-\alpha) j^2_{1,1}) & \text{if } r^2 = \frac{1}{2\pi}. \end{cases} \tag{3.4}$$

The result now follows from a comparison of the values in (3.4). □

Remark 8 Along the curve separating T_3^2 from $T_1^2 \cup T_2^2 \cup T_4^2$, there are two minimizers: one with a single connected component and the other with two connected components.

4 Computational Method and Results

In this section, we consider the minimization of $C^j_{\alpha,\beta}(\Omega)$ over the class $\mathcal{F}_N \sqcup \mathcal{F}_N$, where $\mathcal{F}_N$ is defined in (1.5),

$$C^{j\star}_{\alpha,\beta} = \inf_{\Omega\in\mathcal{F}_N} C^j_{\alpha,\beta}(\Omega) \quad \text{and} \quad \hat{\Omega}^j_{\alpha,\beta} = \left\{\Omega \in \mathcal{F}_N \sqcup \mathcal{F}_N : C^j_{\alpha,\beta}(\Omega) = C^{j\star}_{\alpha,\beta}\right\}. \quad (4.1)$$

We first describe a computational method for the solution of (4.1). In brief, a boundary integral method is used for the solution of the eigenvalue problem (1.1) and a line search-based BFGS method is used for the solution of (4.1). Similar algorithms appear in [1, 2, 11]. We then present some computational results for (4.1).

4.1 Computational Methods

The numerical optimization method is initialized with a choice of Fourier coefficients $\{a_k\}_{k=0}^N$ in (1.4). We use $N = 10$ coefficients and choose the coefficients either randomly or using the results from a previous computation. For a given domain, the first several eigenpairs are computed using the Matlab toolbox `mpspack` [3]. The weighted-Neumann-to-Dirichlet scaling method is chosen with the argument `'ntd'` and $M = 100$ quadrature points are used. For the optimization problem (4.1), we use the line-search-based BFGS algorithm implemented in `HANSO` [14]. This quasi-Newton method has proven to be effective for non-smooth optimization problems such as (4.1) [9]. If λ_j is simple, the derivatives of $\lambda_j(\Omega)$ with respect to the coefficients a_k describing Ω can be found in, e.g., [11] and are given by

$$\frac{\partial \lambda_j}{\partial a_k} = -\int_0^{2\pi} R_N(\theta) e^{\imath k\theta} \left|\nabla\psi_j\big(R_N(\theta),\theta\big)\right|^2 d\theta. \quad (4.2)$$

In our computations, the Neumann data, ∇u_j, is evaluated at the quadrature points and the integral in (4.2) is evaluated via quadrature. We remark that while the derivative of an eigenvalue with higher multiplicity can be computed (see, e.g., [6]), in numerical computations roundoff error causes all eigenvalues to be simple.

To address the questions considered in this paper, we solve the (α,β)-parameterized optimization problem (4.1) for many (≈ 300) values (α,β). We find the method described above to be extremely effective for this. Solving (4.1) requires on the order of 40 BFGS iterations, each requiring approximately 1–7 eigenvalue solutions for the line search. The solution to (4.1) takes approximately 2 minutes using Matlab 2012b on a 2.0 GHz Intel Core i7 Duo desktop computer with 8GB of RAM.

4.2 Computational Results

For $j = 1$ and 2, we use the computational method described above to solve (4.1) for approximately 50 specified values $(\alpha,\beta) \in T$. The results for $j = 1$ are displayed in Table 1 (bottom) and Fig. 3. The results for $j = 2$ are displayed in Table 2 (bottom) and Fig. 4. All reported values are rounded to four significant digits. Note that all values attained for the admissible class $\mathcal{F}_{10} \sqcup \mathcal{F}_{10}$ are at least as small as those for balls, $\mathcal{B} \sqcup \mathcal{B}$. In what follows, we refer to numerically computed solutions as minimizers.

