Approaches to Gambler's Ruin with Catastrophe

Blake Hunter
Alan Krinik
Chau Nguyen
Jenny Switkes
Hubertus von Bremen

Constant Transition Rates

Consider the one-step transition matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b+c & 0 & a & 0 & 0 & \dots & 0 & 0 & 0 \\ c & b & 0 & a & 0 & \dots & 0 & 0 & 0 \\ c & 0 & b & 0 & a & \dots & 0 & 0 & 0 \\ c & 0 & 0 & b & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ c & 0 & 0 & 0 & 0 & \dots & 0 & a & 0 \\ c & 0 & 0 & 0 & 0 & \dots & b & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

on the state space $\{0, 1, 2, \ldots, N+1\}$. This Markov chain represents the "Gambler's Ruin" problem with catastrophe, as shown in Figure 1. Each entry a_{ij} gives the probability of moving from state i to state j in a single step, given that the current state is i. The probability of moving in one step from state 1 to state 0, for instance, is $b+c$ while the probability of moving in one step from state 1 to state 2 is a; there is no probability of moving anywhere else in one step from state 1. By assumption, $a+b+c=1$ since we must move somewhere.

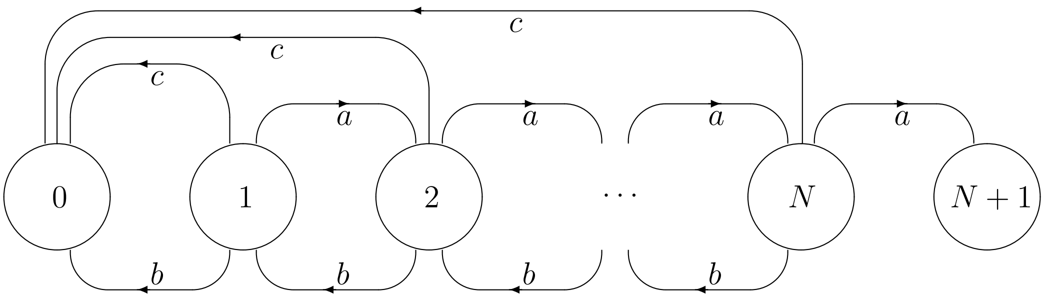
Let

$$p_k = \text{Prob(eventual absorption at state } N+1 \mid \text{initially at state } k).$$

The ruin probabilities are then

$$\pi_k = \text{Prob(eventual ruin} \mid \text{initially at state } k) = 1 - p_k.$$

Figure 1: “Gambler’s Ruin” Markov Chain with Constant Transition Rates.



Difference Equation Approach

Conditioning on the first step,

$$
\begin{aligned}
p_0 &= 0 \\
p_1 &= (b+c)p_0 + ap_2 = bp_0 + ap_2 \\
p_2 &= cp_0 + bp_1 + ap_3 = bp_1 + ap_3 \\
&\vdots \\
p_N &= bp_{N-1} + ap_{N+1} \\
p_{N+1} &= 1
\end{aligned}
$$

where in several places we have used $p_0 = 0$. For instance, p_2 is the probability of eventual absorption at state $N+1$ given that we are initially in state 2. After one step, with probability c we will be in state 0, from which we have no chance to escape; with probability b we will be in state state 1, from which we have probability p_1 of eventual absorption at state $N+1$; and with probability a we will be in state 3, from which we have probability p_3 of eventual absorption at state $N+1$.

Thus, for $k = 1, \ldots, N$,

$$ap_{k+1} - p_k + bp_{k-1} = 0. \tag{1}$$

This is a linear constant-coefficient difference equation. The roots of the characteristic polynomial are

$$r_{1,2} = \frac{1 \pm \sqrt{1-4ab}}{2a},$$

and so

$$p_k = C_1 \left[\frac{1+\sqrt{1-4ab}}{2a}\right]^k + C_2 \left[\frac{1-\sqrt{1-4ab}}{2a}\right]^k.$$

