807 HW2

October 20, 2008

1. Find the eigenvalue and eigenvector of the tridiagonal matrix

$$\begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & a & b & \\ & & & \cdot & \\ & & & c & a \end{bmatrix}.$$

(Hint:) Let λ be the eigenvalue and

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ : \\ v_n \end{bmatrix}$$

be the corresponding eigenvector. If we define $v_0 = v_{n+1} = 0$, we have the difference equation

$$cv_{j-1} + (a - \lambda)v_j + bv_{j+1} = 0, \quad j = 1, 2, ..., N.$$

Solve this difference equation.

2. Consider the model problem for convection diffusion

$$u_t + au_x = \eta u_{xx}, \quad -\infty < x < \infty, \quad 0 \leq t, \quad a, \eta = const > 0$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

with $2\pi-$periodic initial data. Consider the difference approximation

$$v_j^{n+1} = v_j^n + k(\eta D_+ D_- - aD_0)v_j^n, \quad j = 0, 1, 2..., N$$

which is obtained from central discreization in space and forward discretization in time. Let $\lambda = \frac{ak}{h}$ and $\alpha = \frac{2\eta k}{h^2}$. Use Von Neumann's method to derive the condition on α and λ for the scheme to be stable.

3. Consider the model problem

$$u_t = u_{xx}, 0 \leq x \leq 1.$$

Suppose we use the scheme

$$\frac{3}{2}\left(\frac{u_i^{j+1}-u_i^j}{k}\right)-\frac{1}{2}\left(\frac{u_i^j-u_i^{j-1}}{k}\right)=\frac{1}{h^2}\left(u_{i+1}^j-2u_i^j+u_{i-1}^j\right).$$

(a)Show that the scheme is consistent. (b) Investigate the stability by the matrix method.

4. Consider the model problem for convection diffusion

$$u_t = u_x, \quad -\infty < x < \infty, \quad 0 \le t,$$

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

with $2\pi-$periodic initial data. Consider the difference approximation

$$v_j^{n+1} = (1 + kD_0)v_j^n + \sigma kh(D_+D_-)v_j^n, \quad j = 0, 1, 2..., N$$

which is central discretization is space and forward discretization in time. However, there is an extra term (last term in the right hand side) which is so-called artificial viscosity. Show that the truncation error is

$$\tau_j^n = \frac{u_j^{n+1}-u_j^n}{k} - D_0u_j^n - \sigma hD_+D_-u_j^n = \left(\frac{k}{2} - \sigma h\right) u_{xx}(x_j, t_n) + O(h^2 + k^2)$$

We say that the mathod is accurate of order (p,q) if $\tau = O(h^p + k^q)$. For $\sigma \neq \frac{k}{2h}$, the method above is accurate of order (1,1). For $\sigma = \frac{k}{2h}$, it is the so-called Lax-Wendroff method, the order of accuracy is $(2, 2)$.

5 What explicit method could be used for the Schrodinger type equation

$$u_t = iu_{xx}, \quad i = \sqrt{-1}?$$

Derive the stability condition.

6. Write a code that solve

$$u_t = u_{xx} \quad (0 < x < 1)$$

with the initial condition

$$\begin{cases} u = 2x & , \quad 0 \le x \le \frac{1}{2} \\ u = 2(1-x) & , \quad \frac{1}{2} \le x \le 1 \end{cases},$$

and boundary condition

$$u(0,t) = 0, \quad u(1,t) = 0$$

by Dufort-Frankel method. (The first step numerical approximation is given by Crank-Nicolson method). Choose $\delta x = \frac{1}{10}$, $\delta t = \frac{1}{100}$. Provide the solution at $t = 0.01$, 0.1, and 0.2.

7. Write a code to implement ADI method for

$$u_t = (u_{xx} + u_{yy}) \quad (x, y) \in \Omega = (0, 1) \times (0, 1)$$

with the initial condition

$$u(x, y, 0) = 1 \quad \text{for} \quad (x, y) \in \Omega$$

and boundary condition

$$u(x, y, t)|_{\partial\Omega} = 0.$$

Choose $\delta x = \frac{1}{10}$, $\delta t = \frac{1}{100}$. Provide the solution at $t = 0.01$, 0.1, and 0.2.