Table 1 (*top*) The value of $\inf\{C^1_{\alpha,\beta}(\Omega) : \Omega \in \mathcal{B} \sqcup \mathcal{B}\}$ and (*bottom*) $\inf\{C^1_{\alpha,\beta}(\Omega) : \Omega \in \mathcal{F}_{10} \sqcup \mathcal{F}_{10}\}$ for $(\alpha,\beta) \in T$. *Italic* entries are values for which Proposition 2 implies the ball is a local minimizer of $C^1_{\alpha,\beta}(\Omega)$ over $\mathcal{F}_\infty$. See Sect. 4

1	36.34								
0.875	43.33	36.34							
0.75	46.12	42.63	36.34						
0.625	46.12	42.63	39.14	35.64					
0.5	*46.12*	42.63	39.14	35.64	32.15				
0.375	*46.12*	*42.63*	*39.14*	35.64	32.15	28.65			
0.25	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	28.65	25.16		
0.125	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	*28.65*	*25.16*	21.66	
0	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	*28.65*	*25.16*	*21.66*	*18.17*
β/α	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1

1	36.34								
0.875	40.09	35.49							
0.75	43.17	38.80	33.85						
0.625	45.73	41.64	37.12	31.85					
0.5	*46.12*	42.52	38.67	34.44	29.60				
0.375	*46.12*	*42.63*	*39.14*	35.51	31.57	27.12			
0.25	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	28.49	24.42		
0.125	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	*28.65*	*25.16*	21.46	
0	*46.12*	*42.63*	*39.14*	*35.64*	*32.15*	*28.65*	*25.16*	*21.66*	*18.17*
β/α	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1

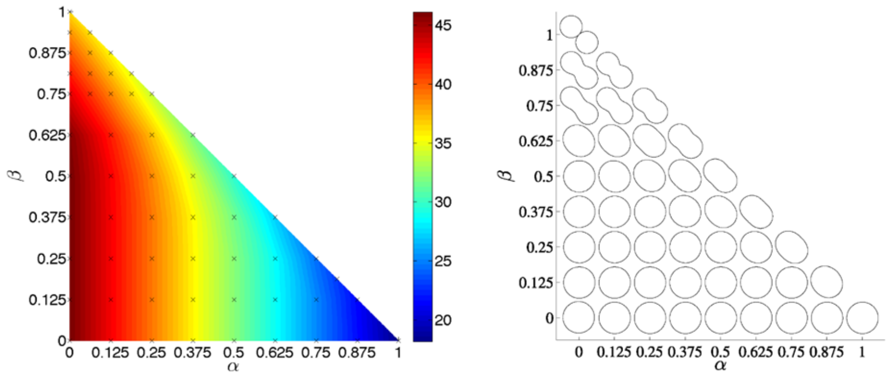

Fig. 3 (*left*) The value of $\inf\{C^1_{\alpha,\beta}(\Omega) : \Omega \in \mathcal{F}_{10} \sqcup \mathcal{F}_{10}\}$ and (*right*) corresponding minimizer for $(\alpha,\beta) \in T$. The values where (4.1) was solved are indicated with an 'x'. Other values are obtained by linear interpolation. See Sect. 4

Table 2 (*top*) The value of $\inf\{C^2_{\alpha,\beta}(\Omega)\colon \Omega \in \mathcal{B} \sqcup \mathcal{B}\}$ and (*bottom*) $\inf\{C^2_{\alpha,\beta}(\Omega)\colon \Omega \in \mathcal{F}_{10} \sqcup \mathcal{F}_{10}\}$ for $(\alpha,\beta) \in T$. Using the identity $C^2_{\beta,1-\beta} = C^1_{0,\beta}$ for $\beta \in [0,1]$, Proposition 2 applies to the *italic* entries, i.e., the ball is a local minimizer. See Sect. 4