Applying the conditions $p_0 = 0$ and $p_{N+1} = 1$,

$$p_k = \frac{\left[\frac{1+\sqrt{1-4ab}}{2a}\right]^k - \left[\frac{1-\sqrt{1-4ab}}{2a}\right]^k}{\left[\frac{1+\sqrt{1-4ab}}{2a}\right]^{N+1} - \left[\frac{1-\sqrt{1-4ab}}{2a}\right]^{N+1}}$$

and so the ruin probabilities are

$$\pi_k = 1 - p_k = 1 - \frac{\left[\frac{1+\sqrt{1-4ab}}{2a}\right]^k - \left[\frac{1-\sqrt{1-4ab}}{2a}\right]^k}{\left[\frac{1+\sqrt{1-4ab}}{2a}\right]^{N+1} - \left[\frac{1-\sqrt{1-4ab}}{2a}\right]^{N+1}}. \tag{2}$$

We can use the Binomial Theorem to re-write this result without using radicals:

$$\begin{aligned}
\left(1+\sqrt{x}\right)^k &= \sum_{i=0}^{k} \binom{k}{i} \left(\sqrt{x}\right)^i \\
\left(1-\sqrt{x}\right)^k &= \sum_{i=0}^{k} (-1)^k \binom{k}{i} \left(\sqrt{x}\right)^i \\
\left(1+\sqrt{x}\right)^k - \left(1-\sqrt{x}\right)^k &= 2 \sum_{\substack{i\,=\,0 \\ i \text{ odd}}}^{k} \binom{k}{i} x^i \\
&= 2\sqrt{x} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} x^j
\end{aligned}$$

where $\lfloor y \rfloor$ denotes the greatest integer $\leq y$. The resulting alternative form for the ruin probabilities is

$$\pi_k = 1 - p_k = 1 - \left[(2a)^{N+1-k} \cdot \frac{\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} (1-4ab)^j}{\sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2j+1} (1-4ab)^j} \right]. \tag{3}$$

System of Linear Equations

In the previous section we found that, for $k = 1, \ldots, N$,

$$ap_{k+1} - p_k + bp_{k-1} = 0,$$

along with $p_0 = 0$ and $p_{N+1} = 1$.

Rather than using difference equation techniques, we can solve for the probabilities p_k by solving the related system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b & -1 & a & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & b & -1 & a & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & b & -1 & a & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & b & -1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & b & -1 & a \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \\ p_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

For any specific value of $N+1$, this system may be solved using a forward substitution technique to solve for $p_2, p_3, \dots, p_{N+1}$ in terms of p_1, after which we can set $p_{N+1} = 1$ to solve for p_1.

For general $N+1$, the pattern in the formula for p_k as a function of p_1 is not easy to find. However, we can find this pattern by our work in the previous section.

Consider our result from the previous section,

$$p_k = (2a)^{N+1-k} \cdot \frac{\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} (1-4ab)^j}{\sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2j+1} (1-4ab)^j}.$$

Upon substituting $k=1$ and simplifying we have that

$$p_1 = \frac{a^N}{\frac{1}{2^N} \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2j+1} (1-4ab)^j}.$$

If we were to use the forward substitution idea to solve the system, we would eventually obtain

$$p_{N+1} = f(N+1) p_1 = 1$$

for some function f. Now we can solve explicitly for f:

$$f(N+1) \cdot \frac{a^N}{\frac{1}{2^N} \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2j+1} (1-4ab)^j} = 1$$

and so

$$f(N+1) = \frac{\frac{1}{2^N} \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N+1}{2j+1} (1-4ab)^j}{a^N}.$$

In general, then,

$$\begin{aligned} p_k &= f(k)p_1 \\ &= \frac{\frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \begin{pmatrix} k \\ 2j+1 \end{pmatrix} (1-4ab)^j}{a^{k-1}} \cdot p_1 \end{aligned}$$

and so the ruin probabilities are related through

$$1 - \pi_k \quad = \quad f(k)(1 - \pi_1)$$

Solving for π_k as a function of π_1,

$$\pi_k = 1 - \frac{\frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \cdot \begin{pmatrix} k \\ 2j+1 \end{pmatrix} (1-4ab)^j}{a^{k-1}} \cdot (1 - \pi_1). \tag{5}$$

Given that we already had a closed-form solution for π_k from our work in the previous section, our reason for solving for π_k as a function of π_1 here was to look at this special case of state-independent transition rates using a structure that will prove to be helpful in investigating state-dependent rates.

State-Dependent Transition Rates

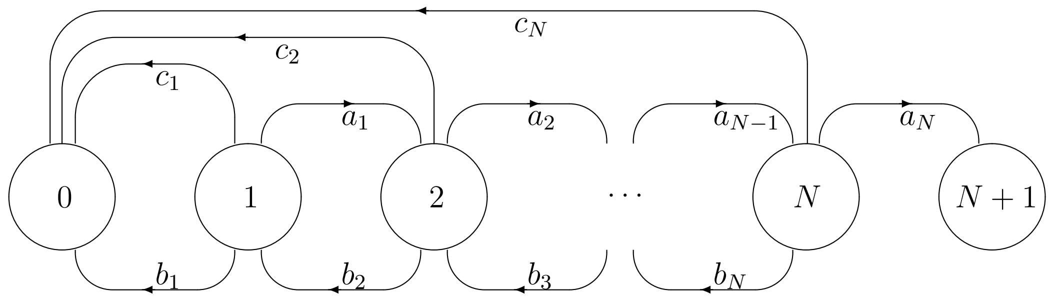
Suppose now that the transition rates are state-dependent. Consider the transition matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_1 + c_1 & 0 & a_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ c_2 & b_2 & 0 & a_2 & 0 & \dots & 0 & 0 & 0 \\ c_3 & 0 & b_3 & 0 & a_3 & \dots & 0 & 0 & 0 \\ c_4 & 0 & 0 & b_4 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ c_{N-1} & 0 & 0 & 0 & 0 & \dots & 0 & a_{N-1} & 0 \\ c_N & 0 & 0 & 0 & 0 & \dots & b_N & 0 & a_N \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

representing the Markov chain shown in Figure 2.

The corresponding difference equation is no longer constant-coefficient. However, the corresponding linear system

Figure 2: “Gambler’s Ruin” with State-Dependent Transition Rates.



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_1 & -1 & a_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & b_2 & -1 & a_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & b_3 & -1 & a_3 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & -1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & a_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & b_N & -1 & a_N \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \\ p_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

can still be solved using a forward substitution technique to solve for p_2, p_3, $\dots$, p_{N+1} in terms of p_1, after which we can again set $p_{N+1} = 1$ to solve for p_1.

Below, we will prove that, for $k \geq 1$,

$$\begin{aligned} p_k \quad = \quad & \Bigg[1 - \sum_{i=1}^{k-2} a_i b_{i+1} + \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3} \right] \\ & - \sum_{i=1}^{k-6} a_i b_{i+1} \left[\sum_{j=i}^{k-6} a_{j+2} b_{j+3} \left[\sum_{l=j}^{k-6} a_{l+4} b_{l+5} \right] \right] + \dots \Bigg] p_1 / \left[\Pi_{i=1}^{k-1} a_i \right], \end{aligned} \tag{6}$$

with the convention that summations are set to zero if they have an upper limit of summation that is less than the lower limit.

Multiplying equation (6) by $\Pi_{i=1}^{k-1} a_i$,

$$\begin{aligned} p_k \cdot \Pi_{i=1}^{k-1} a_i &= p_1 \cdot \left[1 - \sum_{i=1}^{k-2} a_i b_{i+1} + \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3} \right] \right. \\ &\qquad \left. - \sum_{i=1}^{k-6} a_i b_{i+1} \left[\sum_{j=i}^{k-6} a_{j+2} b_{j+3} \left[\sum_{l=j}^{k-6} a_{l+4} b_{l+5} \right] \right] + \dots \right]. \end{aligned}$$

Solving for p_1,

$$\begin{aligned} p_1 &= p_k \cdot \Pi_{i=1}^{k-1} a_i + p_1 \cdot \left[\sum_{i=1}^{k-2} a_i b_{i+1} - \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3} \right] \right. \\ &\qquad \left. + \sum_{i=1}^{k-6} a_i b_{i+1} \left[\sum_{j=i}^{k-6} a_{j+2} b_{j+3} \left[\sum_{l=j}^{k-6} a_{l+4} b_{l+5} \right] \right] - \dots \right]. \end{aligned} \tag{7}$$

We will prove (6) by proving (7). The quantity p_1 on the left-hand side of (7) is the probability of eventual success starting from state 1, where "success" is reaching state $N+1$. I will prove the identity by showing that the right hand side of (7) also gives the probability of eventual success starting from state 1.