1	*46.12*								
0.875	50.72	*46.12*							
0.75	55.31	50.72	*46.12*						
0.625	59.90	55.31	50.72	*46.12*					
0.5	64.29	59.90	55.31	50.72	*46.12*				
0.375	64.29	64.29	59.90	55.31	50.72	46.12			
0.25	64.29	64.29	64.29	59.90	55.31	50.72	46.12		
0.125	64.29	64.29	64.29	63.67	59.90	55.31	50.32	43.33	
0	64.29	64.29	64.29	63.67	61.10	56.68	50.32	43.33	36.34
β/α	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1

1	*46.12*								
0.875	50.55	*46.12*							
0.75	54.69	50.55	*46.12*						
0.625	58.60	54.69	50.55	*46.12*					
0.5	62.29	58.60	54.69	50.55	*46.12*				
0.375	64.29	61.50	57.95	54.16	50.10	45.73			
0.25	64.29	62.02	58.91	55.65	52.17	48.45	43.17		
0.125	64.29	62.02	58.91	55.71	52.43	49.05	45.54	40.09	
0	64.29	62.02	58.91	55.71	52.43	49.05	45.57	41.97	36.34
β/α	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1

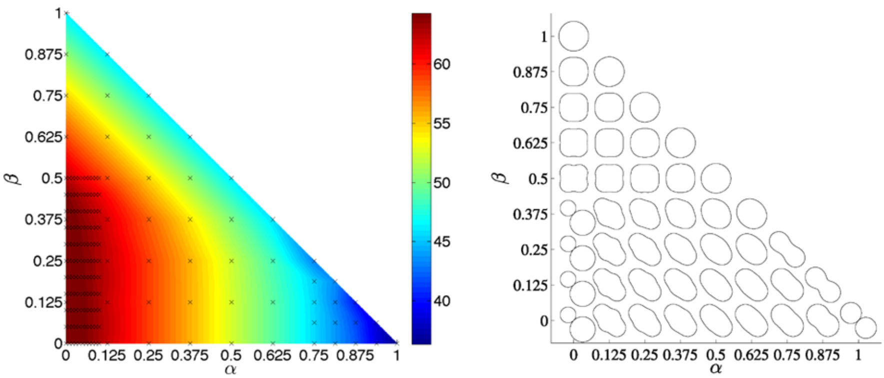

Fig. 4 (*left*) The value of $\inf\{C^2_{\alpha,\beta}(\Omega)\colon \Omega \in \mathcal{F}_{10} \sqcup \mathcal{F}_{10}\}$ and (*right*) corresponding minimizer for $(\alpha,\beta) \in T$. The values where (4.1) was solved are indicated with an 'x'. Other values are obtained by linear interpolation. See Sect. 4

For $j=1$, we observe the following.

(1) For (α,β)-values in the region $\{(\alpha,\beta)\in T:\alpha+2\beta\leq 1\}$, the optimal solution is a ball. The ball is shown to be a local minimizer in Proposition 2 (proven in Sect. 2).
(2) (Connectivity.) We observe that the optimal domain has one connected component except for $(\alpha,\beta)=(0,1)$. This supports a conjecture of Iversen and Mazzoleni [7].
(3) We observe numerically that the minimizer is unique and continuously varies with respect to α and β.
(4) (Symmetry.) For all (α,β) values considered, the minimizer has two axis of symmetry.
(5) (Eigenvalue multiplicity.) For (α,β)-values in the region $\{(\alpha,\beta)\in T:\alpha+2\beta\leq 1\}$, the optimal solution is a ball with $\lambda_1<\lambda_2=\lambda_3<\lambda_4$. For $(\alpha,\beta)=(0,1)$, the optimal solution is two balls of equal measure with $\lambda_1=\lambda_2<\lambda_3=\lambda_4$. For all other (α,β)-values considered, the first four eigenvalues of the optimal domain are each simple.
(6) (Comparison to Proposition 3.) For $\Omega\in\mathcal{B}\sqcup\mathcal{B}$ and $(\alpha,\beta)\in T_1^1$ as defined in Proposition 3, the optimal shape is a ball with $\lambda_1<\lambda_2=\lambda_3<\lambda_4$. For $(\alpha,\beta)\in T_2^1$, the minimizer is the disjoint union of two balls of equal measure with $\lambda_1=\lambda_2<\lambda_3=\lambda_4$.