The term

$$p_k \cdot \Pi_{i=1}^{k-1} a_i$$

gives the probability of eventual success from state 1 with no "up-and-down" before reaching state k. To see this, note that $\Pi_{i=1}^{k-1} a_i$ is the probability of "walking" directly from state 1 to state 2, ..., to state k. The factor p_k then gives the probability of success from state k.

The other terms on the right hand side give the probability of eventual success from state 1 with one or more incidents of "up-and-down" before reaching state k. I am defining "up-and-down from state i," with $i \leq k-2$, as movement from state i directly to state $i+1$ and directly back to state i.

For $i \leq k-2$, the probability that a randomly-chosen path is eventually successful but involves movement from the first occurrence of state i directly to state $i+1$ and directly back to state i is given by $p_1 a_i b_{i+1}$. To see this, note that, starting from state 1,

Prob(a randomly-chosen path reaches state i for the first time by some means, then does an "up-and-down," and then continues on to eventually reach state $N+1$)

is equal to

Prob(a randomly-chosen path reaches state i for the first time by some means and from there reaches state $N+1$ by some means)$\cdot$ Prob(from state i, the first two steps are an "up-and-down")

which is equal to

Prob(a randomly-chosen path reaches state $N+1$ by some means)$\cdot$ Prob(when the path reaches state i for the first time, the first two steps are an "up-and-down")

which is equal to

$$p_1 a_i b_{i+1}$$

since p_1 is the probability of eventual successful movement to state $N+1$ from initial state 1 and $a_i b_{i+1}$ is the probability that from state i the next two steps are an "up-and-down."

Now,

$$\sum_{i=1}^{k-2} a_i b_{i+1} - \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3} \right] + \sum_{i=1}^{k-6} a_i b_{i+1} \left[\sum_{j=i}^{k-6} a_{j+2} b_{j+3} \left[\sum_{l=j}^{k-6} a_{l+4} b_{l+5} \right] \right] - \dots$$

is simply an application of the Inclusion/Exclusion Principle. The summation

$$\sum_{i=1}^{k-2} a_i b_{i+1}$$

gives the sum of the probabilities that from state i, the next two steps are an "up-and-down," for states i from 1 to $k-2$. However, for non-adjacent states i, j, it is possible for a path to have an "up-and-down" occur at the first occurrence of both of these. The double summation

$$\sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3} \right]$$

is subtracted off since it is the sum of probabilities of "up-and-downs" occurring at the first occurrences of two different (non-adjacent) states. The upper limit of summation has been decreased to $k-4$ to allow "space" for both "up-and-downs" to occur at states $\leq k-2$. The triple summation

$$\sum_{i=1}^{k-6} a_i b_{i+1} \left[\sum_{j=i}^{k-6} a_{j+2} b_{j+3} \left[\sum_{l=j}^{k-6} a_{l+4} b_{l+5} \right] \right]$$

then is added back in since it is the sum of probabilities of "up-and-downs" occurring at the first occurrences of three different (pairwise non-adjacent) states. This Inclusion/Exclusion is continued in order to compute correctly a quantity that, when multiplied by p_1, gives the probability of eventual success from state 1 with some "up-and-down" before reaching state k. Adding in the first term on the right hand side of (7), we obtain the total probability of success from initial state 1, with or without "up-and-down" before reaching state k, as given on the left hand side of (7). This completes the proof of (7), and thus (6) is proven.

Now, setting $k = N+1$ in (6) and recalling that $p_{N+1} = 1$,

$$\begin{aligned} p_{N+1} &= \Bigg[1 - \sum_{i=1}^{N-1} a_i b_{i+1} + \sum_{i=1}^{N-3} a_i b_{i+1} \left[\sum_{j=i}^{N-3} a_{j+2} b_{j+3}\right] \\ &\qquad - \sum_{i=1}^{N-5} a_i b_{i+1} \left[\sum_{j=i}^{N-5} a_{j+2} b_{j+3} \left[\sum_{l=j}^{N-5} a_{l+4} b_{l+5}\right]\right] + \ldots\Bigg] p_1 / \left[\Pi_{i=1}^{N} a_i\right] \\ &= 1. \end{aligned}$$

This allows us to solve for p_1, obtaining

$$\begin{aligned} p_1 &= \left[\Pi_{i=1}^{N} a_i\right] / \Bigg[1 - \sum_{i=1}^{N-1} a_i b_{i+1} + \sum_{i=1}^{N-3} a_i b_{i+1} \left[\sum_{j=i}^{N-3} a_{j+2} b_{j+3}\right] \\ &\qquad - \sum_{i=1}^{N-5} a_i b_{i+1} \left[\sum_{j=i}^{N-5} a_{j+2} b_{j+3} \left[\sum_{l=j}^{N-5} a_{l+4} b_{l+5}\right]\right] + \ldots\Bigg] \end{aligned}$$

Finally, this in turn implies that

$$p_k = \left[\frac{1 - \sum_{i=1}^{k-2} a_i b_{i+1} + \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3}\right] - \ldots}{1 - \sum_{i=1}^{N-1} a_i b_{i+1} + \sum_{i=1}^{N-3} a_i b_{i+1} \left[\sum_{j=i}^{N-3} a_{j+2} b_{j+3}\right] - \ldots}\right] \cdot \Pi_{i=k}^{N} a_i.$$

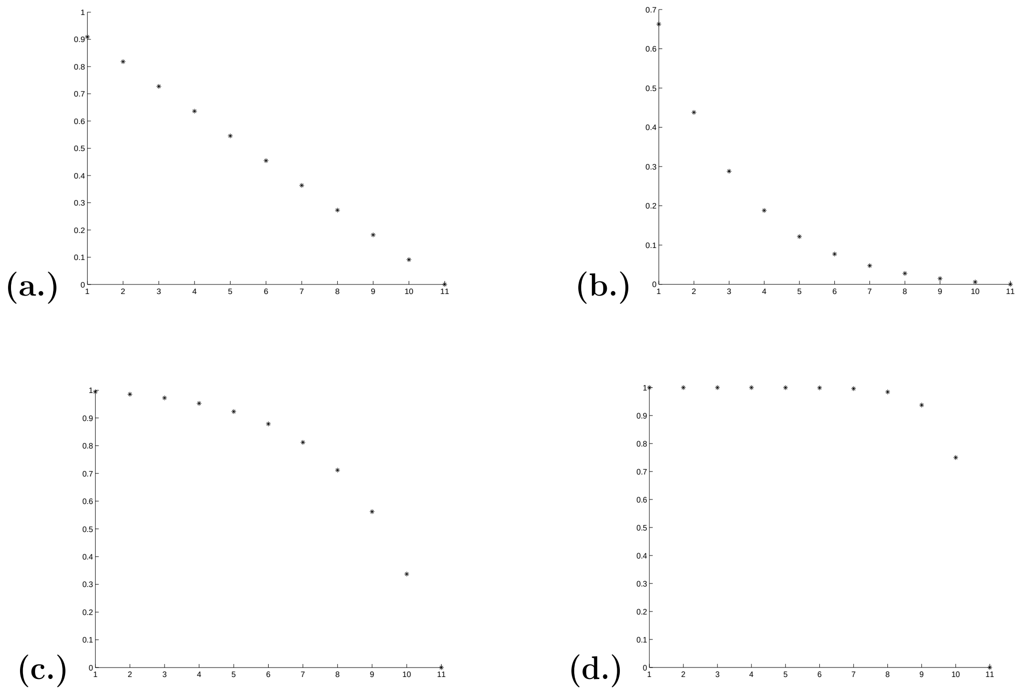
Setting $a_0 = 0$, this formula holds for $k = 0$ as well as for $k = 1, \ldots, N+1$. The ruin probabilities therefore are given by