For $j=2$, we observe the following.

(1) For $\alpha+\beta=1$ and $0\leq\beta\leq\frac{1}{2}$, the ball is a local minimizer. This follows from the identity $C^2_{\beta,1-\beta}=C^1_{0,\beta}$ for $\beta\in[0,1]$ and Proposition 2.
(2) (Connectivity.) We observe that the optimal domain has one connected component except for $(\alpha,\beta)=(1,0)$ and in a neighborhood of $(\alpha,\beta)=(0,0)$. We conjecture that the (α,β)-region containing $(1,0)$ with disconnected minimizer consists only of the isolated point $(1,0)$. To investigate the region near $(\alpha,\beta)=(0,0)$ further, we solve (4.1) 121 additional times for a selection of values $(\alpha,\beta)\in[0,0.1]\times[0,0.5]\subset T$. The optimal values and minimizers are plotted in Fig. 5. The black line is the intersection of the linear interpolation of obj. function values for one- and two-component regions. We observe that the optimal shape has two connected components for $\alpha\lesssim 0.03$ and $\beta\lesssim 0.4$.
(3) We observe numerically that the minimizer is unique except along the (α,β)-curve shown in Fig. 5 (left) separating the minimizers with one and two connected components. For (α,β) values on this curve, the optimal set $\hat{\Omega}^2_{\alpha,\beta}$ consist of a domain with one connected component and a two-connected component domain. Away from the curve, the minimizer varies continuously with respect to α and β.
(4) (Symmetry.) For all (α,β) values considered, connected minimizers have two axis of symmetry. The disconnected minimizers for (α,β) values near $(0,0)$ have only one axis of symmetry.
(5) (Eigenvalue multiplicity.) For (α,β)-values in the region $\{(\alpha,\beta)\in T:\alpha+\beta=1,\ \alpha\leq\frac{1}{2}\}$, the optimal solution is a ball with $\lambda_1<\lambda_2=\lambda_3<\lambda_4=\lambda_5$. For (α,β)-values in the region $\{(\alpha,\beta)\in T:\alpha\lesssim 0.03,\ \beta\lesssim 0.4\}$, the solution is the disjoint union of two balls of different measure with $\lambda_1<\lambda_2=\lambda_3=\lambda_4<\lambda_5$.

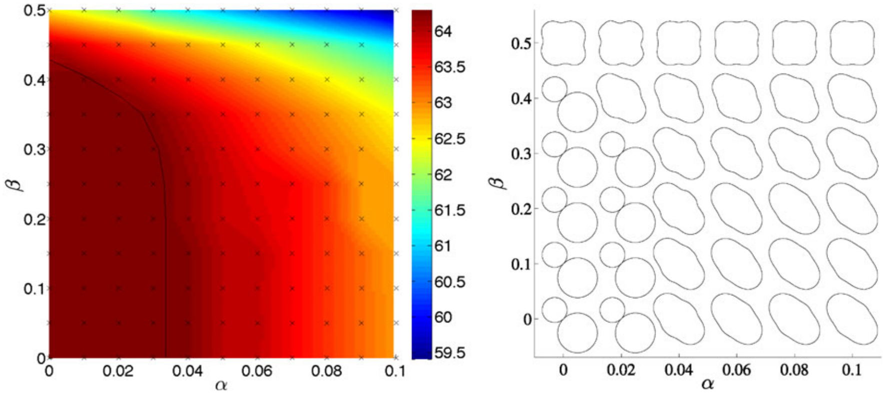

Fig. 5 An enlargement of the (α, β)-region near $(0, 0)$ in Fig. 4

For $(\alpha, \beta) = (1, 0)$, the optimal solution is two balls of equal measure with $\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$. For all other (α, β)-values considered, the first five eigenvalues of the optimal domain are each simple.