$$\pi_k = 1 - \left[\frac{1 - \sum_{i=1}^{k-2} a_i b_{i+1} + \sum_{i=1}^{k-4} a_i b_{i+1} \left[\sum_{j=i}^{k-4} a_{j+2} b_{j+3}\right] - \ldots}{1 - \sum_{i=1}^{N-1} a_i b_{i+1} + \sum_{i=1}^{N-3} a_i b_{i+1} \left[\sum_{j=i}^{N-3} a_{j+2} b_{j+3}\right] - \ldots}\right] \cdot \Pi_{i=k}^{N} a_i.$$

Gallery of Plots for Gambler's Ruin with Catastrophe

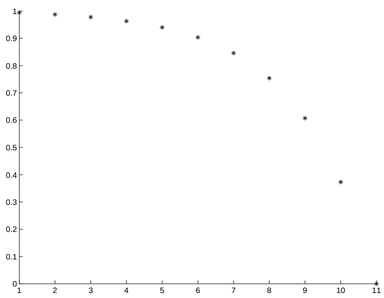
The plots below show some Gambler's Ruin results for various parameter sets. In Figure 3, we assume constant transition rates with no catastrophes.

Figure 3: Gambler's ruin with constant transition rates and no catastrophes. Values on the x-axis represent initial state and values on the y-axis represent corresponding ruin probabilities. In each plot, state 0 represents "ruin" and state 11 represents "success." In plot (a.), the "win" probability is $a = 0.5$ and the "loss" probability is $b = 0.5$. In plot (b.), the "win" probability is $a = 0.6$ and the "loss" probability is $b = 0.4$. In plot (c.), the "win" probability is $a = 0.4$ and the "loss" probability is $b = 0.6$. In plot (d.), the "win" probability is $a = 0.2$ and the "loss" probability is $b = 0.8$.

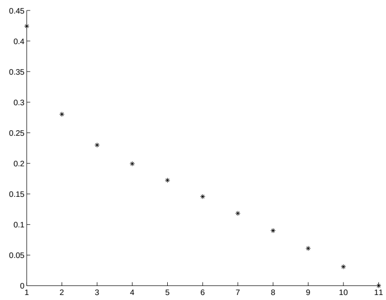


In Figure 4, we assume constant transition rates with catastrophes. Finally, in Figure 5 we give two examples of state-dependent transition rates.

Figure 4: Gambler's ruin with constant transition rates and catastrophes. In plot (a.), the "win" probability is $a = 0.45$, the "loss" probability is $b = 0.45$, and the catastrophe rate is $c = 0.1$. In plot (b.), the "win" probability is $a = 0.8$, the "loss" probability is $b = 0.18$, and the catastrophe rate is $c = 0.02$.

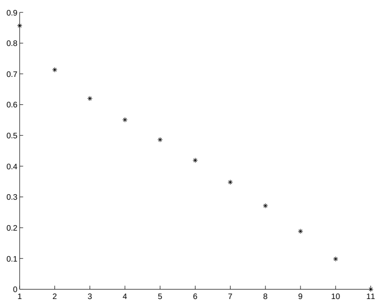


(a.)

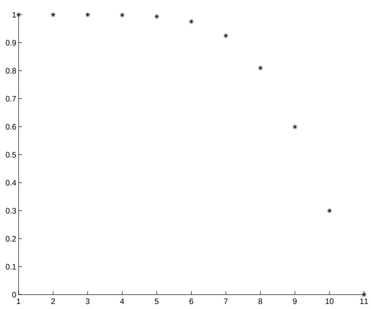


(b.)

Figure 5: Gambler's ruin with state-dependent transition rates. In plot (a.), the "win" probability is $a_j = j/(j+1)$, the "loss" probability is $b_j = 0.9 - j/(j+1)$, and the catastrophe rate is $c_j = 0.1$. In plot (b.), the "win" probability is $a_j = 0.05j$, the "loss" probability is $b_j = 0.05j$, and the catastrophe rate is $c_j = 1 - 0.1j$.



(a.)



(b.)