(6) (Comparison to Proposition 6.) For $\Omega \in \mathcal{B} \sqcup \mathcal{B}$ and $(\alpha, \beta) \in T_1^2 \cup T_2^2 \cup T_4^2$ as defined in Proposition 6, the optimal shape has two connected components. For the more general admissible class $\mathcal{F}_N \sqcup \mathcal{F}_N$ however, the region where the optimal shape has two connected components is relatively small. For example, for $(\alpha, \beta) = (0.5, 0)$, the optimal union of balls has two components while the minimizer over $\mathcal{F}_N \sqcup \mathcal{F}_N$ has just one. In Fig. 2, the minimizer has $\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < \lambda_5$ in region T_1^2, $\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$ in region T_2^2, $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 = \lambda_5$ in region T_3^2, and $\lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < \lambda_5$ in region T_4^2.

When comparison is available, our results agree with those for minimizing single eigenvalues [1, 13], the mean of sequential eigenvalues [2], and convex combinations of two sequential eigenvalues [12, 15]. In particular, we recover the results

$$\lambda_1^\star = \pi j_{0,1}^2 \approx 18.17, \qquad \lambda_2^\star = 2\pi j_{0,1}^2 \approx 36.34, \qquad \lambda_3^\star = \pi j_{1,1}^2 \approx 46.12, \quad \text{and}$$
$$\lambda_4^\star = \pi\left(j_{1,1}^2 + j_{0,1}^2\right) \approx 64.29,$$

where $\lambda_j^\star$ is the optimal j-th eigenvalue. In [12], we numerically observed that for $j = 2:5$, $C_{\alpha,1-\alpha}^{j*}$ is constant on the interval $\alpha \in [0, \delta]$ for some constant $\delta = \delta(j) > 0$. Recalling the identity $C_{\alpha,1-\alpha}^{j+1} = C_{0,\alpha}^{j}$, in the present context, this implies that $C_{\alpha,\beta}^{j*}$ is constant on the line segments $\{(\alpha, \beta) : \alpha + \beta = 1,\ \alpha \in [0, \delta]\}$ and $\{(\alpha, \beta) : \alpha = 0,\ \beta \in [0, \delta]\}$ for $j = 2, 3, 4$. For these larger j-values, it would be interesting to see whether these line segments can be extended to regions $\alpha + \beta < 1$ and $\alpha > 0$ respectively.

5 Discussion and Further Directions

We have studied the shape optimization problem of minimizing the convex combination of three sequential Laplace-Dirichlet eigenvalues, $C^j_{\alpha,\beta}(\Omega) := \alpha\lambda_j(\Omega) + \beta\lambda_{j+1}(\Omega) + (1-\alpha-\beta)\lambda_{j+2}(\Omega)$ over several different admissible sets. In particular, we compare the values and minimizers of $C^j_{\alpha,\beta}(\Omega)$ for $j = 1, 2$ for the two admissible sets: disjoint unions of domains with smooth boundary, $\mathcal{F}_N \sqcup \mathcal{F}_N$, and disjoint unions of balls, $\mathcal{B} \sqcup \mathcal{B}$. We have tried to catalogue properties of the optimizers in hope that our observations stimulate interesting future analytical development in this area.

We conclude with a brief qualitative comparison of the computational method used in the present work and the method recently introduced in [12]. There are two primary differences between these two approaches: (i) in the present work the eigenvalue problem is solved using boundary integral methods, while in [12] it is solved using finite element methods and (ii) in the present work, we have represented the domain using Fourier coefficients, while in [12] the domain is represented using the level set method. We have found the finite element method to be more robust, but much slower and less accurate than the boundary element method. The level set method has the advantage of not fixing the topology of the domain. However, currently available methods for solving the eigenvalue problem require either extracting points on the boundary or a parameterization of the boundary. Thus, each iteration of a gradient-based optimization method requires a rootfinding algorithm to find approximate points on the boundary. We view the problem of finding a method which utilizes the level-set function representation of the domain, but doesn't require such rootfinding at each iteration to be a challenging extension of this work.

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